Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 1, pp. 88–102
https://doi.org/10.31392/MFAT-npu26_1.2021.10

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

TENSOR PRODUCT AND VARIANTS OF WEYL'S TYPE THEOREM FOR p-w-HYPONORMAL OPERATORS

M. H. M. RASHID

Abstract. A Hilbert space operator T is said to be p-w-hyponormal with $0 < p \leq 1$ if $|\widetilde{T}|^p \geq |T|^p \geq |\widetilde{T}^*|^p$, where $\widetilde{T}$ is the Aluthge transform. In this paper we prove basic properties of these operators. Using these results, we also prove that if P is a Riesz idempotent for a non-zero isolated point λ of the spectrum of T, then P is self-adjoint. Among other things, we prove these operators are finitely ascensive and that, for non-zero p-w-hyponormal T and S, $T \otimes S$ is p-w-hyponormal if and only if T and S are p-w-hyponormal. Moreover, it is shown that property (gt) holds for $f(T)$, where $f \in H_{nc}(\sigma(T))$.

Оператор T у гільбертовім просторі називається p-w-гіпонормальним, де $0 < p \leq 1$, якщо $|\widetilde{T}|^p \geq |T|^p \geq |\widetilde{T}^*|^p$, де $\widetilde{T}$ — перетворення Алутге. В цій роботі досліджені основні властивості таких операторів. Показано також, що якщо P — ідемпотент Рісса, який відповідає ненульовій ізольованій точці λ спектру T, то оператор P самоспряжений. Доведено, що ці оператори мають скінченний підйом і що для ненульових p-w-гіпонормальних T і S, $T \otimes S$ є p-w-гіпонормальним тоді й тільки тоді, коли T і S p-w-гіпонормальні. Крім того, доведено, що властивість (gt) має місце для $f(T)$, де $f \in H_{nc}(\sigma(T))$.

1. Introduction

Let $\mathcal{X}$ (or $\mathcal{H}$) be a complex Banach (Hilbert, respectively) space and $\mathcal{B}(\mathcal{X})$ (or $\mathcal{B}(\mathcal{H})$) be the set of all bounded linear operators on $\mathcal{X}$ ($\mathcal{H}$, respectively). Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([27], [32], [41] and [44]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *positive*, $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [7, 22, 25, 28, 29, 38]. An operator T is said to be p-*hyponormal* if$(T^*T)^p \geq (TT^*)^p$ for $p \in (0, 1]$ and an operator T is said to be log-*hyponormal* if T is invertible and $\log |T| \geq \log |T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. Aluthge [6] defined the operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, called the Aluthge transformation of T. An operator T is said to be w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$. The operator $\widetilde{T}(s,t) = |T|^sU|T|^t$ is the generalized Aluthge transformation of T in [6]. The classes of log-hyponormal and w-hyponormal operators were introduced and their properties were studied in [8] and [9]. It is known that the square of a w-hyponormal operator is also w-hyponormal. In [9], Aluthge showed that the class of w-hyponormal operator properly contains the classes of p-hyponormal operators and log-hyponormal.

2020 *Mathematics Subject Classification.* 47B20, 47A63.
Keywords. Furuta inequality, Löwner-Heinz inequality, p-w-hyponormal, Tensor product; Weyl's Type theorems, property (gt).

In [21], [31], and [20] Yang Changsen, Li Haiying introduced a class of p-w-hyponormal $(0 < p < 1)$ which means that if $|\widetilde{T}|^p \geq |T|^p \geq |\widetilde{T^*}|^p$. In [7], they showed that there exists an invertible operator whose integer powers are all p-w-hyponormal. As a generalization of class p-w-hyponormal $(0 < p < 1)$ Li Haiying [30] introduced a new class called (s,p)-w-hyponormal which mean that if $|\widetilde{T}(s,s)|^p \geq |T|^{2sp} \geq |\widetilde{T^*}(s,s)|^p$ $(s > 0,\ 0 < p < 1)$. Clearly, if $s = \frac{1}{2}$, an (s,p)-w-hyponormal operator is p-w-hyponormal. That is to say, the class of (s,p)-w-hyponormal operators contains the class of p-w-hyponormal operators.

Throughout this paper, we shall denote the spectrum, the point spectrum and the isolated points of the spectrum of $T \in \mathcal{B}(\mathcal{H})$ by $\sigma(T), \sigma_p(T)$ and $iso\sigma(T)$, respectively. The range and the kernel of $T \in \mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{R}(T) = T\mathcal{H}$ and $\ker(T)$, respectively. We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The closure of a set S will be denoted by $\overline{S}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$.

In Section 2, we prove basic properties of p-w-hyponormal operators. Among other things, we prove these operators are finitely ascensive. Section 3 is devoted to characterize the quasinilpotent $H_0(T-\lambda) = \{x \in \mathcal{H} : \lim\limits_{n\longrightarrow\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}$ of p-w-hyponormal operators. Using the results established in Section 2, we also prove that if P is a Riesz idempotent for a non-zero isolated point λ of the spectrum of T, then P is self-adjoint and $\ker(T-\lambda) = \mathcal{R}(P) = \ker(T-\lambda)^*$. In Section 4, we prove that for non-zero p-w-hyponormal T and S, $T \otimes S$ is p-w-hyponormal if and only if T and S are p-w-hyponormal. Moreover, in Section 5, it is shown that property (gt) holds for $f(T)$, and f is an analytic function defined on an open neighborhood of the spectrum of T such that f is non constant on each of the components of its domain.

2. Spectral Properties of p-w-hyponormal operators

To prove our main Theorems, we need the following results.

Lemma 2.1. [33, Hansen's Inequality] *If $A, B \in \mathcal{B}(\mathcal{H})$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\alpha \geq B^*A^\alpha B \qquad \text{for all } \alpha \in (0,1].$$

Lemma 2.2. [34, Löwer-Heinz theorem] *$A \geq B \geq 0$ ensure $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

Theorem 2.3. [6] *Let $T \in \mathcal{B}(\mathcal{H})$. If T is p-hyponormal, then the following hold:*

(i) *$\widetilde{T}$ is $(p+\frac{1}{2})$-hyponormal for $0 < p \leq \frac{1}{2}$;*
(ii) *$\widetilde{T}$ is hyponormal for $\frac{1}{2} \leq p \leq 1$.*

Proposition 2.4. ([21]) *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent to each other:*

(i) *T is p-w-hyponormal;*
(ii) *$|T|^p \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{p}{2}}$ and $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{p}{2}} \geq |T^*|^p$;*
(iii) *$|\widetilde{T^*}^*|^p \geq |T^*|^p \geq |\widetilde{T^*}|^p$.*

Theorem 2.5. ([30]) *Let $T \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent:*

(i) *T is (s,p)-w-hyponormal;*
(ii) *$|T|^{2sp} \geq (|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{2}}$ and $(|T^*|^s|T|^{2s}|T^*|^s)^{\frac{p}{2}} \geq |T^*|^{2sp}$;*
(iii) *$|\widetilde{T^*}(s,s)^*| \geq |T^*|^{2sp} \geq |\widetilde{T^*}(s,s)|$.*

Lemma 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is an invertible (s,p)-w-hyponormal operator, then so is T^{-1}.*

Proof. Since $|T^{-1}| = |T^*|^{-1}$, $|T^{-1^*}| = |T|^{-1}$ and $T \geq I \Longleftrightarrow T^{-1} \leq I$. Applying (ii) of Theorem 2.5, we have

$$\begin{aligned}
&(|T^*|^s|T|^{2s}|T^*|^s)^{\frac{p}{2}} \geq |T^*|^{2sp} \\
&\Rightarrow |T^*|^{-sp}(|T^*|^{\frac{1}{2}}|T||T^*|^s)^{\frac{p}{2}}|T^*|^{-sp} \geq I \\
&\Rightarrow \left(|T^*|^{-sp}(|T^*|^s|T|^{2s}|T^*|^s)^{\frac{p}{2}}|T^*|^{-sp}\right)^{-1} \leq I \\
&\Rightarrow |T^*|^{sp}(|T^*|^s|T|^{2s}|T^*|^s)^{-\frac{p}{2}}|T^*|^{sp} \leq I \\
&\Rightarrow (|T^*|^{-s}|T|^{-2s}|T^*|^{-s})^{\frac{p}{2}} \leq |T^*|^{-2sp} \\
&\Rightarrow (|T^{-1}|^s|T^{-1^*}|^{2s}|T^{-1}|^s)^{\frac{p}{2}} \leq |T^{-1}|^{2sp}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&|T|^{2sp} \geq (|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{2}} \\
&\Rightarrow (|T|^s|T^*|^{2s}|T|^s)^{-\frac{p}{4}}|T|^{2sp}(|T|^s|T^*|^{2s}|T|^s)^{-\frac{p}{4}} \geq I \\
&\Rightarrow (|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{4}}|T|^{-2sp}(|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{4}} \leq I \\
&\Rightarrow |T^{*^{-1}}|^{2sp} \leq (|T|^s|T^*|^{2s}|T|^s)^{-\frac{p}{2}} \\
&\Rightarrow |T^{-1^*}|^{2sp} \leq (|T|^{-s}|T^*|^{-2s}|T|^{-s})^{\frac{p}{2}} \\
&\Rightarrow |T^{-1^*}|^{2sp} \leq (|T^{-1^*}|^s|T^{-1}|^{2s}|T^{-1^*}|^s)^{\frac{p}{2}}.
\end{aligned}$$

That is, T^{-1} is (s,p)-w-hyponormal operator. □

Letting $s = \frac{1}{2}$ in Lemma 2.6, we have immediately

Corollary 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is an invertible p-w-hyponormal operator, $0 < p \leq 1$, then so is T^{-1}.*

Lemma 2.8. *If T is p-w-hyponormal, then $\widetilde{T}$ is $\frac{p}{2}$-hyponormal, $\widetilde{\widetilde{T}}$ is $\frac{p+1}{2}$-hyponormal and $\widetilde{\widetilde{\widetilde{T}}}$ is hyponormal.*

Proof. The definition of p-w-hyponormal clearly implies that $\widetilde{T}$ is $\frac{p}{2}$-hyponormal. Since $\widetilde{T}$ is $\frac{p}{2}$-hyponormal , $\widetilde{\widetilde{T}}$ is $\frac{p+1}{2}$-hyponormal by Theorem 2.3, again by Theorem 2.3 $\widetilde{\widetilde{\widetilde{T}}}$ is hyponormal. □

An operator T is said to be normaloid if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T. The equality $\|T\| = r(T)$ was shown to hold in [51] for hyponormal operators, in [6] for p-hyponormal and in [8]. The next theorem shows that the equality holds for p-w-hyponormal operators.

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is p-w-hyponormal, then $\left\|\widetilde{\widetilde{\widetilde{T}}}\right\| = \left\|\widetilde{\widetilde{T}}\right\| = \left\|\widetilde{T}\right\| = \|T\| = r(T)$. That is, T is normaloid.*

Proof. Since $\widetilde{\widetilde{\widetilde{T}}}$ is hyponormal, $\left\|\widetilde{\widetilde{\widetilde{T}}}\right\| = r(\widetilde{\widetilde{\widetilde{T}}})$ by [51, Theorem 1]. The result follows by [8, Corollary 2.3] since $\sigma(T) = \sigma(\widetilde{T}) = \sigma(\widetilde{\widetilde{T}}) = \sigma(\widetilde{\widetilde{\widetilde{T}}})$. □

Theorem 2.10. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal operator. Then T is*

(a) *normal if $\sigma(T)$ is an arc or if $\sigma(T)$ has only a finite number of limit points;*
(b) *self-adjoint if $\sigma(T) \subset \mathbb{R}$;*
(c) *unitary if $\sigma(T)$ is contained in the unit circle.*

Proof. (a) If $\sigma(T)$ is an arc, then $\sigma(\widetilde{\widetilde{\widetilde{T}}})$ is an arc by [8, Corollary 2.3]. The hyponormality of $\widetilde{\widetilde{\widetilde{T}}}$ implies $\widetilde{\widetilde{\widetilde{T}}}$ is normal [51, Theorem 4]. Applying [31, Theorem 4.4] to $\widetilde{\widetilde{T}}$, we obtain $\widetilde{\widetilde{T}} = \widetilde{\widetilde{\widetilde{T}}}$ and thus $\widetilde{\widetilde{T}}$ is normal. Applying the same theorem to $\widetilde{T}$ firstly and to T secondly yields $\widetilde{\widetilde{T}} = \widetilde{T}$ and $T = \widetilde{T}$ and hence T is normal.
(b) If $\sigma(T) \subset \mathbb{R}$, then $\sigma(\widetilde{T}) \subset \mathbb{R}$ by [8, Corollary 2.3]. The p-hyponormality of $\widetilde{T}$ implies that $\widetilde{T}$ is self-adjoint and $\widetilde{T} = T$, so the result.
(c) Both T and T^{-1} are p-w-hyponormal and their spectra are subsets of $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Consequently, by Theorem 2.9 $\|T\| = \left\|T^{-1}\right\| = 1$, and so T is a unitary. □

Corollary 2.11. *A compact p-w-hyponormal operator is normal.*

Theorem 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal operator. If the planar Lebesgue measure $m_2(\sigma(T))$ of $\sigma(T)$ is 0, then T is normal.*

Proof. Since $\widetilde{T}$ is $\frac{p}{2}$-hyponormal by Lemma 2.8, and $\sigma(\widetilde{T}) = \sigma(T)$ by [8, Corollary 2.3], we have $m_2(\sigma(\widetilde{T})) = 0$. Hence $\widetilde{T}$ is normal by Putnam's inequality [54, Corollary]. Thus T is normal by [31, Theorem 4.4]. □

Let $\overline{W}(S)$ denotes the closure of the numerical range of the operator S. In [50] showed that if T is hyponormal, $T = S^{-1}T^*S$ and $0 \notin \overline{W}(S)$, then T is self-adjoint. The next theorem gives an extension of Sheh's result to p-w-hyponormal operators.

Theorem 2.13. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal operator. If $T = S^{-1}T^*S$ and $0 \notin \overline{W}(S)$, then T is self-adjoint.*

Proof. If $T = S^{-1}T^*S$ and $0 \notin \overline{W}(S)$, then it follows from [39, Theorem 1] that $\sigma(T) \subset \mathbb{R}$. Since a p-w-hyponormal operator T with $\sigma(T) \subset \mathbb{R}$ is self-adjoint, the result follows. □

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of an operator T if there is a non-zero vector x for which $(T - \lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then λ to be in the joint point spectrum $\sigma_{jp}(T)$ of T. In general, one has $\sigma_{jp}(T) \subset \sigma_p(T)$. It is known the equality holds for p-hyponormal [6].

If T is hyponormal, it is easy to see [51, Lemma 2] that T posses the property that $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$. This property clearly implies $\sigma_p(T) = \sigma_{jp}(T)$ if T is hyponormal. In the sequel, we show that p-w-hyponormal also possess this property provided $\lambda \neq 0$. Consequently the non-zero points of $\sigma_p(T)$ and $\sigma_{jp}(T)$ are identical if T is p-w-hyponormal.

Theorem 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal operator. If $Tx = \lambda x$, $\lambda \neq 0$, then $T^*x = \overline{\lambda}x$.*

Proof. Since $\widetilde{T}$ is $\frac{p}{2}$-hyponormal by Lemma 2.8, $\widetilde{T}$ possesses the property that $\widetilde{T}x = \lambda x$ implies $\widetilde{T^*}x = \overline{\lambda}x$. It follows from [8, Lemma 3.1] that T possesses the same property. □

Corollary 2.15. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal operator, then $\sigma_p(T) \setminus \{0\} = \sigma_{jp}(T) \setminus \{0\}$.*

Corollary 2.16. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal operator with $Tx = \lambda x$, $Ty = \mu y$ and $\lambda \neq \mu$. Then $\langle x, y\rangle = 0$.*

Proof. Without loss of generality, assume $\mu \neq 0$. Then $T^*y = \overline{\mu}y$ by Theorem 2.14. Thus,

$$\mu \langle x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle = \lambda \langle x, y\rangle .$$

Since $\lambda \neq \mu$, $\langle x, y\rangle = 0$. □

Let $\sigma_a(T)$ denotes the approximate point spectrum of the operator T. In [56], Xia proved that if T is semi-hyponormal, then $\sigma(T) = \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\}$. The next proposition shows that if T is p-w-hyponormal, the non-zero points of $\sigma(T)$ and $\{\lambda : \overline{\lambda} \in \sigma_a(T^*)\}$ are identical.

Proposition 2.17. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal operator, then*

$$\sigma(T) \setminus \{0\} = \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\} \setminus \{0\}.$$

Proof. In [56], it was shown that for any operator T, the equality $\sigma(T) = \sigma_p(T) \cup \{\lambda : \overline{\lambda} \in \sigma_a(T^*)\}$ holds. If T is p-w-hyponormal, then Corollary 2.15 implies $\sigma_p(T) \setminus \{0\} = \sigma_{jp}(T) \setminus \{0\} \subset \{\lambda : \overline{\lambda} \in \sigma_p(T^*)\} \setminus \{0\}$. Since $\sigma_p(T^*) \subset \sigma_a(T^*)$, the result follows. □

Here and elsewhere in this paper, for $A \subset \mathbb{C}$, iso$A$ denotes the set of all isolated points of A and acc$A$ denotes the set of all points of accumulation of A.

A bounded linear operator T is said to be isoloid if iso$\sigma(T) \subset \sigma_p(T)$.

Theorem 2.18. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal. If λ is an isolated point in $\sigma(T)$, then $\lambda \in \sigma_p(T)$. That is, T is isoloid.*

Proof. Since $\sigma(T) = \sigma(\widetilde{\widetilde{\widetilde{T}}})$, λ is an isolated point of $\sigma(\widetilde{\widetilde{\widetilde{T}}})$. the hyponormality of $\widetilde{\widetilde{\widetilde{T}}}$ implies that $\lambda \in \sigma_p(\widetilde{\widetilde{\widetilde{T}}})$ by [51, Theorem 2]. It follows from the fact T is invertible if and only if $\widetilde{T}$ is invertible that $\sigma_p(T) = \sigma_p(\widetilde{T}) = \sigma_p(\widetilde{\widetilde{T}}) = \sigma_p(\widetilde{\widetilde{\widetilde{T}}})$. Thus, $\lambda \in \sigma_p(T)$ and the proof is complete. □

Recall that a complex number λ is said to be in the approximate point spectrum $\sigma_a(T)$ of the operator T if there is a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $(T - \lambda)x_n = 0$. If in addition, $(T^* - \overline{\lambda})x_n = 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of T. Clearly, one has $\sigma_{ja}(T) \subset \sigma_a(T)$. It is known [9] that if T is w-hyponormal, then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$. Here we show that if T is p-w-hyponormal, then the same result holds.

Theorem 2.19. [12] *Given a Hilbert space $\mathcal{H}$, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a map $\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ such that*

(a) *ϕ is a faithful $*$-representation of the algebra $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$,*
(b) *$\phi(A) \geq 0$ for any $A \geq 0$ in $\mathcal{B}(\mathcal{H})$, and*
(c) *$\sigma_a(T) = \sigma_a(\phi(T)) = \sigma_p(\phi(T))$ for any $T \in \mathcal{B}(\mathcal{H})$.*

We also need the following corollary which Xia observed in [56].

Corollary 2.20. *Let $\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ be the Berberian's faithful $*$-representation. For any operator $T \in \mathcal{B}(\mathcal{H})$, $\sigma_{jp}(\phi(T)) = \sigma_{ja}(T)$.*

Theorem 2.21. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal, then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$.*

Proof. Let ϕ be the representation of Berberian. First, we show that $\phi(T)$ is p-w-hyponormal. In view of Proposition 2.4, we need only establish

$$|\phi(T)|^p \geq (|\phi(T)|^{\frac{1}{2}}|\phi(T)^*||\phi(T)|^{\frac{1}{2}})^{\frac{p}{2}}$$

and

$$(|\phi(T)^*|^{\frac{1}{2}}|\phi(T)||\phi(T)^*|^{\frac{1}{2}})^{\frac{p}{2}} \geq |\phi(T)^*|^p.$$

Part (a) and (b) of Theorem 2.19 imply

$$|\phi(T)|^p = \phi(|T|^p) \geq \phi((|T|^{\frac{1}{2}}|T^*||T||T|^{\frac{1}{2}})^{\frac{p}{2}}) = (|\phi(T)|^{\frac{1}{2}}|\phi(T)^*||\phi(T)|^{\frac{1}{2}})^{\frac{p}{2}},$$

and similarly,

$$(|\phi(T)^*|^{\frac{1}{2}}|\phi(T)||\phi(T)^*|^{\frac{1}{2}})^{\frac{p}{2}} \geq |\phi(T)^*|^p.$$

Thus, $\phi(T)$ is p-w-hyponormal. Now, by part(c) of Theorem 2.19, we have

$$\begin{aligned}\sigma_a(T)\setminus\{0\} &= \sigma_a(\phi(T))\setminus\{0\}\\ &= \sigma_p(\phi(T))\setminus\{0\}\\ &= \sigma_{jp}(\phi(T))\setminus\{0\} \qquad \text{by Corollary 2.15}\\ &= \sigma_{ja}(T)\setminus\{0\}\end{aligned}$$

where the last equality follows from Corollary 2.20. The proof is complete. □

Corollary 2.22. *If $T \in \mathcal{B}(\mathcal{H})$ is an invertible p-w-hyponormal, then $\sigma_a(T) = \sigma_{ja}(T)$.*

Lemma 2.23. (Hölder-McCarthy Inequality) Let $A \geq 0$. Then the following assertions hold.

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r \|x\|^{2(1-r)}$ for $r > 1$ and $x \in \mathcal{H}$.
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \|x\|^{2(1-r)}$ for $r \in [0,1]$ and $x \in \mathcal{H}$.

Theorem 2.24. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal, $0 < p \leq 1$. Then*

$$\ker(T-\lambda)^2 = \ker(T-\lambda) \textit{ for all } \lambda \in \mathbb{C}\setminus\{0\}.$$

Proof. Since $\ker(T-\lambda) \subset \ker(T-\lambda)^2$ is clear, we need only show that $\ker(T-\lambda)^2 \subset \ker(T-\lambda)$. For simplicity, write $K = \ker(T-\lambda)^2$ and denote by F the closure of $(T-\lambda)K$. Let $x \in K$. The hypothesis implies

$$(T-\lambda)^*(T-\lambda)x = 0,$$

and consequently,

$$(T-\lambda)^*F = 0$$

If $z \in \mathcal{H}$, write $z = w + y$, where $w \in F$ and $y \in F^\perp$. Then $(T-\lambda)^*z = (T-\lambda)^*y$, and hence

$$\langle (T-\lambda)^*z, x\rangle = \langle (T-\lambda)^*y, x\rangle = \langle y, (T-\lambda)x\rangle .$$

for all $x \in K$. Therefore, $\mathcal{R}(T-\lambda)^* \subset K^\perp$, and consequently,

$$\ker(T-\lambda)^2 = K^{\perp\perp} \subset (\mathcal{R}(T-\lambda)^*)^\perp = \ker(T-\lambda).$$

□

Corollary 2.25. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal for $0 < p \leq 1$, then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}\setminus\{0\}$.*

3. Quasinilpotent part of p-w-hyponormal operators

Lemma 3.1. [48, Lemma 2.22] *Let $T \in \mathcal{B}(\mathcal{H})$ be a p-w-hyponormal operator for some $0 < p \leq 1$ and let $\mathcal{M}$ an invariant subspace of T. Then the restriction $T|_\mathcal{M}$ is also a p-w-hyponormal operator.*

Lemma 3.2. [48, Lemma 2.24] *Let $T \in \mathcal{B}(\mathcal{H})$ be a p-w-hyponormal operator, let $\mathcal{M}$ be an invariant subspace for T and a reduced subspace for $\widetilde{T}$ such that $\widetilde{T}|_\mathcal{M}$ the restriction of $\widetilde{T}$ to $\mathcal{M}$ is an injective normal operator, then $T|_\mathcal{M} = \widetilde{T}|_\mathcal{M}$ and $\mathcal{M}$ reduces T.*

Two important subspaces in local spectral theory and Fredholm theory are defined in the sequel. The quasi-nilpotent part of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set

$$H_0(T) = \{x \in \mathcal{H} : \lim_{n\longrightarrow\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$. If $T \in \mathcal{B}(\mathcal{H})$, the analytic core $K(T)$ is the set of all $x \in \mathcal{H}$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathcal{H}$

such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$. Note that by Theorem 2.2 of [4], $T \in \mathcal{B}(\mathcal{H})$ is polaroid if and only if there exists $p := p(\lambda - T) \in \mathbb{N}$ such that

$$H_0(\lambda - T) = \ker(\lambda - T)^p \quad \text{for all} \quad \lambda \in \operatorname{iso}\sigma(T). \tag{3.1}$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyper-invariant subspaces of $T-\lambda$ such that $\ker(T-\lambda)^p \subseteq H_0(T-\lambda)$ for all $p = 0, 1, \cdots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$. Recall that if $\lambda \in \operatorname{iso}\sigma(T)$, then $H_0(T-\lambda) = \mathcal{X}_T(\{\lambda\})$, where $\mathcal{X}_T(\{\lambda\})$ is the global spectral subspace consisting of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow \mathcal{H}$ that satisfies $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$, see [1].

Let $H_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in H_{nc}(\sigma(T))$. Following [26] We say that $T \in \mathcal{B}(\mathcal{H})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathcal{H}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathcal{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathcal{B}(\mathcal{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [42, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Definition 3.3. [19] An operator T is said to have *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if for every open neighborhood G of λ, the function $f_n \in H_{nc}(G)$ with $(T-\lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G,where $Hol(G)$ means the space of all analytic functions on G. When T has Bishop's property (β) at each $\lambda \in \mathbb{C}$, simply say that T has property (β).

Theorem 3.4. [20] *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal, then T has property (β) and hence has SVEP.*

Theorem 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal. Then $H_0(T-\lambda I) = \ker(T-\lambda I$ for $\lambda \in \mathbb{C}$.*

Proof. Let $F \subset \mathbb{C}$ be closed set. Define the global spectral subspace by

$$\mathcal{X}_T(F) = \{x \in \mathcal{H} : \text{there is analytic } f(z) : (T-z)f(z) = x \text{ on } \mathbb{C} \setminus F\}.$$

It is known that $H_0(T-\lambda) = \mathcal{X}_T(\{\lambda\})$ by [1, Theorem 2.20]. As T has Bishop's property (β) by Theorem 3.4, $\mathcal{X}_T(F)$ is closed and $\sigma(T|_{\mathcal{X}_T(F)}) \subset F$ by [43, Proposition 1.2.19]. Hence $H_0(T-\lambda I)$ is closed and $T|_{H_0(T-\lambda I)}$ is p-w-hyponormal by Theorem 3.1. Since $\sigma(T|_{H_0(T-\lambda I)}) \subset \{\lambda\}$, $T|_{H_0(T-\lambda I)}$ is normal by Corollary 2.12. If $\sigma(T|_{H_0(T-\lambda I)}) = \emptyset$, then $H_0(T-\lambda I) = \{0\}$ and $\ker(T-\lambda I) = \{0\}$. If $\sigma(T|_{H_0(T-\lambda I)}) = \{\lambda\}$, then $T|_{H_0(T-\lambda I)} = \lambda I$ and $H_0(T-\lambda I) \subset \ker(T-\lambda I)$. □

Remark 3.6. If $\lambda \neq 0$, then $H_0(T-\lambda I) = \ker(T-\lambda I) \subset \ker(T-\lambda I)^*$. Moreover, if $\lambda \in \sigma(T) \setminus \{0\}$ is an isolated point then $H_0(T-\lambda I) = \ker(T-\lambda I) = \ker(T-\lambda I)^*$.

Lemma 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal. Let $\lambda \in \mathbb{C}$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$*

Proof. We consider two cases:
Case (I). $(\lambda = 0)$: Since T is p-w-hyponormal, T is normaloid. Therefore $T = 0$.
Case (II). $(\lambda \neq 0)$: Here T is invertible, and since T is p-w-hyponormal, we see that T^{-1} is also p-w-hyponormal. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda$. □

Two classical quantities associated with a linear operator T are the ascent $p := p(T)$, defined as the smallest non-negative integer p (if it does exist) such that $\ker T^p = \ker T^{p+1}$, and the descent $q := q(T)$, defined as the smallest non-negative integer q (if it does exists) such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$. It is well-known that if $p(T-\lambda)$ and $q(T-\lambda)$ are both finite then $p(T-\lambda) = q(T-\lambda)$ and λ is a pole of the the function resolvent $\lambda \longrightarrow (T-\lambda)^{-1}$, in particular λ is an isolated point of the spectrum $\sigma(T)$, see Proposition 38.3 and Proposition 50.2 of Heuser [35].

A bounded operator $T \in \mathcal{B}(\mathcal{H})$ defined on a Banach space is said to be polaroid if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent. The following result has been proved in [3, Theorem 2.4].

Theorem 3.8. *For an operator $T \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:*

(i) *T is polaroid;*
(ii) *there exists $f \in H_{nc}(\sigma(T))$ such that $f(T)$ is polaroid;*
(iii) *$f(T)$ is polaroid for every $f \in H_{nc}(\sigma(T))$.*

Theorem 3.9. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal operator, $0 < p \leq 1$, then T is polaroid.*

Proof. We show that for every isolated point λ of $\sigma(T)$ we have $p(T-\lambda) = q(T-\lambda) < \infty$. Let λ be an isolated point of $\sigma(T)$, and denote by P_λ denote the spectral projection associated with $\{\lambda\}$. Then $\mathcal{M} := K(T-\lambda) = \ker P_\lambda$ and $\mathcal{N} := H_0(T-\lambda) = P_\lambda(\mathcal{H})$, see [1, Theorem 3.74]. Therefore, $H = H_0(T-\lambda) \oplus K(T-\lambda)$. Furthermore, since $\sigma(T|_{\mathcal{N}}) = \{\lambda\}$, while $\sigma(T|_{\mathcal{M}}) = \sigma(T) \setminus \{\lambda\}$, so the restriction $T|_{\mathcal{N}} - \lambda$ is quasi-nilpotent and $T|_{\mathcal{N}} - \lambda$ is invertible. Since $T|_{\mathcal{N}} - \lambda$ is p-w-hyponormal, then Lemma 3.7 implies that $T|_{\mathcal{N}} - \lambda$ is nilpotent. In other words, $T|_{\mathcal{N}} - \lambda$ is an operator of Kato Type.

Now, both T and the dual T^* have SVEP at λ, since λ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.16 and Theorem 3.17 of [1], that both $p(T-\lambda)$ and $q(T-\lambda)$ are finite. Therefore, λ is a pole of the resolvent. □

4. Riesz idempotent of w-hyponormal

Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \sigma(T)$ be an isolated of $\sigma(T)$. then there exists a closed disc $\mathbf{D}_\lambda$ centered λ which satisfies $\mathbf{D}_\lambda \cap \sigma(T) = \{\lambda\}$. The operator

$$P = \frac{1}{2\pi i} \int_{\partial \mathbf{D}_\lambda} (T - \lambda I)^{-1} \, d\lambda$$

is called the Riesz idempotent with respect to λ which has properties that

$$P^2 = P, PT = TP, \ker(T-\lambda I) \subset \mathcal{R}(P) \quad \text{and} \quad \sigma(T|_{\mathcal{R}(P)}) = \{\lambda\}.$$

In [51], Stampfli proved that if T is hyponormal and $\lambda \in \sigma(T)$ is isolated, then the Riesz idempotent P with respect to λ is self-adjoint and satisfies

$$\mathcal{R}(P) = \ker(T-\lambda I) = \ker(T-\lambda I)^*.$$

In this paper we extend these result to the case of ap-w-hyponormal operator.

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a p-w-hyponormal operator and λ be a non-zero isolated point of $\sigma(T)$. Let $\mathbf{D}_\lambda$ denote the closed disc which centered λ such that $\mathbf{D}_\lambda \cap \sigma(T) = \{\lambda\}$. Then the Riesz idempotent $P = \frac{1}{2\pi i} \int_{\partial \mathbf{D}_\lambda} (T-\lambda I)^{-1} \, d\lambda$ satisfies that*

$$\mathcal{R}(P) = \ker(T-\lambda I) = \ker(T-\lambda I)^*.$$

In particular P is self-adjoint.

Proof. Since p-w-hyponormal operators are isoloid by Corollary 2.18. Then every isolated point of $\sigma(T)$ of T is an eigenvalue of T. Then the range of Riesz idempotent $P =$

$\frac{1}{2\pi i}\int_{\partial\mathbf{D}_\lambda}(T-\lambda I)^{-1}\,d\lambda$ is an invariant closed subspace of T and $\sigma(T|_{\mathcal{R}(P)})=\{\lambda\}$. Here $\mathbf{D}_\lambda$ is a closed disc with its center λ such that $\mathbf{D}_\lambda\cap\sigma(T)=\{\lambda\}$.

If $\lambda=0$, then $\sigma(T|_{\mathcal{R}(P)})=\{0\}$. Since $T|_{\mathcal{R}(P)}$ is p-w-hyponormal by Theorem 3.1, $T|_{\mathcal{R}(P)}=0$ by Lemma 3.7. Therefore, 0 is an eigenvalue of T.

If $\lambda\neq 0$, then $T|_{\mathcal{R}(P)}$ is an invertible p-w-hyponormal operator and hence $(T|_{\mathcal{R}(P)})^{-1}$ is also w-hyponormal. We see that $\left\|T|_{\mathcal{R}(P)}\right\|=|\lambda|$ and $\left\|(T|_{\mathcal{R}(P)})^{-1}\right\|=\frac{1}{|\lambda|}$. Let $x\in\mathcal{R}(P)$ be arbitrary vector. Then

$$\|x\|\leq\left\|(T|_{\mathcal{R}(P)})^{-1}\right\|\left\|T|_{\mathcal{R}(P)}x\right\|=\frac{1}{|\lambda|}\left\|T|_{\mathcal{R}(P)}x\right\|\leq\frac{1}{|\lambda|}|\lambda|\,\|x\|=\|x\|.$$

This implies that $\frac{1}{\lambda}T|_{\mathcal{R}(P)}$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T|_{\mathcal{R}(P)})=\{1\}$. Hence $T|_{\mathcal{R}(P)}=\lambda I$ and λ is an eigenvalue of T. Therefore, $\mathcal{R}(P)=\ker(T-\lambda I)$. Since $\ker(T-\lambda I)\subset\ker(T-\lambda I)^*$ by Proposition 2.14, it suffices to show that $\ker(T-\lambda I)^*\subset\ker(T-\lambda I)$. Since $\ker(T-\lambda I)$ is a reducing subspace of T by Proposition 2.14 and the restriction of a p-w-hyponormal to its reducing subspace is also p-w-hyponormal operator, we see that T is of the form $T=T'\oplus\lambda I$ on $\mathcal{H}=\ker(T-\lambda I)\oplus\ker(T-\lambda I)^\perp$, where T' is a p-w-hyponormal operator with $\ker(T'-\lambda I)=\{0\}$. since $\lambda\in\sigma(T)=\sigma(T')\cup\{\lambda\}$ is isolated, the only two cases occur. One is $\lambda\notin\sigma(T')$ and the other is that λ is an isolated point of $\sigma(T')$. The latter case, however, does not occur otherwise we have $\lambda\in\sigma_p(T')$ and this contradicts the fact that $\ker(T'-\lambda I)=\{0\}$. $\ker(T-\lambda I)=\ker(T-\lambda I)^*$ is immediate from the injectivity of $T'-\lambda I$ as an operator on $\ker(T-\lambda I)^\perp$.

Next, we show that P is self-adjoint. Since $\mathcal{R}(P)=\ker(T-\lambda I)=\ker(T-\lambda I)^*$, we have $((T-zI)^*)^{-1}P=\overline{(z-\lambda)^{-1}}P$. Hence

$$\begin{aligned}P^*P&=-\frac{1}{2i\pi}\int_{\partial\mathbf{D}_\lambda}((T-zI)^*)^{-1}P\,d\bar{z}\\&=-\frac{1}{2i\pi}\int_{\partial\mathbf{D}_\lambda}\overline{(z-\lambda)^{-1}}P\,d\bar{z}\\&=\overline{\left(\frac{1}{2i\pi}\int_{\partial\mathbf{D}_\lambda}\frac{1}{z-\lambda}\,d\bar{z}\right)}P\\&=PP^*.\end{aligned}$$

□

In the following we give an example T of p-w-hyponormal operator which has properties that 0 is an isolated point of $\sigma(T)$, the Riesz idempotent with respect to 0 is not self-adjoint and $\ker(T)\neq\ker(T^*)$.

Example 4.2. Let $\mathcal{H}=\oplus_{n=0}^{\infty}\mathbb{C}^2$ and define an operator T on $\mathcal{H}$ by

$$T(\cdots\oplus x_{-2}\oplus x_{-1}\oplus x_0^{(0)}\oplus x_1\oplus\cdots)=\cdots\oplus Ax_{-2}\oplus A^{(0)}x_{-1}\oplus Bx_0\oplus Bx_1\oplus\cdots,$$

where $A=\begin{pmatrix}1/8&1/8\\1/8&1/8\end{pmatrix}$ and $B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Then T is p-w-hyponormal with $0<p\leq 1$ and $\sigma(T)=\{0\}\cup\{z:\frac{1}{4}\leq|z|\leq 1\}$. Moreover $P\mathcal{H}=\ker(T)$, P is not self-adjoint and $\ker(T)\neq\ker(T^*)$, where P is the Riesz idempotent with respect to 0.

Proof. Let $x=\cdots\oplus x_{-2}\oplus x_{-1}\oplus x_0^{(0)}\oplus x_1\oplus\cdots$, we have

$$\begin{aligned}T^*x&=(\cdots\oplus Ax_0\oplus B^{(0)}x_1\oplus Bx_2\oplus\cdots),\\|T|x&=(\oplus_{n<0}Ax_n)\oplus(\oplus_{n\geq 0}Bx_n),\\|\widetilde{T}|x&=(\oplus_{n<-1}Ax_n)\oplus(A^{1/2}BA^{1/2})^{1/2}x_{-1}\oplus(\oplus_{n\geq 0}Bx_n),\\|(\widetilde{T})^*|x&=(\oplus_{n<0}Ax_n)\oplus(A^{1/2}BA^{1/2})^{1/2}x_0\oplus(\oplus_{n\geq 1}Bx_n).\end{aligned}$$

Since $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{p}{2}} = 2^{\frac{p}{2}}A$ and $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{p}{2}} = (BAB)^{\frac{p}{2}} = \frac{1}{8^{\frac{p}{2}}}B$,

$$\left\langle (|\widetilde{T}|^p - |T|^p)x, x \right\rangle = \left\langle ((A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{p}{2}} - A)x_{-1}, x_{-1} \right\rangle \geq 0$$
$$\left\langle (|T|^p - |(\widetilde{T})^*|^p)x, x \right\rangle = \left\langle (B - (BAB)^{\frac{p}{2}})x_0, x_0 \right\rangle \geq 0.$$

Hence T is p-w-hyponormal.

(i) Let $\mathcal{H}_+ = \{(T-\lambda)x | x \in \mathcal{H}, x = \cdots \oplus 0 \oplus x_0 \oplus x_1 \oplus x_2 \oplus \cdots\}$, $\mathcal{H}_- = \{(T-\lambda)x | x \in \mathcal{H}, x = \cdots \oplus x_{-4} \oplus x_{-3} \oplus 0 \oplus \cdots\}$, and $\mathcal{H}_0 = \{(T-\lambda)x | x \in \mathcal{H}, x = \cdots \oplus 0 \oplus x_{-2} \oplus x_{-1} \oplus 0 \oplus \cdots\}$. Then $\mathcal{H}_+ \perp \mathcal{H}_-$. We remark that $4A$ is unitary equivalent to B. By Lemma 12 of [55], $\mathcal{H}_+$ and $\mathcal{H}_-$ are closed for $\lambda < \frac{1}{4}$. Since $\mathcal{H}_0$ is finite dimensional, $\mathcal{R}(T-\lambda) = (\mathcal{H}_+ \oplus \mathcal{H}_-) + \mathcal{H}_0$ is closed.

(ii) It is easy to check that

$$\ker(T) = \left\{ [\oplus_{n\leq -1} c_n \begin{pmatrix} 1 \\ -1 \end{pmatrix}] \oplus [\oplus_{n\geq 0} c_n \begin{pmatrix} 0 \\ 1 \end{pmatrix}] | \{c_n\} \in \ell^2(\mathbb{Z}) \right\},$$
$$\ker(T^*) = \left\{ [\oplus_{n\leq 0} c_n \begin{pmatrix} 1 \\ -1 \end{pmatrix}] \oplus [\oplus_{n\geq 1} c_n \begin{pmatrix} 0 \\ 1 \end{pmatrix}] | \{c_n\} \in \ell^2(\mathbb{Z}) \right\}.$$

Hence, $\ker(T) \neq \ker(T^*)$.

(iii) If $0 < \lambda < 1/4$, it easy to check that $\ker(T-\lambda) = \ker(T-\lambda)^* = \{0\}$. Since $\mathcal{R}(T-\lambda)$ is closed by [55, Lemma 12], we have $\mathcal{R}(T-\lambda) = \overline{\mathcal{R}(T-\lambda)} = [\ker(T-\lambda)^*]^{\perp} = \mathcal{H}$ and therefore $\lambda \notin \sigma(T)$.

(iv) If $\frac{1}{4} < \lambda < 1$, we have

$$\ker(T-\lambda)^* = \mathbb{C}\left([\oplus_{n<0} \frac{1}{2(4\lambda)^{|n|}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}] \oplus [\oplus_{n\geq 0} \lambda^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}] \right).$$

(v) It follows from (iii) and (iv) that $\sigma(T) = \{0\} \cup \{\lambda \in \mathbb{C} | \frac{1}{4} \leq |\lambda| \leq 1\}$.

(vi) Since T is paranormal, we have $\mathcal{R}(P) = \ker(T)$ by the proof of [55, Lemma 6]. Suppose that P is self-adjoint. Then $\mathcal{R}(P) \perp \mathcal{R}(I-P)$, so that $T = 0 \oplus S$ for some paranormal operator on $\mathcal{R}(I-P)$ with $\mathcal{R}(S) = \{0\}$. Since S is isoloid, $0 \notin \sigma(S)$. Hence $\ker(T) = \mathcal{R}(P) = \ker(T^*)$. This contradicts (ii).

□

5. Tensor Product

Let $\mathcal{H}$ and $\mathcal{K}$ denote the Hilbert spaces. For given non-zero operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, $T \otimes S$ denotes the tensor product on the product space $\mathcal{H} \otimes \mathcal{K}$. The normaloid property is invariant under tensor products [53]. $T \otimes S$ is normal if and only if T and S are normal [23, 40]. There exist paranormal operators T and S such that $T \otimes S$ is not paranormal [52]. In [37], I.H.Kim showed that for non-zero $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, $T \otimes S$ is log-hyponormal if and only if T and S are log-hyponormal. This result was extended to p-quasihyponormal operators, w-hyponormal operators , class A operators and class $A(k)$ in [37], [36], and [47] respectively. In this section, we prove an analogous result for p-w-hyponormal operators.

Remark 5.1. Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be non-zero operators, then we have

(i) $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$
(ii) $|T \otimes S|^t = |T|^t \otimes |S|^t$ for any positive real t.

Lemma 5.2. *([40]) Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$, $S_1, S_2 \in \mathcal{B}(\mathcal{K})$ be non-negative operators. If T_1 and S_1 are non-zero, then the following assertions are equivalent:*

(a) $T_1 \otimes S_1 \leq T_2 \otimes S_2$
(b) *there exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$.*

Theorem 5.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be non-zero operators and let $0 < p \leq 1$. Then $T \otimes S$ is p-w-hyponormal if and only if T and S is p-w-hyponormal.*

Proof. We shall use the fact that the function $T \longrightarrow \widetilde{T}$ has the property $\widetilde{T \otimes S} = \widetilde{T} \otimes \widetilde{S}$. It follows from Remark 5.1 that

$$|\widetilde{T \otimes S}|^p = |\widetilde{T}|^p \otimes |\widetilde{S}|^p \geq |T \otimes S|^p = |T|^p \otimes |S|^p \geq |(\widetilde{T \otimes S})^*|^p = |\widetilde{T}^*|^p \otimes |\widetilde{S}^*|^p. \quad (5.2)$$

Inequality 5.2 holds if and only if

$$\left(|\widetilde{T}|^p - |T|^p\right) \otimes |\widetilde{S}|^p + |T|^p \left(|\widetilde{S}|^p - |S|^p\right) \geq 0 \text{ and}$$
$$\left(|T|^p - |\widetilde{T}^*|^p\right) \otimes |S|^p + |\widetilde{T}^*|^p \left(|S|^p - |\widetilde{S}^*|^p\right) \geq 0, \quad (5.3)$$

or, equivalently, if and only if

$$\left(|\widetilde{T}|^p - |T|^p\right) \otimes |S|^p + |\widetilde{T}|^p \left(|\widetilde{S}|^p - |S|^p\right) \geq 0 \text{ and}$$
$$\left(|T|^p - |\widetilde{T}^*|^p\right) \otimes |\widetilde{S}^*|^p + |T|^p \left(|S|^p - |\widetilde{S}^*|^p\right) \geq 0. \quad (5.4)$$

So, the sufficency is clear.

To prove the necessity, suppose that $T \otimes S$ is p-w-hyponormal. Then

$$|\widetilde{T}|^p \otimes |\widetilde{S}|^p \geq |T|^p \otimes |S|^p.$$

Therefore, by Lemma 5.2, there exists a $c \in \mathbb{R}^+$ such that

$$c|\widetilde{T}|^p \geq |T|^p \text{ and } c^{-1}|\widetilde{S}|^p \geq |S|^p.$$

Consequently,

$$\left\||T|^p\right\|^2 = \sup_{\|x\|=1} \left\langle |T|^{2p}x, x\right\rangle \leq \sup_{\|x\|=1} \left\langle c|\widetilde{T}|^{2p}x, x\right\rangle \leq c \left\||T|^p\right\|^2$$

and

$$\left\||S|^p\right\|^2 = \sup_{\|x\|=1} \left\langle |S|^{2p}x, x\right\rangle \leq \sup_{\|x\|=1} \left\langle c^{-1}|\widetilde{S}|^{2p}x, x\right\rangle \leq c^{-1} \left\||S|^p\right\|^2$$

Thus, $c = 1$ and

$$|\widetilde{T}|^p \geq |T|^p \text{ and } |\widetilde{S}|^p \geq |S|^p. \quad (5.5)$$

Now we just to show that $|T|^p \geq |\widetilde{T}^*|^p$ and $|S|^p \geq |\widetilde{S}^*|^p$. Let $x \in \mathcal{H}$ and $y \in \mathcal{K}$ be arbitrary. Then, from inequalities (5.3) and (5.4), we have

$$\left\langle \left(|T|^p - |\widetilde{T}^*|^p\right) x, x\right\rangle \langle |S|^p y, y\rangle + \left\langle |\widetilde{T}^*|^p x, x\right\rangle \left\langle \left(|S|^p - |\widetilde{S}^*|^p\right) y, y\right\rangle \geq 0 \quad (5.6)$$

and

$$\left\langle \left(|T|^p - |\widetilde{T}^*|^p\right) x, x\right\rangle \left\langle |\widetilde{S}^*|^p y, y\right\rangle + \langle ||T|^p x, x\rangle \left\langle \left(|S|^p - |\widetilde{S}^*|^p\right) y, y\right\rangle \geq 0. \quad (5.7)$$

Suppose that $|T|^p - |\widetilde{T}^*|^p$ is not a positive operator. Then there is a $x_0 \in \mathcal{H}$ such that

$$\left\langle \left(|T|^p - |\widetilde{T}^*|^p\right) x_0, x_0\right\rangle = \alpha < 0 \text{ and } \left\langle |\widetilde{T}^* x_0, x_0\right\rangle = \beta > 0.$$

From inequality (5.6) we get

$$(\alpha + \beta) \left\||S|^p y\right\| \geq \beta \left\||\widetilde{S}^*|^p y\right\|.$$

That is,

$$(\alpha + \beta) \left\||S|^p\right\| \geq \beta \left\||\widetilde{S}^*|^p\right\|.$$

Since, by inequality (5.5), $|\widetilde{S}|^p \geq |S|^p$, we have also

$$(\alpha + \beta) \|S\|^p = (\alpha + \beta) \left\||S|^p\right\| \geq \beta \left\||\widetilde{S}^*|^p\right\| = \beta \left\||\widetilde{S}|^p\right\| \geq \beta \|S\|^p.$$

This is a contradiction. Hence, $|T|^p \geq |\widetilde{T}^*|^p$. A similar argument shows, by using inequality (5.7), that $|S|^p \geq |\widetilde{S}^*|^p$. □

6. Weyl's Type theorems

Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that the operator $T \in \mathcal{B}(\mathcal{X})$ is said to be *upper semi-Fredholm*, $T \in SF_+(\mathcal{X})$, if the range of $T \in \mathcal{B}(\mathcal{X})$ is closed and $\alpha(T) < \infty$, while $T \in \mathcal{B}(\mathcal{X})$ is said to be *lower semi-Fredholm*, $T \in SF_-(\mathcal{X})$, if $\beta(T) < \infty$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be *semi-Fredholm* if $T \in SF_+(\mathcal{X}) \cup SF_-(\mathcal{X})$ and Fredholm if $T \in SF_+(\mathcal{X}) \cap SF_-(X)$. If T is semi-Fredholm then the *index* of T is defined by $\mathrm{ind\,(T)} = \alpha(\mathrm{T}) - \beta(\mathrm{T})$.

A bounded linear operator T acting on a Banach space $\mathcal{X}$ is *Weyl* if it is Fredholm of index zero and Browder if T is Fredholm of finite ascent and descent. The Weyl spectrum $\sigma_W(T)$ and *Browder spectrum* $\sigma_b(T)$ of T are defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

Let $E^0(T) = \{\lambda \in \mathrm{iso}\,\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ and let $\pi^0(T) := \sigma(T) \setminus \sigma_b(T)$ all *Riesz points* of T. According to Coburn [24], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_W(T) = E^0(T)$, and that *Browder's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_W(T) = \pi^0(T)$.

Let $SF_+^-(\mathcal{X}) = \{T \in SF_+ : \mathrm{ind\,}(T) \leq 0\}$. The *upper semi Weyl spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathcal{X})\}$. According to Rakočević [49], an operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \mathrm{iso}\,\sigma_\mathrm{a}(\mathrm{T}) : 0 < \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty\}.$$

It is known [49] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \ \mathcal{B}(\mathcal{X})$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ to $\mathcal{R}(T^n)$(in particular $T_{[0]} = T$). If for some integer n the range space $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm* operator. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. A *semi-B-Fredholm operator* is an upper or lower semi-Fredholm operator [18]. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator [13]. An operator T is called a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not B-Weyl operator $\}$ [14]. Let $E(T)$ be the set of all eigenvalues of T which are isolated in $\sigma(T)$. According to [15], an operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy *generalized Weyl's theorem*, if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. In general, generalized Weyl's theorem implies Weyl's theorem but the converse is not true [17]. Following [14], we say that T satisfies *generalized Browders's theorem*, if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, where $\pi(T)$ is the set of poles of T.

Let $SBF_+^-(\mathcal{X})$ denote the class of all is *upper B-Fredholm* operators such that $\mathrm{ind\,(T)} \leq 0$. The *upper B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{X})\}.$$

Following [17], we say that *generalized a-Weyl's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$ if $\Delta_a^g(S) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in iso\sigma_a(T) : \alpha(T - \lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathcal{B}(\mathcal{X})$ obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [10, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [17, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption

$E_a(T) = \pi_a(T)$ it is proved in [16, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem. Following [46], we say that $T \in \mathcal{B}(\mathcal{X})$ possesses *property* (t) if $\Delta_+(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. In Proposition 2.7 of [46], it is shown that property (t) implies Weyl's theorem, but the converse is not true in general. We say that $T \in \mathcal{B}(\mathcal{X})$ possesses *property* (gt) if $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gt) has been introduced and studied in [46]. Property (gt) extends property (t) to the context of B-Fredholm theory, and it is proved in [46] that an operator possessing property (gt) possesses property (t) but the converse is not true in general.

Lemma 6.1. *If $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal operator with $0 < p \leq 1$, then T and T^* satisfy Weyl's theorem.*

Proof. Since T is p-w-hyponormal, then T has SVEP by Theorem 3.4. Then T satisfies Browder's theorem if and only if T^* satisfies Browder's theorem if and only if $\pi^0(T) = \sigma(T) \setminus \sigma_w(T) \subseteq E^0(T)$ and $\pi^0(T^*) = \sigma(T^*) \setminus \sigma_w(T^*) \subseteq E^0(T^*)$. If $\overline{\lambda} \in E^0(T^*)$, then T has SVEP at λ and T^* has SVEP at $\overline{\lambda}$ and $0 < p(T-\lambda)^* = q(T-\lambda) < \infty$. Thus the ascent and descent of $T - \lambda$ are finite and hence equal, see [35]. Then $T - \lambda$ is a Fredholm of index 0 and also $(T-\lambda)^*$ is a Fredholm of index 0, then $E^0(T) \subseteq \pi^0(T)$ and $E^0(T^*) \subseteq \pi^0(T^*)$. This implies that both T and T^* satisfy Weyl's theorem. □

Lemma 6.2. *If T or T^* is p-w-hyponormal operator with $0 < p \leq 1$, then both T and T^* satisfy generalized Weyl's theorem.*

Proof. If T or T^* is p-w-hyponormal, then T is polaroid by Theorem 3.9 also T^* is polaroid, and generalized Weyl's theorem for T, or T^* are equivalent, see [2, Theorem 3.7]. The assertion then follows from [2, Theorem 3.3]. □

Definition 6.3. Let $T \in \mathcal{B}(\mathcal{X})$. Then we say that

(i) T possess property (w) if $\sigma_a(T) \setminus \sigma_{SF_-^+}(T) = E^0(T)$ [5];
(ii) T possess property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_-^+}(T) = E(T)$ [11];
(iii) T possess property (b) if $\sigma_a(T) \setminus \sigma_{SF_-^+}(T) = \pi^0(T)$ [45];
(iv) T possess property (gb) if $\sigma_a(T) \setminus \sigma_{SBF_-^+}(T) = \pi(T)$ [45].

Theorem 6.4. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal with $0 < p \leq 1$ and $f \in H_{nc}(\sigma(T))$. Then*

(i) *property (t) holds for $f(T^*)$, or equivalently property (w), property (R), property (b), Weyl's theorem, a-Weyl's theorem hold for $f(T^*)$.*
(ii) *property (gt) holds for $f(T^*)$, or equivalently property (gw), property (gb), generalized Weyl's theorem, generalized a-Weyl's theorem hold for $f(T^*)$.*

Proof. Since T has SVEP by Theorem 3.4 and polaroid by 3.9. The assertions then follows from Theorem 3.6 (ii) and Theorem 3.7 (ii) of [46]. □

Theorem 6.5. *Suppose that $T^* \in \mathcal{B}(\mathcal{H})$ is p-w-hyponormal with $0 < p \leq 1$ and $f \in H_{nc}(\sigma(T))$. Then*

(i) *property (t) holds for $f(T)$, or equivalently property (w), property (R), property (b), Weyl's theorem, a-Weyl's theorem hold for $f(T)$.*
(ii) *property (gt) holds for $f(T)$, or equivalently property (gw), property (gb), generalized Weyl's theorem, generalized a-Weyl's theorem hold for $f(T)$.*

Proof. Since T has SVEP by Theorem 3.4 and polaroid by Theorem 3.9. The assertions then follows from Theorem 3.6 (i) and Theorem 3.7 (i) of [46]. □

References

[1] P. Aiena, *Fredholm and local spectral theory, with applications to multipliers*, Functional Analysis Mathematics, vol. 1, Springer Netherlands, 2004, doi:10.1007/1-4020-2525-4.
[2] P. Aiena, E. Aponte, and E. Bazan, *Weyl type theorems for left and right polaroid operators*, Integr. Equ. Oper. Theory **66** (2010), no. 1, 485–495, doi:10.1007/s00020-009-1738-2.
[3] P. Aiena, J.R. Guillen, and P. Peña, *A unifying approch to Weyl type theorems for Banach space operators*, Integr. Equ. Oper. Theory **77** (2013), 371—384, doi:10.1007/s00020-013-2097-6.
[4] P. Aiena, M. Chö, and M. Gonzălez, *Polaroid type operator under quasiaffinities*, J. Math. Anal. Appl. **371** (2010), no. 2, 485–495, doi:10.1016/j.jmaa.2010.05.057.
[5] P. Aiena and P. Peña, *Variations on Weyl's theorem*, J. Math. Anal. Appl. **324** (2006), no. 1, 566–579, doi:10.1016/j.jmaa.2005.11.027.
[6] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equ. Oper. Theory **13** (1990), 307–315, doi:10.1007/BF01199886.
[7] A. Aluthge and D. Wang, *An operator inequality which implies paranormality*, Math. Ineq. Appl. **2** (1999), no. 1, 113–119, doi:10.7153/mia-02-09.
[8] A. Aluthge and D. Wang, *w-hyponormal operators*, Integr. Equ. Oper. Theory **36** (2000), 1–10.
[9] A. Aluthge and D. Wang, *w-hyponormal operators II*, Integr. Equ. Oper. Theory **37** (2000), 324—-331, doi:10.1007/BF01194481.
[10] M. Amouch and H. Zguitti, *On the equivalence of Browder's and generalized Browder's theorem*, Glasg. Math. J. **48** (2006), 179–185, doi:10.1017/S0017089505002971.
[11] M. Amouch and M. Berkani, *on the property* (gw), Mediterr. J. Math **5** (2008), 371–378, doi:10.1007/s00009-008-0156-z.
[12] S. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13** (1962), no. 1, 111–114, doi:10.2307/2033783.
[13] M. Berkani, *On a class of quasi-Fredholm operators*, Proc. Amer. Math. Soc. **34** (1999), no. 2, 244–249, doi:10.1007/BF01236475.
[14] M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl. **272** (2002), no. 2, 596–603, doi:10.1016/S0022-247X(02)00179-8.
[15] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1717–1723, doi:10.2307/2699766.
[16] M. Berkani, *On the equivalence of Weyl's theorem and generalized Weyl's theorem*, Acta Math Sinica **23** (2007), 103–110, doi:10.1007/s10114-005-0720-4.
[17] M. Berkani and J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), no. 1, 359–376.
[18] M. Berkani and M. Sarih, *On semi B-Fredholm operators*, Glasg. Math. J. **43** (2001), 457–465, doi:10.1017/S0017089501030075.
[19] E. Bishop, *A duality theorem for an arbitrary operator*, Pacific J. Math. **9** (1959), no. 2, 379–397, doi:euclid.pjm/1103039262.
[20] Y. Changsen and L. Haiying, *On p-w-hyponormal operators*, Chinese Quart. J. Math **20** (2005), no. 1, 79–84.
[21] Y. Changsen and L. Haiying, *A note on p-w-hyponormal operators*, Acta Math Sinica (Chinese) **49** (2006), no. 1, 19–28.
[22] M. Chō and T. Yamazaki, *An operator transform from class A to the class of hyponormal operators and its application*, Integr. equ. oper. theory **53** (2005), 497–508, doi:10.1007/s00020-004-1332-6.
[23] J. chuan Hou, *On tensor products of operators*, Acta Mathematica Sinica **9** (1993), 195–202, doi:10.1007/BF02560050.
[24] L. Coburn, *Weyl's theorem for non-normal operators*, Michigan. Math. J. **13** (1966), no. 3, 285–288, doi:10.1307/mmj/1031732778.
[25] J.B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, vol. 96, Springer-Verlag New York, 1990, doi:10.1007/978-1-4757-3828-5.
[26] J.K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), no. 1, 61–69, doi:euclid.pjm/1102905839.
[27] M. Fujii, S. Izumino, and R. Nakamoto, *classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality*, Nihonkai Math. J. **5** (1994), no. 1, 61–67.
[28] T. Furuta, *On the class of paranormal operators*, Proc. Jaban. Acad. **43** (1967), no. 7, 594–598, doi:10.3792/pja/1195521514.
[29] T. Furuta, M. Ito, and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. Jpn. **1** (1999), 389–403.
[30] L. Haiying, *Powers of an invertible* (s, p)*-w-hyponormal operator*, Acta Math. Sci. **28** (2008), no. 2, 282–288, doi:10.1016/S0252-9602(08)60028-4.

[31] L. Haiying and Y. Chang-sen, *Several properties of p-w-hyponormal operator*, Chin. Quart. J. Math. **23** (2008), no. 2, 195–201.
[32] P.R. Halmos, *A Hilbert space problem book*, Graduate Texts in Mathematics, vol. 19, Springer-Verlag New York, 1982, doi:10.1007/978-1-4615-9976-0.
[33] F. Hansen, *An equality*, Math. Ann. **246** (1980), 249–250.
[34] E. Heinz, *Beiträge zur störungstheorie der spektralzerlegung*, Math. Ann. **123** (1951), 415–438.
[35] H. Heuser, *Functional analysis*, vol. 1, Wiley, 1982.
[36] I.H.Jeon and I.H.Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T$, Linear Algebra Appl. **418** (1993), no. 2-3, 854–862, doi:10.1016/j.laa.2006.02.040.
[37] I.H.Kim, *Tensor products of* log*-hyponormal operators*, Bull. Korean Math. Soc. **42** (2005), no. 2, 269–277, doi:10.4134/BKMS.2005.42.2.269.
[38] I.H. Jeon, J. Lee, and A. Uchiyama, *On p-quasihyponormal operators and quasisimilarity*, Math. Ineq. Appl. **6** (2003), no. 2, 309–315, doi:10.7153/mia-06-29.
[39] J.P.Williams, *Operators similar to their adjoints*, Proc. Amer. Math. Soc. **20** (1969), no. 1, 121–23, doi:10.2307/2035972.
[40] J.Stochel, *Seminormality of operators from their tensor product*, Proc. Amer. Math. Soc. **124** (1996), no. 1, 135–140, doi:10.1090/S0002-9939-96-03017-1.
[41] I. Jung, E. Ko, and C. Pearcu, *Aluthge transforms of operators*, Integr equ oper theory **37** (2000), 437–448, doi:10.1007/BF01192831.
[42] K.B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), no. 2, 323–336, doi:10.2140/pjm.1992.152.323.
[43] K.B. Laursen and M.M. Neumann, *An introduction to local spectral theory*, London Mathematical Society Monographs, Clarendon Press, 2000.
[44] M.Y. Lee and S.H. Lee, *On a class of operators related to paranormal operators*, J. Korean Math. Soc. **44** (2007), no. 1, 25–34, doi:10.4134/JKMS.2007.44.1.025.
[45] M.H.M.Rashid, *Property* (gb) *and perturbations*, J. Math. Anal. Appl. **383** (2011), no. 1, 82–94, doi:10.1016/j.jmaa.2011.05.001.
[46] M.H.M.Rashid, *Properties* (t) *and* (gt) *for bounded linear operators*, Mediterr. J. Math **11** (2014), no. 2, 729–744, doi:10.1007/s00009-013-0313-x.
[47] M.H.M.Rashid, *On operators satisfying* $T^m(T^*|T|^{2k}T)^{1/(k+1)}T^m \geq T^{m*}|T|^2T^m$, Commun. Korean Math. Soc. **32** (2017), no. 3, 661–676, doi:10.4134/CKMS.c160191.
[48] M.H.M.Rashid, *Quasinormality and Fuglede-Putnam theorem for* (s,p)*-w-hyponormal operators*, Linear Multilinear Algebra **65** (2017), no. 8, 1600–1616, doi:10.1080/03081087.2016.1248346.
[49] V. Rakočević, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **34** (1989), 915–919.
[50] I. Sheth, *On hyponormal operators*, Proc. Amer. Math. Soc. **17** (1966), no. 5, 998–1000, doi:10.2307/2036076.
[51] J. Stampfli, *Hyponormal operatorsand spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476, doi:10.2307/1994219.
[52] T.Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged) **33** (1972), no. 3-4, 169–178.
[53] T.Saito, *Hyponormal operators and related topics*, Lecture Notes in Mathematics, vol. 247, Springer, Berlin, Heidelberg, 1971, doi:10.1007/BFb0058557.
[54] A. Uchiyama, *Berger-Shaw's theorem for p-hyponormal operators*, Integr equ oper theory **33** (1999), 221–230, doi:10.1007/BF01233965.
[55] A. Uchiyama and K. Tanahashi, *On the Riesz idempotent of class A*, Math. Ineq. Appl. **5** (2002), no. 2, 291–298, doi:10.7153/mia-05-32.
[56] D. Xia, *Spectral theory of hyponormal operators*, Operator Theory: Advances and Applications, vol. 10, Birkhäuser Basel, 1983, doi:10.1007/978-3-0348-5435-1.

M. H. M. Rashid: malik_okasha@yahoo.com
Dept. of Mathematics & Statistics, Faculty of Science P.O. Box(7), Mu'tah University, Mu'tah-Jordan

Received 07/09/2020; Revised 29/09/2020