Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 2, pp. 151–156
https://doi.org/10.31392/MFAT-npu26_2.2021.04

A REMARK ON THE RANGE CLOSURES OF AN ELEMENTARY OPERATOR

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Abstract. Let $L(H)$ denote the algebra of operators on a complex infinite dimensional Hilbert space H into itself. For $A, B \in L(H)$, the elementary operator $\tau_{A,B} \in L(L(H))$ is defined by $\tau_{A,B}(X) = AXB - X$. An operator $A \in L(H)$ is said to be generalized quasi-adjoint if $ATA = T$ implies $A^*TA^* = T$ for every $T \in C_1(H)$ (trace class operators). In this paper, we give an extension of generalized quasi-adjoint operators. We consider the class of pairs of operators $A, B \in L(H)$ such that $\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}$, where $\overline{R(\tau_{A,B})}^{W^*}$ denotes the ultra-weak closure of the range $R(\tau_{A,B})$ of $\tau_{A,B}$. Such pairs of operators are called generalized quasi-adjoint. We establish some basic properties of those pairs of operators.

Нехай $L(H)$ – алгебра операторів у комплексному нескінченновимірному гільбертовому просторі H. Для $A, B \in L(H)$, елементарний оператор $\tau_{A,B} \in L(L(H))$ визначається як $\tau_{A,B}(X) = AXB - X$. Кажуть, що оператор $A \in L(H)$ є узагальненим квазіспряженим, якщо з $ATA = T$ випливає, що $A^*TA^* = T$ для кожного $T \in C_1(H)$ (клас ядерних операторів). У статті дається розширення класу узагальнених квазіспряжених операторів. Розглядається клас пар операторів $A, B \in L(H)$, таких, що $\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}$, де через $\overline{R(\tau_{A,B})}^{W^*}$ позначене ультраслабке замикання області значень $R(\tau_{A,B})$ of $\tau_{A,B}$. Такі пари операторів звуться узагальненими квазіспряженими. Встановлені основні властивості таких пар операторів.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators acting on H into itself.

For $n \geq 1$, let $A = (A_1, \cdots, A_n)$ and $B = (B_1, \cdots, B_n)$ be n-tuples of operators in $L(H)$. The corresponding the elementary operator $R_{A,B}$ is the operator

$$\begin{aligned} R_{A,B}: \quad L(H) &\longrightarrow \quad L(H) \\ X &\longmapsto \quad R_{A,B}(X) = \Sigma_{i=1}^n A_i X B_i. \end{aligned}$$

Let A and B be operators in $L(H)$, the most important special examples of elementary operators are the left and right multiplications $L_A : X \longmapsto AX$ and $R_B : X \longmapsto XB$, the generalized derivation $\delta_{A,B} : X \longmapsto AX - XB$, the elementary multiplication operator $S_{A,B} : X \longmapsto AXB$ and the elementary operator $\tau_{A,B} : X \longmapsto AXB - X$, we simply write τ_A for $\tau_{A,A}$.

Given two n-tuples $A = (A_1, \cdots, A_n)$ and $B = (B_1, \cdots, B_n)$ of bounded linear operators on H, the elementary operator $R_{A,B}$ has the form:

$$R_{A,B} = \Sigma_{i=1}^n S_{A_i,B_i} = \Sigma_{i=1}^n L_{A_i} R_{B_i}.$$

The elementary operators on $L(H)$ are thus elements of the subalgebras of $L(L(H))$ generated by the left and the right multiplications L_U and R_V with arbitrary $U, V \in L(H)$. Elementary operators were introduced by Lumer and Rosenblum [12], who emphasized spectral properties and applications to systems of operator equations.

2020 *Mathematics Subject Classification.* 47A30, 47A63, 47B15, 47B20, 47B47, 47B10.

Keywords. Elementary operator, Fuglede-Putnam property, trace class operator, quasi-adjoint operator.

The elementary operator $R_{A,B}$ has been studied extensively, and many of its spectral, algebraic, metric and structural properties are known ([5, 6, 7, 10, 11, 13, 14, 15, 17, 18]). But several problems concerning the range of $R_{A,B}$ remains also open ([1, 4, 8, 9, 19]).

An operator $A \in L(H)$ is called quasi-adjoint if the norm closure of the range $R(\tau_A)$ of τ_A is closed under taking the adjoint, i.e,

$$\overline{R(\tau_A)} = \overline{R(\tau_{A^*})}.$$

Clearly, A is quasi-adjoint if and only if $\overline{R(\tau_A)}$ is a self adjoint subspace of $L(H)$. In [2] it is proved that if A is quasi-adjoint, then the pair (A, A) satisfies the property $(FP)(\tau, C_1(H))$, that is, $ATA = T$ implies $A^*TA^* = T$ for every $T \in C_1(H)$ (trace class operators). Operators $A \in L(H)$ for which the pair (A, A) satisfies the property $(FP)(\tau, C_1(H))$ are termed generalized quasi-adjoint operators. Examples of generalized quasi-adjoint operators include the normal operators and contractions.

S.Bouali and Y.Bouhafsi introduced the class of generalized quasi-adjoint operators, and they gave some basic properties of those operators ([3]).

In this paper, we would like to explore this class of operators. We initiate the study of a more general classes of generalized quasi-adjoint operators. The pair (A, B) of operators $A, B \in L(H)$ is called generalized quasi-adjoint if

$$\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*},$$

where $\overline{R(\tau_{A,B})}^{W^*}$ is the ultra-weak closure of $R(\tau_{A,B})$. We use different arguments to generalize some results on this class of operators. We establish a characterization and some basic properties of pairs generalized quasi-adjoint of operators A and B. We conclude this section with some notations.

Let $K(H)$, $C_1(H)$ and $F(H)$ be respectively the ideal of compact operators, the ideal of trace class operators and the ideal of finite rank on H. The trace function is defined on $C_1(H)$ by $tr(T) = \sum_n (Te_n, e_n)$, where (e_n) is any complete orthonormal sequence in H. The weakly continuous linear functionals on $L(H)$ are those of the form $f_T(X) = tr(XT)$, where $T \in F(H)$. The ultra-weakly continuous linear functionals on $L(H)$ are those of the form $f_T(X) = tr(XT)$, where $T \in C_1(H)$.

Given $X \in L(H)$, we shall denote the kernel, the orthogonal complement of the kernel, the closure of the range of X, the restriction of X to an invariant subspace M by $\ker(X)$,$\ker^\perp X$, $\overline{R(X)}$ and $X|M$ respectively. The spectrum and the point spectrum will be denoted by $\sigma(X)$ and $\sigma_p(X)$. Let $\mathcal{B}$ be a Banach and let $\mathcal{S}$ be a subspace of $\mathcal{B}$. By $\mathcal{B}'$ we denote the dual of $\mathcal{B}$, the set

$$\mathcal{S}^\circ = \{f \in \mathcal{B}' : f(x) = 0 \text{ for every } x \in \mathcal{S}\}$$

denotes the annihilator of S.

2. Preliminaries

Theorem 2.1 ([16]). *Let E, F be Banach spaces and let $S \in L(E, F)$, then*

$$R(S^{**})^\circ = R(S^{**})^\circ \cap F^\circ \oplus \ker(S^*).$$

Let $A, B \in L(H)$. If we consider $E = F = K(H)$ and $S = \tau_{A,B} : K(H) \longrightarrow K(H)$. By duality we have $S^* = \tau_{B,A} : C_1(H) \longrightarrow C_1(H)$. Then we get the following result.

Theorem 2.2. *Let $A, B \in L(H)$, then*

$$R(\tau_{A,B})^\circ = R(\tau_{A,B})^\circ \cap K(H)^\circ \oplus \ker(\tau_{B,A}) \cap C_1(H).$$

Definition 2.3. Let $A \in L(H)$. The operator A is said to be quasi-adjoint if

$$\overline{R(\tau_A)} = \overline{R(\tau_{A^*})}.$$

Remark 2.4. Let $A \in L(H)$, then A is quasi-adjoint if and only if $\overline{R(\tau_A)}$ is a self-adjoint subspace of $L(H)$. Equivalently,$R(\tau_A)^\circ$ the annihilator of $R(\tau_A)$ is a self-adjoint subspace of $L(H)$, in the sense that, $f \in R(\tau_A)^\circ$ implies $f^* \in R(\tau_A)^\circ$, where $f^*(X) = \overline{f(X^*)}$ for all $X \in L(H)$.

Theorem 2.5 ([2]). *Let $A \in L(H)$. Then the following statements are equivalent:*
(1) *A is quasi-adjoint.*
(2) (i) *The element $\pi(A)$ of the Calkin algebra is quasi-adjoint, and*
(ii) *$ATA = T$ and $T \in C_1(H)$ implies $A^*TA^* = T$.*

3. Main Results

Definition 3.1. Let $A, B \in L(H)$. The pair (A, B) is called quasi-adjoint If

$$\overline{R(\tau_{A,B})} = \overline{R(\tau_{A^*,B^*})}.$$

Definition 3.2. Let $A, B \in L(H)$ and $\mathcal{J}$ be a two-sided ideal of $L(H)$. The pair (A, B) is said to possess the Fuglede-Putnam property $(FP)(\tau, \mathcal{J})$ if $ATB = T$ and $T \in \mathcal{J}$ implies $A^*TB^* = T$. i.e. $\ker(\tau_{A,B}|\mathcal{J}) \subseteq \ker(\tau_{A^*,B^*}|\mathcal{J})$.

Definition 3.3. Let $A, B \in L(H)$. The pair (A, B) of operators A and B is called generalized quasi-adjoint if:

$$\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}.$$

Theorem 3.4. *Let $A, B \in L(H)$. The pair (A, B) is generalized quasi-adjoint if and only if $(A, B) \in (FP)_{C_1(H)}$ and $(B, A) \in (FP)_{C_1(H)}$.*

Proof. Observe that the assertion $\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}$, is equivalent to

$$R(\tau_{A,B})^\circ \cap \overline{L^{'}(H)}^{W^*} = R(\tau_{A^*,B^*})^\circ \cap \overline{L^{'}(H)}^{W^*}.$$

It is known from Theorem 2.2. that

$$R(\tau_{A,B})^\circ \simeq R(\tau_{A,B})^\circ \cap K(H)^\circ \oplus \ker(\tau_{B,A}) \cap C_1(H).$$

Hence, it follows that

$$R(\tau_{A,B})^\circ \cap \overline{L^{'}(H)}^{W^*} = \ker(\tau_{B,A}) \cap C_1(H).$$

Consequently, we have $\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}$ if and only if

$$\ker(\tau_{B,A}) \cap C_1(H) = \ker(\tau_{B^*,A^*}) \cap C_1(H).$$

This completes the proof. □

Remark 3.5. (1) Let $A, B \in L(H)$. Then (A, B) is generalized quasi-adjoint if and only if (A^*, B^*) is generalized quasi-adjoint.
(2) (A, B) is generalized quasi-adjoint in each of the following cases:
(i). A, B are normal operators.
(ii). A, B are contractions.

Theorem 3.6. *Let $A, B \in L(H)$. If there exist $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta = 1$ and nonzero vectors $f, g \in H$ such that:*
(1) *$Bf = \alpha f$, $\|B^*f\| \neq \|\alpha f\|$ and*
(2) *$A^*g = \overline{\beta}g$.*
Then the pair (A, B) is not generalized quasi-adjoint.

Proof. We must show that $\overline{R(\tau_{A,B})}^{W^*} \neq \overline{R(\tau_{A^*,B^*})}^{W^*}$. Then, it is easy to see that $\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}$ if and only if for every $T \in C_1(H)$ we have

$$f_T \in R(\tau_{A,B})^\circ \Longleftrightarrow f_T \in R(\tau_{A^*,B^*})^\circ.$$

It suffices to exhibit a trace class operator T for which $f_T \in R(\tau_{A,B})^\circ$ but $f_T \notin R(\tau_{A^*,B^*})^\circ$. Let us define the rank one operator $T = f \otimes g$. Hence for any operator Y in $L(H)$, we obtain

$$\begin{aligned} f_T(\tau_{A,B}(Y)) &= tr[(AYB - Y)T] \\ &= tr[\{(A-\beta)YB + \beta Y(B-\alpha)\}T] \\ &= tr[(A-\beta)YBT] + tr[\beta Y(B-\alpha)T] \\ &= < YBf, (A^* - \overline{\beta})g > + < \beta Y(B-\alpha)f, g > \\ &= 0. \end{aligned}$$

Define an operator $X \in L(H)$ by $X = g \otimes (B-\alpha)^* f$. Then, it follows that

$$\begin{aligned} f_T(\tau_{A^*,B^*}(X)) &= tr[(A^*XB^* - X)T] \\ &= tr[\{(A^* - \overline{\beta})XB^* + \overline{\beta}X(B^* - \overline{\alpha})\}T] \\ &= tr[(A^* - \overline{\beta})XB^*T] + tr[\overline{\beta}X(B^* - \overline{\alpha})T] \\ &= tr[\{(A^* - \overline{\beta})(g \otimes (B-\alpha)^* f)B^*\}T] \\ &\qquad + tr[\{\overline{\beta}(g \otimes (B-\alpha)^* f)(B^* - \overline{\alpha})\}T] \\ &= \overline{\beta}\|(B-\alpha)^* f\|^2.\|g\|^2 \neq 0. \end{aligned}$$

Which completes the proof. □

Example 3.7. Let $(e_n)_n$ be an orthonormal basis for H. Let $H_\circ = vect\{e_1, e_2, e_3\}$ and set

$$B_\circ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in L(H_\circ).$$

We define the operator $B = B_\circ \oplus I$ with respect to decomposition $H = H_\circ \oplus H_\circ^\perp$ and $A = e_2 \otimes e_2$. It is easily seen that, $Be_1 = e_1$, $B^*e_1 = e_1 + e_2$, and $A^*(e_2) = e_2$. Hence, It follows that the pair (A, B) is not generalized quasi-adjoint.

Remark 3.8. Let $A, B \in L(H)$, if the pair (A, B) is generalized quasi-adjoint, then for all $T \in \ker(\tau_{A,B}|C_1(H))$, $\overline{R(T)}$ reduces A, $\ker^\perp T$ reduces B, and the restrictions $A|\overline{R(T)}$ and $B|\ker^\perp T$ are unitarily equivalent to normal operators.

Proposition 3.9. *Let $A, B \in L(H)$. If A and B are generalized quasi-adjoint operators such that $1 \notin \sigma(A)\sigma(B)$, then the pair (A, B) is generalized quasi-adjoint.*

Proof. Assume that A and B are generalized quasi-adjoint operators such that $1 \notin \sigma(A)\sigma(B)$. Let $T \in \overline{R(\tau_{A,B})}^{W^*}$, then there exists a generalized sequence $(X_\alpha)_\alpha$ of elements in $L(H)$ such that $AX_\alpha B - X_\alpha \longrightarrow T$. On $H \oplus H$, define the operators L, Y_α and S by

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \ Y_\alpha = \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

It follows that

$$\tau_L(Y_\alpha) = \begin{pmatrix} 0 & \tau_{A,B}(X_\alpha) \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} = S.$$

Hence, we get $S \in \overline{R(\tau_L)}^{W^*}$. Since A and B are generalized quasi-adjoint operators such that $1 \notin \sigma(A)\sigma(B)$, then it results from Proposition 3.12 [3], that L is generalized quasi-adjoint. Thus, there exists a generalized sequence $(Z_\alpha)_\alpha$ in $L(H \oplus H)$ for which $\tau_{L^*}(Z_\alpha) \longrightarrow S$. An elementary calculation shows that there exists a generalized sequence $(U_\alpha)_\alpha$ in $L(H)$, which satisfies $\tau_{A^*,B^*}(U_\alpha) \longrightarrow T$. Consequently, we conclude that $\overline{R(\tau_{A,B})}^{W^*} \subseteq \overline{R(\tau_{A^*,B^*})}^{W^*}$. Arguing as above, it is elementary to prove the reverse inclusion. □

Proposition 3.10. *Let $A, B \in L(H)$. If the pair (A, B) is generalized quasi-adjoint then*

$$A^*R(\tau_{A,B}) + R(\tau_{A,B})B^* \subseteq \overline{R(\tau_{A,B})}^{W^*},$$

$$AR(\tau_{A^*,B^*}) + R(\tau_{A^*,B^*})B \subseteq \overline{R(\tau_{A,B})}^{W^*}.$$

Proof. Suppose that the pair (A, B) is generalized quasi-adjoint, then we have

$$\overline{R(\tau_{A,B})}^{W^*} = \overline{R(\tau_{A^*,B^*})}^{W^*}.$$

Since

$$A^*\tau_{A^*,B^*}(X) = \tau_{A^*,B^*}(A^*X) \quad \text{and} \quad \tau_{A^*,B^*}(X)B^* = \tau_{A^*,B^*}(XB^*),$$

it follows that

$$A^*R(\tau_{A,B}) \subseteq A^*\overline{R(\tau_{A,B})}^{W^*} = A^*\overline{R(\tau_{A^*,B^*})}^{W^*} \subseteq \overline{R(\tau_{A^*,B^*})}^{W^*} = \overline{R(\tau_{A,B})}^{W^*},$$

$$R(\tau_{A,B})B^* \subseteq \overline{R(\tau_{A,B})}^{W^*}B^* = \overline{R(\tau_{A^*,B^*})}^{W^*}B^* \subseteq \overline{R(\tau_{A^*,B^*})}^{W^*} = \overline{R(\tau_{A,B})}^{W^*}.$$

A similar argument using $A\tau_{A,B}(X) = \tau_{A,B}(AX)$ and $\tau_{A,B}(X)B = \tau_{A,B}(XB)$, gives

$$AR(\tau_{A^*,B^*}) + R(\tau_{A^*,B^*})B \subseteq \overline{R(\tau_{A,B})}^{W^*}.$$

This completes the proof. □

Acknowledgements

The authors would like to thank the referee for his careful reading of the paper and for helpful suggestions.

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Received 29/09/2020; Revised 07/04/2021