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AN OPERATOR APPROACH TO EXTREMAL PROBLEMS ON HARDY AND BERGMAN SPACES

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Abstract. S. Abbott and S. Abbott and B. Hanson developed an operator-theoretic approach to solve some extremal problems. We give a different proof of a theorem of S. Abbott and B. Hanson in the case when the corresponding operator is unbounded. We apply our theorem to the classical Kolmogorov and Szegö infimum problems. We also consider Kolmogorov and Szegö type infima, when integration over the unit circle is replaced by integration over the unit disk.

С. Аббот і Б. Хенсон розвинули теоретико-операторний підхід до розв'язанні деяких екстремальних задач. Ми даємо нове доведення теореми С. Аббота і Б. Хенсона для випадку, коли відповідний оператор необмежений. Теорема застосовується для класичних задач Колмогорова і Сеге про інфімум. Також розглянуті задачі Колмогорова і Сеге про інфімум для випадку, коли інтегрування ведеться не по колу, а по кругу.

1. Introduction

In 1995 S. Abbott published an article [1] in which, in particular, he proved the following theorem:

Theorem 1.1. *Let W be a bounded nonnegative and invertible operator on a Hilbert space $\mathfrak{H}$ and let P be the orthogonal projection onto a closed subspace $\mathcal{L} \subset \mathfrak{H}$. Then for any $k \in \mathcal{L}$*

$$\inf\{< W(k-f), k-f >: f \in \mathcal{L}^{\perp}\} = < [PW^{-1}P]^{-1}k, k > .$$

Remark 1.2. If we do not assume that the operator W is invertible then the right-hand side of the last formula is equal to

$$\lim_{\epsilon \downarrow 0} < [PW_{\epsilon}^{-1}P]^{-1}k, k >,$$

where $W_\epsilon = W + \epsilon I$, $\epsilon > 0$.

In that article the author applied Theorem 1.1 to some extremal problems. In particular, the author developed an operator-theoretic approach to the problem of finding Kolmogorov and Szego infima (see definitions (3.3) and (5.9), respectively). To achieve his results S. Abbott must assume the absolutely continuous parts of the measures appearing in (3.3) and (5.9) have bounded densities. The condition of boundedness of the operator W was removed in article [2]. It allowed the authors to consider problems of finding Kolmogorov and Szego infima for unbounded weights w.

In the present paper we also remove the condition of boundedness of the operator W in Theorem 1.1 and give a new proof of the main result of S. Abbott and B. Hanson (Theorem 2.3 of [2]).

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As in [1] and [2] we prove the formula for the Kolmogorov infimum with the assumption that $w \in L^1(\mathbb{T})$ and the formula for the Szegö infimum with assumption that $w \in L^2(\mathbb{T})$. We also apply our theorem to the problem of finding the Kolmogorov type and the Szegö type infima, when integration over the unit circle $\mathbb{T}$ is replaced by integration over the unit disk $\mathbb{D}$.

The article is organized as follows. In Section 2 we prove our generalization of Theorem 1.1. In Section 3 we discuss Kolmogorov infimum for the circle. In Section 4 we consider Kolmogorov type infimum for the disk. Surprisingly, this consideration can be done in an elementary way. In Section 5 we consider Szego infimum on the unit circle and in Section 6 we discuss Szegö type infimum for the unit disk. We evaluate the values of the infimum for two classes of functions $w(z)$, namely when $\log w(z)$ is harmonic and when w is radial. We also provide an estimate for the value of the Szegö type infimum for harmonic w.

2. Main Theorem

We start with the following lemma.

Lemma 2.1. *Let $\mathfrak{D}$ be a linear set dense in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M} \subset \mathfrak{D}$ be a (closed) subspace of $\mathfrak{H}$. Then $\mathfrak{M}^\perp \cap \mathfrak{D}$ is dense in $\mathfrak{M}^\perp$.*

Proof. Pick $x \in \mathfrak{M}^\perp$. Then there exists a sequence $\{d_n\}_{n=1}^\infty \subset \mathfrak{D}$ such that $x = \lim\limits_{n\to\infty} d_n$. Denote by P the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{M}^\perp$ and put $Q = I - P$, that is Q is the orthogonal projection onto $\mathfrak{M}$. We have

$$x = Px = \lim_{n\to\infty} Pd_n = \lim_{n\to\infty} (d_n - Qd_n).$$

The vectors $Pd_n \in \mathfrak{M}^\perp$. We also have $Pd_n = d_n - Qd_n \in \mathfrak{D}$ since $\mathfrak{M}$ is closed and $\mathfrak{M} \subset \mathfrak{D}$. □

Remark 2.2. Without the condition $\mathfrak{M} \subset \mathfrak{D}$ it is possible that $\mathfrak{M}^\perp \cap \mathfrak{D}=\{0\}$. But if $\dim \mathfrak{M} < \infty$ then it follows, see [4], that $\overline{\mathfrak{D}(W) \cap \mathfrak{M}^\perp} = \mathfrak{M}^\perp$

The next statement is a generalization of Theorem 1.1 for the case when the operator W is unbounded. It is equivalent to Theorem 2.3 of [2] but our proof is different.

Theorem 2.3. *Let W be a non-negative self-adjoint operator in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{D}(W)$ be the domain of W ($\overline{\mathfrak{D}(W)} = \mathfrak{H}$). Let $\mathfrak{M}$ be a finite-dimensional subspace of $\mathfrak{H}$ such that $\mathfrak{M} \subset \mathfrak{D}(W)$ and let $P = P_{\mathfrak{M}}$ be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}$. Then for any $k \in \mathfrak{M}$*

$$\begin{gathered}\inf \{< W(k-f), k-f >: f \in \mathfrak{D}(W), Pf = 0\} = \\ < [PW^{-1}P]^{-1}k, k > .\end{gathered} \tag{2.1}$$

If the operator W is not boundedly invertible, the right side of (2.1) is understood as

$$\lim_{\epsilon\downarrow 0} < [P(W+\epsilon I)^{-1}P]^{-1}k, k > .$$

Proof. Since W is a non-negative operator, the operator $W^{1/2}$ exists and well defined. The operator $W^{1/2}$ is self-adjoint and $\mathfrak{D}(W) \subset \mathfrak{D}(W^{1/2})$.

Assume first that the operator W has a bounded inverse W^{-1}. Then the operator $W^{-1/2}$ is also bounded. Put

$$\mathfrak{L} = \left\{ W^{1/2}f : f \in \mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^\perp \right\}.$$

Then $\mathfrak{L}$ is closed in $\mathfrak{H}$. Indeed, let $\{h_n\} \subset \mathfrak{L}$, $h_n \to h$. Then the sequence $\{h_n\}$ is fundamental. If $f_n = W^{-1/2}h_n$, then the sequence $\{f_n\}$ is also fundamental and there exists $\lim_{n\to\infty} f_n = f \in \mathfrak{M}^\perp$. Since $W^{1/2}$ is a closed operator, $f \in \mathfrak{D}(W^{1/2})$ and $h = W^{1/2}f \in \mathfrak{L}$.

Denote by Q the orthogonal projection onto the subspace $\mathfrak{L}^{\perp}$. Then we have

$$\inf\{< W(k-f), k-f >: f \in \mathfrak{D}(W), Pf = 0\} = \\ \inf\left\{\|W^{1/2}k - l\|^2 : l \in \mathfrak{L}\right\} = \|QW^{1/2}k\|^2 = \\ \left[\sup\left\{| < W^{1/2}k, g > | : g \in \mathfrak{L}^{\perp}, \|g\| \le 1\right\}\right]^2.$$

Now, $g \in \mathfrak{L}^{\perp}$ means

$$< W^{1/2}f, g >= 0 \qquad \forall f \in \mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^{\perp}.$$

We write $g = g_1 + g_2$ where $g_1 \in \mathfrak{M}$, $g_2 \in \mathfrak{M}^{\perp}$ and obtain

$$< W^{1/2}f, g_1 > + < W^{1/2}f, g_2 >= 0 \qquad \forall f \in \mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^{\perp}.$$

Since $\mathfrak{M} \in \mathfrak{D}(W) \subset \mathfrak{D}(W^{1/2})$, the first term of the last expression is $< f, W^{1/2}g_1 >$. Thus

$$< W^{1/2}f, g_2 >=< f, -W^{1/2}g_1 > \qquad \forall f \in \mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^{\perp}.$$

Because $\mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^{\perp}$ is dense in $\mathfrak{M}^{\perp}$, the right side of the last expression is a continuous linear functional on $\mathfrak{M}^{\perp}$, consequently so is the left side. Thus there exists $g_2^* \in \mathfrak{M}^{\perp}$ such that

$$< W^{1/2}f, g_2 >=< f, g_2^* > . \tag{2.2}$$

Note that $< W^{1/2}f, g_2 >=< P_{\perp}W^{1/2}fP_{\perp}, g_2 >$, where $P_{\perp}$ is the orthogonal projection onto the subspace $\mathfrak{M}^{\perp}$. According to Stenger's lemma (see [6]) $P_{\perp}W^{1/2}P_{\perp}$ is a self-adjoint operator in $\mathfrak{M}^{\perp}$. The equality (2.2) means that g_2 belongs to the domain of the operator $P_{\perp}W^{1/2}P_{\perp}$ and $P_{\perp}W^{1/2}P_{\perp}g_2 = g_2^*$. Thus we can write

$$< W^{1/2}f, g_2 >=< f, P_{\perp}W^{1/2}P_{\perp}g_{>} =< f, W^{1/2}g_2 > .$$

In summary, we have

$$0 =< W^{1/2}f, g_1 > + < W^{1/2}f, g_2 >=< f, W^{1/2}g_1 > + < f, W^{1/2}g_2 >= \\ < f, W^{1/2}g >, \quad \forall f \in \mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^{\perp}.$$

Therefore $g \in \mathfrak{L}^{\perp}$ implies $g \in \mathfrak{D}(W^{1/2})$ and $W^{1/2}g \in \mathfrak{M}$. Conversely, if $g \in \mathfrak{D}(W^{1/2})$ and $W^{1/2}g \in \mathfrak{M}$, then for any $f \in \mathfrak{D}(W^{1/2}) \cap \mathfrak{M}^{\perp}$ one has

$$0 =< f, W^{1/2}g >=< W^{1/2}f, g >,$$

i.e. $g \in \mathfrak{L}^{\perp}$.

We have shown that $g \in \mathfrak{L}^{\perp}$ if and only if $g \in \mathfrak{D}(W^{1/2})$ and $W^{1/2}g \in \mathfrak{M}$.

Since the operator $W^{-1/2}$ is bounded, for any $f \in \mathfrak{M}$ we may put $g = W^{-1/2}f$. Then $g \in \mathfrak{L}^{\perp}$, hence $g \in \mathfrak{D}(W^{1/2})$. Consequently we have

$$\inf\{< W(k-f), k-f >: f \in \mathfrak{D}(W), Pf = 0\} \\ = \left[\sup\left\{| < W^{1/2}k, g > | : g \in \mathfrak{L}^{\perp}, \|g\| \le 1\right\}\right]^2 \\ = \left[\sup\left\{|\frac{< W^{1/2}k, g >}{\|g\|}| : g \in \mathfrak{L}^{\perp}, g \neq 0\right\}\right]^2 \\ = \left[\sup\left\{|\frac{< W^{1/2}k, W^{-1/2}f >}{\|W^{-1/2}f\|}| : f \in \mathfrak{M}, f \neq 0\right\}\right]^2 \\ = \left[\sup\left\{|\frac{< k, f >}{\|W^{-1/2}f\|}| : f \in \mathfrak{M}, f \neq 0\right\}\right]^2.$$

Let A be an operator on $\mathfrak{M}$ defined as follows: for $f \in \mathfrak{M}$ set $Af = PW^{-1}f = PW^{-1}Pf$. Then A is a non-negative and invertible. Also, for $f \in \mathfrak{M}$,

$$\|W^{-1/2}f\|^2 =< W^{-1}f, f >=< PW^{-1}Pf, f >=< Af, f >= \|A^{1/2}f\|^2.$$

We have

$$\sup\left\{|\frac{<k,f>}{\|W^{-1/2}f\|}|: f\in\mathfrak{M}, f\neq 0\right\}$$
$$=\sup\left\{\frac{|<A^{-1/2}k, A^{1/2}f>|}{\|A^{1/2}f\|}: f\in\mathfrak{M}, f\neq 0\right\}$$
$$=\sup\left\{|<A^{-1/2}k,f>|: f\in\mathfrak{M}, \|f\|=1\right\}=\|A^{-1/2}f\|.$$

Now combining the equalities above we obtain

$$\inf\{<W(k-f),k-f>: f\in\mathfrak{D}(W), Pf=0\}=\|A^{-1/2}k\|^2$$
$$=<A^{-1}k,k>=<[PW^{-1}P]^{-1}k,k>.$$

If the operator W is not boundedly invertible we use the same argument that was used in [1] to finish the proof. □

Remark 2.4. We may slightly relax the hypotheses of Theorem 2.3 and assume that the finite-dimensional subspace $\mathfrak{M}$ satisfies condition

$$\mathfrak{M}\subset\mathfrak{D}(W^{1/2}).$$

3. The Kolmogorov Infimum

Denote by $\mathcal{Q}$ the set of trigonometric polynomials of the form $\sum_{-N}^{M} c_k e^{ik\theta}$ and by $\mathcal{Q}_0\subset\mathcal{Q}$ the set of trigonometric polynomials with $c_0=0$. Let μ be a finite positive measure on the unit circle $\mathbb{T}$ and let τ be the normalized Lebesgue measure on the unit circle. Denote by $\mathbb{1}$ the constant function from $L^2(\mathbb{T},d\tau)$ which assumes the value 1 at each point of the circle.

The Kolmogorov infimum for the circle, $K_{\mathbb{T}}(\mu)$, is defined as follows:

$$K_{\mathbb{T}}(\mu)=\inf\{\int_{\mathbb{T}}|1-q(e^{i\theta})|^2 d\mu(\theta): q\in\mathcal{Q}_0\}. \tag{3.3}$$

If the measure μ is absolutely continuous with respect to the Lebesgue measure, $d\mu=w(\theta)d\tau$, $w\geq 0$, $w\in L^1(\mathbb{T},d\tau)$ we write $K_T(w)$ instead of $K_{\mathbb{T}}(\mu)$.

Using the same arguments that are given in [5], pp.44-45, one deduces that $K_{\mathbb{T}}(\mu)$ depends only on the absolutely continuous part of μ.

Theorem 3.1. *Let μ be a positive measure on the unit circle $\mathbb{T}$. If $d\mu(\theta)=w(\theta)d\tau+d\mu_s$ (μ_s is the singular part of μ), then*

$$K_{\mathbb{T}}(\mu)=K_{\mathbb{T}}(w)=\Big[\int_{\mathbb{T}}\frac{1}{w(\theta)}d\tau\Big]^{-1}. \tag{3.4}$$

If $w\notin L^1(\mathbb{T},d\tau)$, then $K_{\mathbb{T}}(\mu)=0$.

Proof. We use Theorem 2.1 and Remark 2.4. Put $\mathfrak{H}=L^2(\mathbb{T},d\tau)$ and denote by $W^{1/2}=M_{w^{1/2}}$ the operator of multiplication by $w^{1/2}$. The domain $\mathfrak{D}(W^{1/2})$ of $W^{1/2}$ is the set of all $f\in L^2(\mathbb{T},d\tau)$ such that $\sqrt{w(\theta)}f(\theta)\in L^2(\mathbb{T},d\tau)$. Since $L^\infty(\mathbb{T},d\tau)\subset\mathfrak{D}(W^{1/2})$, the operator $W^{1/2}$ is densely defined and clearly self-adjoint. Then the operator $W=M_w$ is also densely defined and self-adjoint.

Also, put $\mathfrak{M}=\text{l.h.}\{\mathbb{1}\}$ (l.h. means linear hull), the subspace generated by the constant function $\mathbb{1}$, and the vector k is that constant function $\mathbb{1}$. Then according to Theorem 2.1 we have

$$K(w)=<[PM_w^{-1}P]^{-1}\mathbb{1},\mathbb{1}>,$$

where P is the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{M}$.

Put $[PM_w^{-1}P]^{-1}\mathbb{1} = h$. Then $K(\omega) =< h, \mathbb{1} >= h_0$. We have

$$\mathbb{1} = PM_w^{-1}Ph = h_0 PM_\omega^{-1}\mathbb{1} = h_0 \left(\int_{\mathbb{T}} \frac{d\tau}{w(\theta)}\right) P\mathbb{1} = h_0 \left(\int_{\mathbb{T}} \frac{d\tau}{w(\tau)}\right)\mathbb{1}.$$

Therefore, $h_0 = \big(\int_{\mathbb{T}} \frac{d\tau}{w(\theta}\big)^{-1}$. The theorem is proved. □

4. The Kolmogorov Type Infimum

Denote by $\mathcal{P} = \mathcal{P}(z, \bar{z})$ the set of polynomials $p(z, \bar{z})$ of variables z and $\bar{z}$, $z \in \mathbb{D}$, and by $\mathcal{P}_0 \subset \mathcal{P}$ polynomials satisfying the condition $p(0, 0) = 0$.

Under the Kolmogorov type infimum we understand the quantity $K(\mu)$ defined as follows:

$$K_{\mathbb{D}}(\mu) = \inf\Big\{\int_{\mathbb{D}} |1 - p(z, \bar{z})|^2 d\mu(z) : f \in \mathcal{P}_0)\Big\}, \tag{4.5}$$

where μ is a positive measure on the unit disk $\mathbb{D}$. The quantity $K_{\mathbb{D}}(\mu)$ is the square of the distance from $\mathbb{1}$ to the closure of $\mathcal{P}_0$ in the Hilbert space $L^2(\mathbb{D}, d\mu)$.

Denote by $\mu_0(z)$ the Dirac measure at the origin.

Lemma 4.1. *Let μ be a finite measure on the unit disk $\mathbb{D}$ which annihilates all $p \in \mathcal{P}_0$. Then $\mu(z) = \lambda\mu_0(z)$, where λ is a constant.*

Proof. Put $\lambda = \int_{\mathbb{D}} d\mu(z)$ and $d\mu_1(z) = d\mu(z) - \lambda d\mu_0(z)$. For any $p \in \mathcal{P}$ we have

$$\int_{\mathbb{D}} p(z, \bar{z})d\mu_1(z) = \int_{\mathbb{D}} [p(z, \bar{z}) - p(0, 0)]d\mu_1 + p(0, 0)\int_{\mathbb{D}} d\mu_1(z) = 0.$$

Hence, the measure μ_1 annihilates all polynomials $p \in \mathcal{P}$ and consequently is the zero measure. □

Let μ be a finite positive measure on the unit disk $\mathbb{D}$. Denote by $\mathfrak{N}$ the closure of $\mathcal{P}_0$ in the the space $L^2(\mathbb{D}, d\mu)$ and suppose that $\mathbb{1} \notin \mathfrak{N}$. Let F be the orthogonal projection of $\mathbb{1}$ onto $\mathfrak{N}$. Then

$$\int_{\mathbb{D}} |1 - F(z, \bar{z})|^2 d\mu(z) > 0. \tag{4.6}$$

The function $1 - F$ is orthogonal to $\mathfrak{N}$. But $(1 - F)p$ is also orthogonal to $\mathfrak{N}$ for every $p \in \mathcal{P}_0$. Indeed, F is the limit in $L^2(\mathbb{D}, d\mu)$ of the sequence of elements p_n from $\mathcal{P}_0$ and if p is a fixed element from $\mathcal{P}_0$, then $(1 - p_n)p \in \mathcal{P}_0$ and converges to $(1 - F)p$. The statement that $(1 - F)$ is orthogonal to $(1 - F)p$ for each $p \in \mathcal{P}_0$ means

$$\int_{\mathbb{D}} p(z, \bar{z})|1 - F(z, \bar{z})|^2 d\mu(z) = 0. \tag{4.7}$$

Now Lemma 4.1 gives

$$|1 - F|^2 d\mu(z) = \lambda d\mu_0(z), \qquad \lambda \in \mathbb{R}. \tag{4.8}$$

Denote by $\mathbb{D}_0$ the punctured unit disk, $\mathbb{D}_0 = \mathbb{D} \setminus \{0\}$. Then from (4.8) it follows

$$\int_{\mathbb{D}_0} |1 - F|^2 d\mu(z) = 0.$$

Therefore $F = 1$ $\mu-$ almost everywhere on $\mathbb{D}_0$. If, in addition, $\mu(\{0\}) = 0$ (μ has no mass at the origin), then

$$\int_{\mathbb{D}} |1 - F|^2 d\mu(z) = 0$$

which contradicts (4.6).

Suppose now that $\mu(\{0\}) > 0$. Since $1 - F = 0$ $\mu-$ almost everywhere on $\mathbb{D}_0$ we have

$$\int_{\mathbb{D}} |1 - F(z,\bar{z})| d\mu(z) = \lim_{n\to\infty} \int_{\mathbb{D}} |1 - p_n(z,\bar{z})|^2 d\mu = \mu(\{0\}),$$

where $p_n \in \mathcal{P}_0$ are polynomials that converge to F in $L^2(\mathbb{D}, d\mu)$.

Thus we obtained the following result:

Theorem 4.2. . *Let μ be a finite positive measure on the unit disk $\mathbb{D}$. Then*

$$K_{\mathbb{D}}(\mu) = \mu(\{0\}).$$

Remark 4.3. Instead of the unit disk $\mathbb{D}$ we may consider the unit ball B in $\mathbb{C}^n$ and a positive finite measure ν on B. The definition of the Kolmogorov type infimum $K_B(\nu)$ is clear. Using the same arguments as above one obtains

$$K_B(\nu) = \nu(\{0\}).$$

5. The Szegö Infimum

First we recall the classical Szegö-Kolmogorov-Krein theorem:

Theorem 5.1. *Let μ be a finite positive measure on the unit circle $\mathbb{T}$. Put $d\mu = w(\theta)d\tau + d\mu_s$, where τ is the normalized Lebesgue measure on the unit circle and μ_s is the singular component of the measure μ. Then*

$$\begin{aligned} S_{\mathbb{T}}(\mu) &= \inf\Big\{ \int_{\mathbb{T}} |1 - p(e^{i\theta})|^2 d\mu(\theta) : p(e^{i\theta}) = \sum_{j=1}^{N} c_j e^{i\theta j} \Big\} \\ &= \inf\Big\{ \int_{\mathbb{T}} |1 - p(e^{i\theta})|^2 w(\theta) d\tau : p(e^{i\theta}) = \sum_{j=1}^{N} c_j e^{i\theta j} \Big\} = \exp\Big[\int_{\mathbb{T}} \log w(\theta) d\tau \Big]. \qquad (5.9) \end{aligned}$$

In particular,

$$S_{\mathbb{T}}(\mu_s) = \inf\Big\{ \int_{\mathbb{T}} |1 - p(e^{i\theta})|^2 d\mu_s(\theta) : p(e^{i\theta}) = \sum_{j=1}^{N} c_j e^{i\theta j} \Big\} = 0.$$

For the proof of Theorem 5.1 we refer for example, to the book [5]. $S_{\mathbb{T}}(\mu)$ is called the Szegö infimum. The operator-theoretic proof of formula (5.9) under the assumption $w \in L^\infty(\mathbb{T}, d\tau)$ was presented in [1]. Below we give a proof of formula (5.9) using Theorem 2.3 under the assumption that $w \in L^2(\mathbb{T}, d\tau)$.

Put $\mathfrak{H} = H^2(\mathbb{T})$ (the Hardy space on the unit circle) and for $\epsilon > 0$ denote $W = T_{w+\epsilon} = T_w + \epsilon I$, a (generally speaking unbounded) Toeplitz operator on $\mathfrak{H}$ with symbol $w + \epsilon$, given by

$$T_{w+\epsilon} f = P_S (w + \epsilon) f, \quad f \in \mathfrak{D}(T_w) \qquad (5.10)$$

where P_S is the (Szegö) projection from $L^2(\mathbb{T}, d\tau)$ onto H^2. The domain of $T_{w+\epsilon}$ is the set of all $f \in H^2$ for which right-hand side of (5.10) is in H^2. From our assumption about the function w it follows that $H^\infty \subset \mathfrak{D}(T_{w+\epsilon})$. Hence the operator $T_{w+\epsilon}$ is densely defined. Since w is a nonnegative real-valued function, $T_{w+\epsilon}$ is self-adjoint, $T_{w+\epsilon} \geq 0$, and $T_{w+\epsilon}$ has bounded inverse. Also, put $\mathfrak{M} = \text{l.h.}\{\mathbb{1}\}$. Note, that

$$\int_{\mathbb{T}} |1 - p(e^{i\theta})|^2 w(\theta) d\theta =< T_w(1-p), (1-p) >, \qquad p(e^{i\theta}) = \sum_{j=1}^{N} c_j e^{i\theta j}.$$

consequently, according to Theorem 2.3 one obtains

$$S_{\mathbb{T}}(\mu) = \lim_{\epsilon \downarrow 0} < [P T_{w+\epsilon}^{-1} P]^{-1} \mathbb{1}, \mathbb{1} >,$$

where P is the orthogonal projection of H^2 onto $\mathbb{1}$. Now we proceed in the way similar to that one in [1]. Put

$$g_\epsilon(z) = \exp\left[\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(w(\theta)+\epsilon\right)^{1/2} d\tau\right]$$

Then g_ϵ is an outer function from H^4, $w(\theta)+\epsilon = |g_\epsilon(e^{i\theta})|^2$,
$|g_\epsilon(0)|^2 = \exp\left[\int_{\mathbb{T}} \log\left(w(\theta)+\epsilon\right)d\tau\right]$, and $T_{w+\epsilon} = T_{\bar{g}_\epsilon}T_{g_\epsilon}$. Hence

$$T_{w+\epsilon}^{-1} = T_{g_\epsilon}^{-1}T_{\bar{g}_\epsilon}^{-1} = T_{1/g_\epsilon}^{-1}(T_{1/g_\epsilon}^{*})^{-1}.$$

Now we proceed in the same way as in the proof of Theorem 3.4 and obtain

$$< [PT_{w+\epsilon}^{-1}P^{-1}]^{-1}\mathbb{1}, \mathbb{1} >= |g_\epsilon(0)|^2 = \exp\left[\int_{\mathbb{T}} \log\left(w(\theta)+\epsilon\right)d\tau\right].$$

Therefore

$$S_{\mathbb{T}}(\mu) = \lim_{\epsilon\downarrow 0} \exp\left[\int_{\mathbb{T}} \log\left(w(\theta)+\epsilon\right)d\tau\right].$$

Existence of the limit is guaranteed by the monotone convergence theorem.
□

6. The Szegö Type Infimum

Let μ be a positive finite measure on the unit disk $\mathbb{D}$. We define a Szegö type infimum $S_{\mathbb{D}}(\mu)$ by the following formula

$$S_{\mathbb{D}}(\mu) = \inf\{\int_{\mathbb{D}} |1-p(z)|^2 d\mu(z) : p(z) = \sum_{1}^{N} c_j z^j, N \text{ is finite}\} \tag{6.11}$$

Note the essential difference between $S_{\mathbb{T}}(\mu)$ and $S_{\mathbb{D}}(\mu)$. *There are singular measures μ_s on $\mathbb{D}$ such that $S(\mu_s) > 0$ (compare with Theorem 5.1!).*

Indeed, let $\mu_s(z) = c\mu_0(z)$, $\mu_0(z)$ is the Dirac measure at the origin. Then for any analytic polynomial $p(z)$, $p(0) = 0$ we have

$$\int_{\mathbb{D}} |1-p(z)|^2 d\mu_s(z) = c.$$

But there are singular measures μ_s such that $S(\mu_s) = 0$. Indeed, if the support of the measure μ_s consists of a finite number of points z_k, $k = 1, \ldots, N$, off the origin, we can always find an analytic polynomial $p(z)$, $p(0) = 0$, such that $p(z_k) = 1$. Then for such polynomial p we have

$$\int_{\mathbb{D}} |1-p(z)|^2 d\mu_s(z) = \sum_{k=1}^{N} |1-p(z_k)|^2 \mu_s(\{z_k\}) = 0.$$

Later on we assume that the measure μ is absolutely continuous with respect to Lebesgue measure, that is $d\mu(z) = w(z)d\sigma(z)$, $w \geq 0$, $\omega \in L^1(\mathbb{D}, d\sigma)$ and write $S_{\mathbb{D}}(w)$ instead of $S_{\mathbb{D}}(wd\sigma)$. Here $d\sigma$ is the normalized Lebesgue measure of the unit disk $\mathbb{D}$.

For a nonnegative function $w(z)$ we denote by T_w a Toeplitz operator on the Bergman space $A^2(\mathbb{D}, d\sigma)$. Recall that T_w is defined by the formula

$$T_w f = P_B w f = \int_{\mathbb{D}} \frac{w(\zeta)f(\zeta)}{(1-z\bar{\zeta})^2} d\sigma(\zeta), \qquad f \in \mathfrak{D}(T_w),$$

where P_B is the Bergman projection from $L^2(\mathbb{D}, d\sigma)$ onto $A^2(\mathbb{D}, d\sigma)$. We assume that the domain $\mathfrak{D}(T_w)$ of the operator T_w contains function $\mathbb{1}$, T_w is densely defined and selfadjoint (this condition, $w \in L^2(\mathbb{D}, d\sigma)$, guarantees this, and of course there are other functions $w \in L^1(\mathbb{D}, d\sigma)$ for which these conditions are satisfied). If we now set in Theorem 2.3, $\mathfrak{H} = A^2(\mathbb{D}, d\sigma)$, $\mathfrak{M} = \text{l.h.}\{\mathbb{1}\}$, and $W = T_w$ we obtain

$$S_{\mathbb{D}}(w) =< [PT_w^{-1}P]^{-1}\mathbb{1}, \mathbb{1} > . \tag{6.12}$$

Below we give two applications of formula (6.12).

Theorem 6.1. *Suppose that a function w is defined on the unit disk $\mathbb{D}$ and satisfies the following conditions:*

(1) $w(z) > 0$ *for* $z \in \mathbb{D}$*;*
(2) $w \in L^2(\mathbb{D}, \sigma)$*;*
(3) $\log w$ *is harmonic in* $\mathbb{D}$*.*

Then

$$S_{\mathbb{D}}(w) = w(0) = \exp\,[\int_{\mathbb{D}} \log w(z) d\sigma(z)].$$

Proof. At first we note that condition 3 of the above theorem is necessary and sufficient for w to admit a factorization $w(z) = |g(z)|^2$, where g is analytic in $\mathbb{D}$. Since $w \in L^2(\mathbb{D}, d\sigma)$, the function g belongs to the Bergman space $A^4(\mathbb{D}, d\sigma)$. Therefore, the Toeplitz operator T_w admits a factorization $T_w = T_{\bar{g}}T_g$. Now direct calculations gives $S_{\mathbb{D}}(w) = |g(0)|^2 = w(0)$. □

Theorem 6.2. *Let a nonnegative function $w(z)$, $z \in \mathbb{D}$ be of the form*

$$w(z) = \varphi(r), \quad z = re^{i\theta},$$

where $\varphi \in L^1([0, 1], dr)$. Then

$$S_{\mathbb{D}}(w) = 2\int_0^1 \varphi(r) r dr = \int_{\mathbb{D}} w d\sigma$$

Proof. For a Toeplitz operator T_w on Bergman space $A^2(\mathbb{D}, d\sigma)$ with a radial symbol w, each of the functions z^k, $k = 0, 1, \ldots$ is an eigenfunction,

$$T_w z^k = \lambda_k z^k$$

where the corresponding eigenvalue λ_k is given by

$$\lambda_k = 2(k+1)\int_0^1 \varphi(r) r^{2k+1} dr = (k+1)\int_{\mathbb{D}} w r^{2k} d\sigma.$$

Therefore the domain $\mathfrak{D}(T_w)$ of the operator T_w is dense in $A^2(\mathbb{D})d\sigma$ and $T_w^{\mathbb{1}}\mathbb{1} = \lambda_0^{-1}\mathbb{1}$. Now the proof completed as in the proof of Theorem 3.1. □

Corollary 6.3. *Let w be a positive harmonic function. Then*

$$\int_D w(z) d\sigma(z) \geq S(w) \geq (3 - 4\log 2) w(0) \approx 0.2274 w(0).$$

Proof. Indeed, according to the Harnack's inequality

$$w(z) \geq \frac{1-r}{1+r} w(0), \qquad z = re^{i\theta}.$$

Now we have

$$S(w) \geq w(0) \inf\Big\{\int_{\mathbb{D}} |1 - p(z)|^2 \frac{1-r}{1+r} d\sigma(z) : p = \sum_{j=1}^N c_j z^j\Big\}$$
$$= 2w(0)\int_0^1 r\frac{1-r}{1+r} dr = (3 - 4\log 2) w(0) \approx 0.2274 w(0).$$

It is also obvious that $S(w) \leq \int_{\mathbb{D}} \omega(z) d\sigma(z) = w(0)$. □

Expression (6.11) is the square of the distance in the Hilbert space $L^2(\mathbb{D}, d\mu)$ from the function $\mathbb{1}$ to the closed subspace generated by analytic polynomials $p(z)$, $p(0) = 0$. The following statement is well known (see, for example [3]).

Let $h_0, h_1, \ldots,$ be vectors from a Hilbert space $\mathfrak{H}$. Let $\delta \geq 0$ be the distance in $\mathfrak{H}$ from h_0 to the subspace generated by vectors $h_1, h_2 \ldots,$. Then

$$\delta^2 = \lim_{n\to\infty} \frac{\det G(h_0, h_1, \ldots, h_n)}{\det G(h_1, h_2, \ldots, h_n)} \tag{6.13}$$

where $G(h_0, h_1, \ldots, h_n)$ is the Gram matrix of vectors $h_0, h_1, \ldots, h_n$ and similarly for the denominator.

In the case of a radial function w the corresponding Gram matrix is diagonal and the statement of Theorem 6.2 is immediate. Theorem 6.1 can be reformulated as follows: *Let $w(z)$, $z \in \mathbb{D}$ be a nonnegative function, $w \in L^2(\mathbb{D}, d\sigma)$ and define*

$$\gamma_{jk} = \int_D z^k \bar{z^j} w(z) d\sigma(z), \qquad j, k = 0, 1, \ldots$$

Suppose that $\log w(z)$ is harmonic in $\mathbb{D}$. Then

$$\lim_{n\to\infty} \frac{\det\,(\gamma_{jk})_{j,k=0}^n}{\det\,(\gamma_{jk})_{j,k=1}^n} = w(0).$$

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