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ON SET-VALUED FUNCTIONAL INTEGRAL EQUATIONS OF HAMMERSTEIN-STIELTJES TYPE: EXISTENCE OF SOLUTIONS, CONTINUOUS DEPENDENCE, AND APPLICATIONS

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Abstract. We study the existence of continuous solutions of a nonlinear functional integral inclusion of Hammerstein-Stieltjes type. The continuous dependence of the solutions on the set of selections and on some other functions will be proved. Nonlinear set-valued functional integral equations of Chandrasekhar type and nonlinear set-valued fractional-orders functional integral equations will be given as applications. An initial value problem of fractional-orders set-valued integro-differential equation will be considered.

Досліджується існування неперервних розв'язків нелінійного функціонального інтегрального включення типу Гамерштейна-Стілтьєса. Доведена неперервна залежність розв'язку від множини виборок і деяких інших функцій. Як застосування, розглядаються нелінійні багатозначні функціональні інтегральні рівняння типу Чандрасекара і нелінійні багатозначні функціональні інтегральні рівняння дробових порядків, а також задачі з початковими умовами для останнього класу рівнянь.

1. Introduction

The integral equations of Hammerstein-Stieltjes type have been studied by some authors, for example, see [8, 12]. In this paper, we investigate the existence and uniqueness results for the functional integral equation of Hammerstein-Stieltjes type

$$x(t) = p(t) + \int_0^1 k(t,s) f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta)))d_\theta g_2(s,\theta))d_s g_1(t,s), \ \ t,s \ \in [0,1]. \qquad (1.1)$$

Our results will be generalized for functional integral inclusion of Hammerstein-Stieltjes type

$$x(t) \in p(t) + \int_0^1 k(t,s) \ F_1(s, \int_0^s f_2(\theta, x(\varphi(\theta)))d_\theta g_2(s,\theta))d_s g_1(t,s), \ \ t,s \ \in [0,1] \qquad (1.2)$$

where $F_1 : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map and $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The Chandrasekhar integral equation has been studied in some papers (for example, see [1, 4, 6, 9, 17]) and references therein. As applications of (1.2) the nonlinear Chandrasekhar set-valued functional integral equation

$$x(t) \in a(t) + \int_0^1 \ \frac{t}{t+s} F_1\Big(s, \int_0^s \frac{s}{s+\theta} b_1(\theta) x(\theta) d\theta\Big) ds, \ \ t \in [0,1], \qquad (1.3)$$

the Set-valued fractional-orders integral equation

$$x(t) \in p(t) + \int_0^1 \ k(t,s) \ F_1(s, \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} f_2(\theta, x(\varphi(\theta)))d\theta) ds, \ \ s,\theta \in [0,1], \qquad (1.4)$$

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and the set-valued fractional-orders integro-differential equation

$$\frac{dx(t)}{dt} \in \int_0^1 F_1(s, D^\gamma x(s))ds,\ t \in (0,1],\ \gamma \in (0,1], \tag{1.5}$$

$$x(0) = x_\circ, \tag{1.6}$$

where α , γ $\in (0,1)$, will be considered.

The paper is organized as follows. In Section 2, we recall some useful preliminaries. Section 3 is devoted to a study of existence, uniqueness and continuous dependence of solutions on the functions $g_i, (i = 1,2)$ for Single-valued nonlinear Hammerstein-Stieltjes integral equation (1.1). While in Section 4, we discuss existence results for the set-valued equation (1.2) with continuous dependence on the set S_{F_1}, we also discuss some special cases of inclusion by present the existence of solutions for the set-valued Chandrasekhar nonlinear functional integral equation. As an application on the previous results for inclusion (1.2), we also discuss the nonlinear Hammerstein functional integral inclusion of fractional order, the set-valued fractional-order integro-differential equations will be also considered.

2. Preliminaries

Here we recall some theorem, definitions, and preliminary facts.

First, denote by $I = [0,1]$ a fixed interval, and by $C(I, \mathbb{R})$ the Banach space of continuous functions from the interval I into $\mathbb{R}$ with the standard norm

$$\|x\|_C = \sup_{t \in I} |x(t)|.$$

The product space $X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ turns out to be a Banach space with the norm

$$\|(x, y)\|_X = \|x\|_C + \|y\|_C.$$

Definition 2.1. Let F be a set-valued map defined on a Banach space E, f is called a selection of F if $f(x) \in F(x)$, for every $x \in E$ and we denote by

$$S_F = \{f : f(x) \in F(x), x \in E\}$$

the set of all selections of F (for properties of the selection of F see [7, 18, 19]).

Definition 2.2. A set-valued map F from $I \times E$ to family of all nonempty closed subsets of E is called Lipschitzian if there exists $k > 0$ such that for all $t, s \in I$ and all $x_1, x_2 \in E$, we have

$$h(F(t, x_1), F(s, x_2)) \leq k(|t - s| + |x_1 - x_2|), \tag{2.7}$$

where $h(A, B)$ is the Hausdorff distance between subsets $A, B \subset I \times E$. (For properties of the Hausdorff distance see [3]).

The following Theorem [3, Sect. 9, Chap. 1, Th. 1] assumes the existence of a Lipschitzian selection.

Theorem 2.3. *Let M be a metric space and F be Lipschitzian set-valued function from M into nonempty compact convex subsets of R^n. Assume, moreover, that for some $\lambda > 0$, $F(x) \subset \lambda B$ for all $x \in M$ where B is the unit ball on R^n. Then there exists a constant c and a single-valued function $f : M \to R^n$, $f(x) \in F(x)$ for $x \in M$; this function is Lipschitzian with a constant k.*

Next, we recall some properties of the Stieltjes integral that will be used in our considerations (cf. [2]).

Lemma 2.4. *Assume that x is Stieltjes integrable on the interval $[a,b]$ with respect to a function ϕ of bounded variation. Then*

$$|\int_a^b x(t)d\phi(t)| \leq \int_a^b |x(t)|d(\bigvee_a^t \phi).$$

Lemma 2.5. *Let x_1 and x_2 be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function ϕ such that $x_1(t) \leq x_2(t)$ for $t \in [a,b]$. Then the following inequality is satisfied:*

$$\int_a^b x_1(t)d\phi(t) \leq \int_a^b x_2(t)d\phi(t).$$

Finally, we recall some basic facts related to fractional calculus.

Definition 2.6. The Riemann-Liouville of fractional integral of a function $f \in L^1(I, \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by (see [21, 22, 23])

$$I_a^\alpha \ f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ f(s) \ ds,$$

where $\Gamma(.)$ is the Euler gamma function and when $a = 0$, we have $I^\alpha \ f(t) = I_0^\alpha \ f(t)$.

Definition 2.7. The Liouville-Caputo fractional derivative of $f(t)$ of o $\alpha \in (0,1]$ is defined as follows

$$D^\alpha f(t) = I^{1-\alpha} \ \frac{d}{dt}\{f(t)\} = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \ \frac{d}{ds}\{f(s)\} \ ds.$$

For further properties of fractional calculus operator (see [10, 22, 23])

3. Existence and uniqueness results for single-valued problem

Here, we are regarding the integral nonlinear functional integral value Hammerstein-Stieltjes type (1.1)

$$x(t) = p(t) + \int_0^1 k(t,s) f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) d_s g_1(t,s), \ \ t, s \ \in [0,1].$$

This equation will be studied under the following assumptions:

(i) $p : I \to I$ is a continuous function, with $p^* = \sup_{t \in I} |p(t)|$.

(ii) $\varphi : I \to I$ is a continuous function.

(iii) $f_1 : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist two constants k and f_1^* such that

$$|f_1(t,x)| \leq f_1^* + k|x|.$$

(iv) $f_2 : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist two constants a and b such that

$$|f_2(t,x)| \leq \ a + \ b|x| \qquad \forall \ t \ \in I \text{ and } x \in \mathbb{R}.$$

(v) $k(t,s)$ is a continuous function such that

$$K = \sup_{t \in I} |k(t,s)|, \qquad \text{and } K \text{ is a positive constant.}$$

(vi) The functions g_i are continuous on the triangle $\triangle_i$ for $i = 1, 2$, where

$$\triangle_1 = \{(t,s) : 0 \leq s \leq t \leq T\},$$

$$\triangle_2 = \{(s,\theta) : 0 \leq \theta \leq s \leq T\}.$$

(vii) The functions $s \to g_i(t,s)$ are of bounded variation on $[0,t]$ for each $t \in I$, (i=1,2).

(viii) For any $\epsilon > 0$ there exists $\delta > 0$ such that, for all $t_1; t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$, the following inequality holds:

$$\bigvee_{0}^{t_1}[g_i(t_2, s) - g_i(t_1, s)] \leq \epsilon$$

for $i = 1, 2$.

(ix) $g_i(t, 0) = 0$ for any $t \in I$ (i=1,2).

In the sequel we will need the following lemmas.

Lemma 3.1 ([13]). *The function $z \to \bigvee_{s=0}^{z} g(t, s)$ is continuous on $[0, t]$ for any $t \in I$.*

Lemma 3.2 ([13]). *Let the assumptions (vi)-(ix) be satisfied. Then, for arbitrary fixed number $0 < t_2 \in I$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1 \in I$, $t_1 < t_2$ and $t_2 - t_1 \leq \delta$, we have*

$$\bigvee_{s=t_1}^{t_2} g(t_2; s) \leq \epsilon.$$

Lemma 3.3 ([13]). *Under the the assumptions (vi)-(ix), the function $t \to \bigvee_{s=0}^{t} g(t, s)$ is continuous on I.*

Moreover, let us note that based on Lemma 3.3, we conclude that there is a positive fixed constant H,

$$H = \sup_{t \in I} \left\{ \bigvee_{s=0}^{t} g_2(t, s) \right\},$$

In what follows we will assume that the functions $t \to g_1(t, 1)$ and $t \to g_1(t, 0)$ are continuous on I. Let us put

$$\mu = \sup_{t \in I} |g_1(t, 1)| + \sup_{t \in I} |g_1(t, 0)|,$$

$$N(\epsilon) = \sup\{ \bigvee_{s=0}^{t_1} (g_2(t_2, s) - g_2(t_1, s)) : t_1, t_2 \in I,\ t_1 < t_2;\ t_2 - t_1 \leq \epsilon\}.$$

Now, let

$$y(t) = \int_0^t f_2(s, x(\varphi(s))) d_s g_2(t, s)\ t \in I.$$

Then (1.1) can be written in the form of the following coupled system

$$x(t) = p(t) + \int_0^1 k(t, s) f_1(s, y(s)) d_s g_1(t, s), \qquad t \in I, \tag{3.8}$$

$$y(t) = \int_0^t f_2(s, x(\varphi(s)) d_s g_2(t, s), \qquad t \in I. \tag{3.9}$$

Definition 3.4. By solutions of the coupled system (3.8), (3.9), we mean functions $x, y \in C(I, \mathbb{R})$ satisfying (3.8), (3.9).

Theorem 3.5. *Let the assumptions $(i) - (ix)$ be satisfied. Then there exists at least one continuous solution $u = (x, y)$, $x, y \in C(I, \mathbb{R})$ of the coupled system (3.8), (3.9).*

Proof. Let the set Q_r be defined as

$$Q_r = \{u = (x, y) : (x, y) \in X, \|x\| \leq r_1,\ \|y\| \leq r_2,\ \|u\| \leq r = r_1 + r_2\},$$

where $r = \frac{p^* + f_1^* K\mu}{1 - Kk\mu} + \frac{aH}{1 - bH}$. It is clear that the set Q_r is nonempty, bounded, closed, and convex.

Let A be any operator defined by

$$Au(t) = A(x(t), y(t)) = (A_1 y(t), A_2 x(t)),$$

$$A_1 y(t) \;=\; p(t) + \int_0^1 k(t,s)\; f_1(s, y(s)) d_s g_1(t,s), \qquad t \in I,$$

and

$$A_2 x(t) \;=\; \int_0^t f_2(s, x(\phi(s)) d_s g_2(t,s), \qquad t \in I,$$

where for $u = (x, y) \in Q_r$,

$$\begin{aligned} |A_1 y(t)| &= |p(t) + \int_0^1 k(t,s) f_1(s, y(s))\; d_s g_1(t,s)| \\ &\leq |p(t)| \;+ \int_0^1 |k(t,s)||f_1(s, y(s))|\; |d_s g_1(t,s)| \\ &\leq p^* + \int_0^1 K(k|y| + f_1^*) d_s (\bigvee_{p=0}^{s} g_1(t,p)). \end{aligned}$$

Taking the supremum over $t \in I$, we get

$$\begin{aligned} \|A_1 y\| \leq &\; p^* + K(kr_1 + f_1^*) \int_0^1 d_s g_1(t,s) \\ \leq &\; p^* + K(kr_1 + f_1^*)[g_1(t,1) - g_1(t,0)] \\ \leq &\; p^* + K(kr_1 + f_1^*)[|g_1(t,1)| + |g_1(t,0)|] \\ \leq &\; p^* + K(kr_1 + f_1^*)[\sup_{t\in I} |g_1(t,1)| + \sup_{t \in I} |g_1(t,0)|] \\ \leq &\; p^* + K(kr_1 + f_1^*)\mu = r_1, \qquad r_1 = \frac{p^* + f_1^* K\mu}{1 - kK\mu}. \end{aligned}$$

Also,

$$\begin{aligned} |A_2 x(t)| \;&=\; |\int_0^t f_2(s, x(\varphi(s))) d_s g_2(t,s)| \\ &\leq\; \int_0^t |f_2(s, x(\varphi(s)))|\; |d_s g_2(t,s)| \\ &\leq\; \int_0^t [a + b\; |x(\varphi(s))|] d_s \left(\bigvee_{p=0}^{s} g_2(t,p) \right). \end{aligned}$$

Taking the supremum over $t \in I$, we get

$$\begin{aligned} \|A_2 x\| \;&\leq\; (a + br_2) \left(\bigvee_{s=0}^{t} g_2(t,s) \right) \\ &\leq\; (a + br_2) \sup_{t \in I} \left(\bigvee_{s=0}^{t} g_2(t,s) \right) \\ &\leq\; (a + br_2) H = r_2, \qquad r_2 = \frac{aH}{1 - bH}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \|Au\|_X \;&=\; \|A_1 y\|_C + \|A_2 x\|_C \;\leq\; r_1 + r_2 \\ &\leq \frac{p^* + f_1^* K\mu}{1 - Kk\mu} + \frac{aH}{1 - bH} = r. \end{aligned}$$

Then $AQ_r \subset Q_r$ and the class $\{Au\}$, $u \in Q_r$ is uniformly bounded.

Now, for $u = (x, y) \in Q_r$, for all $\epsilon > 0$, $\delta > 0$ and for each $t_1, t_2 \in I$, $t_1 < t_2$ such that $|t_2 - t_1| < \delta$, we have

$$\begin{aligned}
|A_1y(t_2) - A_1y(t_1)| &\leq |p(t_2) - p(t_1)| \\
&\quad + \Big|\int_0^1 k(t_2,s)f_1(s,y(s))d_sg_1(t_2,s) - \int_0^1 k(t_1,s)f_1(s,y(s))d_sg_1(t_1,s)\Big| \\
&\leq |p(t_2) - p(t_1)| \\
&\quad + \Big|\int_0^1 k(t_2,s)f_1(s,y(s))d_sg_1(t_2,s) - \int_0^1 k(t_1,s)f_1(s,y(s))d_sg_1(t_2,s) \\
&\quad + \int_0^1 k(t_1,s)f_1(s,y(s))d_sg_1(t_2,s) - \int_0^1 k(t_1,s)f_1(s,y(s))d_sg_1(t_1,s)\Big| \\
&\leq |p(t_2) - p(t_1)| + \Big|\int_0^1 [k(t_2,s) - k(t_1,s)]f_1(s,y(s))d_sg_1(t_2,s)\Big| \\
&\quad + \Big|\int_0^1 k(t_1,s)f_1(s,y(s))[d_sg_1(t_2,s) - d_sg_1(t_1,s)]\Big| \\
&\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2,s) - k(t_1,s)||f_1(s,y(s))||d_sg_1(t_2,s)| \\
&\quad + \int_0^1 |k(t_1,s)||f_1(s,y(s))||d_sg_1(t_2,s) - d_sg_1(t_1,s)| \\
&\leq |p(t_2) - p(t_1)| + \int_0^1 |k(t_2,s) - k(t_1,s)|[k|y(s)| + f_1^*]d_s(\bigvee_{p=0}^{s} g_1(t_2,p)) \\
&\quad + \int_0^1 |k(t_1,s)|[k|y(s)| + f_1^*]d_s(\bigvee_{p=0}^{s} [g_1(t_2,p) - d_sg_1(t_1,p)]| \\
&\leq |p(t_2) - p(t_1)| + [kr_1 + f_1^*]\int_0^1 |k(t_2,s) - k(t_1,s)|d_sg_1(t_2,s)) \\
&\quad + K[kr_1 + f_1^*]\int_0^1 d_s[g_1(t_2,s) - d_sg_1(t_1,s)] \\
&\leq |p(t_2) - p(t_1)| + [kr_1 + f_1^*]\int_0^1 |k(t_2,s)| + k|(t_1,s)|d_sg_1(t_2,s)) \\
&\quad + K[kr_1 + f_1^*]\int_0^1 d_s[g_1(t_2,s) - d_sg_1(t_1,s)] \\
&\leq |p(t_2) - p(t_1)| + \ 2K[kr_1 + f_1^*][[g_1(t_2,1) - g_1(t_2,0)] \\
&\quad + K[kr_1 + f_1^*])[|g_1(t_2,1) - g_1(t_1,1)| + |g_1(t_2,0) - g_1(t_1,0)|]
\end{aligned}$$

and

$$\begin{aligned}
|A_2x(t_2) - A_2x(t_1)| &\leq |\int_0^{t_2} f_2(s,x(\varphi(s)))d_sg_2(t_2,s) - \int_0^{t_1} f_2(s,x(\varphi(s)))d_sg_2(t_1,s)| \\
&\quad + |\int_0^{t_1} f_2(s,x(\varphi(s))))d_sg_2(t_2,s) - \int_0^{t_1} f_2(s,x(\varphi(s))))d_sg_2(t_1,s)| \\
&\leq |\int_{t_1}^{t_2} f_2(s,x(\varphi(s))))d_sg_2(t_2,s)| \\
&\quad + |\int_0^{t_1} f_2(s,x(\varphi(s))))[d_sg_2(t_2,s) - d_sg_2(t_1,s)]|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2}[a+b|x(\varphi(s))|]d_s(\bigvee_{p=0}^{s} g_2(t_2,p)) \\
&\quad + \int_0^{t_1}[a+b|x(\varphi(s))|]d_s(\bigvee_{p=0}^{s}[g_2(t_2,p)-g_2(t_1,p)] \\
&\leq (a+br_2)[\int_{t_1}^{t_2} d_s(\bigvee_{p=0}^{s} g_2(t_2,p))] + \int_0^{t_1} d_s(\bigvee_{p=0}^{s}[g_2(t_2,p)-g_2(t_1,p)] \\
&\leq (a+br_2)[\bigvee_{s=0}^{t_2} g_2(t_2,s) - \bigvee_{s=0}^{t_1} g(t_2,s) + \bigvee_{s=0}^{t_1}[g_2(t_2,s)-g_2(t_1,s)]] \\
&\leq (a+br_2)[\bigvee_{s=t_1}^{t_2} g_2(t_2,s) + N(\epsilon)].
\end{aligned}$$

For the operator A and $u \in Q_r$ we have

$$\begin{aligned}
Au(t_2) - Au(t_1) &= A(x,y)(t_2) - A(x,y)(t_1) \\
&= (A_1y(t_2), A_2x(t_2)) - (A_1y(t_1), A_2x(t_1)) \\
&= (A_1y(t_2) - A_1y(t_1), A_2x(t_2) - A_2x(t_1)),
\end{aligned}$$

and consequently, we obtain

$$\begin{aligned}
\|Au(t_2) - Au(t_1)\|_X &= \|(A_1y(t_2) - A_1y(t_2), A_2x(t_2) - A_2x(t_1))\|_X \\
&= \|A_1y(t_2) - A_1y(t_2)\|_C + \|A_2x(t_2) - A_2x(t_1)\|_C, \\
&= |p(t_2) - p(t_1)| + \ 2K[kr_1 + f_1^*][[g_1(t_2,1) - g_1(t_2,0)] \\
&\quad + K[kr_1 + f_1^*])[|g_1(t_2,1) - g_1(t_1,1)| + |g_1(t_2,0) - g_1(t_1,0)|] \\
&\quad + (a+br_2)[\bigvee_{s=t_1}^{t_2} g_2(t_2,s) + N(\epsilon)].
\end{aligned}$$

This means that the class of functions $\{Au\}$ is equicontinuous on Q_r. Then by the Arzela-Ascoli Theorem [11], the operator A is compact.

It remains to prove the continuity of $A : Q_r \to Q_r$. Let $u_n = (x_n, y_n)$ be a sequence in Q_r with $x_n \to x$ and $y_n \to x$, and since $f_1(t, y(t))$ and $f_2(t, x(t))$ are continuous in X, $f_1(t, y_n(t))$ and $f_2(t, x_n(t))$ converges to $f_1(t, y(t))$ and $f_2(t, x(t))$, we have that $f_2(t, x_n(\varphi(t)))$ converges to $f_2(t, x(\varphi(t)))$ (see assumption (ii))-(iv)) and applying the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^t f_2(s, x_n(\varphi(s)))d_s g_2(t,s) = \int_0^t f_2(s, x(\varphi(s)))d_s g_2(t,s),$$

and

$$\lim_{n\to\infty} \int_0^1 k(t,s) f_1(s, y_n(s))d_s g_1(t,s) = \int_0^1 k(t,s) f_1(s, y(s))d_s g_1(t,s),$$

then

$$\begin{aligned}
\lim_{n\to\infty} A_1y_n(t) &= p(t) + \lim_{n\to\infty} \int_0^1 k(t,s) f_1(s, y_n(s))d_s g_1(t,s) \\
&= p(t) + \int_0^1 k(t,s) f_1(s, y(s))d_s g_1(t,s) = A_1y(t), \qquad t \in I,
\end{aligned}$$

and

$$\lim_{n\to\infty} A_2x_n(t) = \lim_{n\to\infty} \int_0^t f_2(s, x_n(\varphi(s)))d_s g_2(t,s)$$

$$= \int_0^t f_2(s, x(\varphi(s))) d_s g_2(t, s) = A_2 x(t), \qquad t \in I.$$

Hence,

$$\lim_{n\to\infty} Au_n(t) = \lim_{n\to\infty} (A_1 y_n(t), A_2 x_n(t)),$$
$$= (\lim_{n\to\infty} A_1 y_n(t), \lim_{n\to\infty} A_2 x_n(t)) = (A_1 y(t), A_2 x(t)) = Au(t).$$

Since all conditions of the Schauder fixed-point theorem [20] hold, we concluded that A has a fixed point $u \in Q_r$, and then the system (3.8), (3.9) has at least one continuous solution $u = (x, y) \in Q_r$, $x, y \in C(I, \mathbb{R})$.

Consequently, the functional integral equation (1.1) has at least one solution $x \in C(I, \mathbb{R})$. □

3.1. **Uniqueness of the solution.** In this subsection, we give a sufficient condition for uniqueness of the solution of the Hammerstein-Stieltjes functional integral equation (1.1). To this end we replace assumptions (iii) and (iv), respectively, by the following ones:

$(iii)^*$ $f_1 : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition

$$|f_1(t, x_1) - f_1(t, x_2)| \leq k|x_1 - x_2|.$$

$(iv)^*$ $f_2 : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition

$$|f_2(t, x) - f_2(t, y)| \leq c|x - y|.$$

From this assumption $(iii)^*$, we have

$$|f_1(t, , x(t))| - |f_1(t, 0)| \leq |f_1(t, x(t)) - f_1(t, 0)| \leq k|x|,$$

then

$$|f_1(t, , x(t))| \leq k|x| + |f_1(t, 0)|,$$

and

$$|f_1(t, , x(t))| \leq k|x| + f_1^*,$$

where $f_1^* = \sup_{t\in I} |f_1(t, 0)|$.

Similarly, from assumption $(iv)^*$, we have

$$|f_2(t, x(t))| \leq c|x(t)| + |f_2(t, 0)|,$$

and

$$|f_2(t, x(t))| \leq c|x| + f_2^*,$$

where $f_2^* = \sup_{t\in I} |f_2(t, 0)|$.

Theorem 3.6. *Let the assumptions of Theorem 3.5 be satisfied with replacing condition (iv) by $(iv)^*$, if $\mu ckKH < 1$. Then the Hammerstein-Stieltjes functional integral equation (1.1) has a unique solution on I.*

Proof. Let x_1, x_2 be two solution of equation (1.1), then

$$|x_1(t) - x_2(t)| \leq \int_0^1 |k(t, s)| [|f_1(s, \int_0^s f_2(\theta, x_1(\varphi(\theta))) d_\theta g_2(s, \theta))$$
$$- f_1(s, \int_0^s f_2(\theta, x_2(\varphi(\theta))) d_\theta g_2(s, \theta))|] d_s g_1(t, s).$$

Hence, using Lipschitz condition for f_1, we obtain

$$|x_1(t) - x_2(t)|$$
$$\leq kK| \int_0^1 [\int_0^s f_2(\theta, x_1(\varphi(\theta)) d_\theta g_2(s, \theta)) - \int_0^s f_2(\theta, x_2(\varphi(\theta)) d_\theta g_2(s, \theta))]| d_s g_1(t, s)$$

$$\leq kK \int_0^1 \int_0^s |f_2(\theta, x_1(\varphi(\theta))) - f_2(\theta, x_2(\varphi(\theta)))|d_\theta g_2(s,\theta))||d_s g_1(t,s)|.$$

Further, using Lipschitz condition for f_2, we obtain

$$\begin{aligned}
|x_1(t) - x_2(t)| &\leq kKc \int_0^1 \int_0^s |x_1(\varphi(\theta)) - x_2(\varphi(\theta))|d_\theta(\bigvee_{p=0}^{\theta} g_2(s,p))d_s(\bigvee_{p=0}^{s} g_1(t,p)) \\
&\leq ckK\|x_1 - x_2\| \int_0^1 \int_0^s d_\theta(\bigvee_{p=0}^{\theta} g_2(s,p))d_s(\bigvee_{q=0}^{s} g_1(t,q)) \\
&\leq ckK\|x_1 - x_2\| \bigvee_{\theta=0}^{s} g_2(s,\theta) \int_0^1 d_s g_1(t,s) \\
&\leq ckK\|x_1 - x_2\|[g_1(t,1) - g_1(t,0)] \sup_{s\in[0,T]} \bigvee_{\theta=0}^{s} g_2(s,\theta) \\
&\leq ckK\|x_1 - x_2\|[|g_1(t,1)| + |g_1(t,0)|]H \\
&\leq ckKH\|x_1 - x_2\|[\sup_t |g_1(t,1)| + \sup_t |g_1(t,0)|].
\end{aligned}$$

Taking the supremum over $t \in I$, we get

$$\|x_1 - x_2\| \leq \mu ckKH\|x_1 - x_2\|.$$

Thus

$$(1 - \mu ckKH)\|x_1 - x_2\| < 0.$$

This proves the uniqueness of the solution of equation (1.1) on I. □

3.1.1. *Continuous dependence on the functions $g_i(t,s)$.*

Definition 3.7. The solutions of the functional integral equation (1.1), depends continuously on the functions $g_i(t,s)$, i=1,2, if $\forall\epsilon > 0$ $\exists\delta > 0$ such that

$$|g_i(t,s) - g_i^*(t,s)| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon.$$

Theorem 3.8. *Let the assumptions of Theorem 3.6 be satisfied. Then the solution of the equation (1.1) depends continuously on functions $g_i(t,s)$.*

Proof. Let $\delta > 0$ be given such that $|g_i(t,s) - g_i^*(t,s)| \leq \delta$, $\forall t \geq 0$. Then

$$\begin{aligned}
|x(t) - x^*(t)| \leq \Big| &\int_0^1 k(t,s) f_1\Big(s, \int_0^s f_2(\theta, x(\varphi(\theta)))d_\theta g_2(s,\theta)\Big)d_s g_1(t,s) \\
&- \int_0^1 k(t,s) f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta)))d_\theta g_2(s,\theta)\Big)d_s g_1(t,s) \\
&+ \int_0^1 k(t,s) f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta)))d_\theta g_2(s,\theta)\Big)d_s g_1(t,s)) \\
&- \int_0^1 k(t,s) f_1\Big(s, \int_0^s f(\theta, x^*(\varphi(\theta)))d_\theta g_2^*(s,\theta)\Big)d_s g_1^*(t,s)\Big| \\
\leq \int_0^1 |k(t,s)|\Big| &f_1\big(s, \int_0^s f_2(\theta, x(\varphi(\theta)))d_\theta g_2(s,\theta)\big) \\
&- f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta)))d_\theta g_2(s,\theta)\Big)\Big|d_s g_1(t,s) \\
&+ \Big|\int_0^1 k(t,s)\big[f_1\big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta)))d_\theta g_2(s,\theta)\big)
\end{aligned}$$

$$
\begin{aligned}
&- f_1\big(s, \int_0^s f(\theta, x^*(\varphi(\theta))) d_\theta g_2^*(s,\theta)\big)\big] d_s g_1^*(t,s)\Big| \\
\leq{}& Kk \int_0^1 \int_0^s |f_2(\theta, x(\varphi(\theta))) - f_2(\theta, x^*(\varphi(\theta)))| |d_\theta g_2(s,\theta)| |d_s g_1(t,s)| \\
&+ \Big| \int_0^1 k(t,s) f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta)\Big) d_s g_1(t,s)) \\
&- \int_0^1 k(t,s) f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2^*(s,\theta)\Big) d_s g_1(t,s)) \\
&+ \int_0^1 k(t,s) f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2^*(s,\theta)\Big) d_s g_1(t,s)) \\
&- \int_0^1 k(t,s) f_1\Big(s, \int_0^s f(\theta, x^*(\varphi(\theta))) d_\theta g_2^*(s,\theta)) d_s g_1^*(t,s))\Big| \\
\leq{}& Kk \int_0^t \int_0^s c|x(\theta) - x^*(\theta)| d_\theta g_2(s,\theta)\Big) d_s g_1(t,s) \\
&+ \int_0^1 k(t,s) \Big| f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta)\Big) \\
&- f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2^*(s,\theta)\Big)\Big| d_s g_1(t,s) \\
&+ \int_0^1 k(t,s) \Big| f_1\Big(s, \int_0^s f_2(s,\theta, x^*(\theta)) d_\theta g_2^*(s,\theta)\Big)\Big| [d_s g_1(t,s) - d_s g_1^*(t,s)] \\
\leq{}& Kkc \int_0^1 \int_0^s |x(\theta) - x^*(\theta)| |d_\theta g_2(s,\theta)| |d_s g_1(t,s)| \\
&+ K \int_0^1 k\Big(\Big| \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta) \\
&\quad - \int_0^s f_2(s,\theta, x^*(\theta)) d_\theta g_2^*(s,\theta)\Big|\Big) |d_s g_1(t,s)| \\
&+ K \int_0^1 \Big| f_1\Big(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2^*(s,\theta)\Big)\Big| [d_s g_1(t,s) - d_s g_1^*(t,s)] \\
\leq{}& Kkc\|x - x^*\| \int_0^1 \int_0^s d_\theta(\bigvee_{p=0}^{\theta} g_2(s,p)) d_s(\bigvee_{p=0}^{s} d_s g_1(t,p)) \\
&+ K \int_0^1 k\Big(\int_0^s \big|f_2(s,\theta, x^*(\theta))\big| [d_\theta(\bigvee_{p=0}^{\theta} [g_2(s,p) - g_2^*(s,p)]\Big)] d_s(\bigvee_{p=0}^{s} g_1(t,p)) \\
&+ K \int_0^1 \Big(|f_1^*| + k \int_0^s |f_2(\theta, x^*(\varphi(\theta)))| d_\theta(\bigvee_{p=0}^{\theta} g_2^*(s,p)\Big) d_s(\bigvee_{p=0}^{s} [g_1(t,p) - g_1^*(t,p)] \\
\leq{}& Kkc\|x - x^*\|(\bigvee_{\theta=0}^{s} g_2(s,\theta))[g_1(t,1) - g_1(t,0)] \\
&+ K \int_0^1 k\Big(\int_0^s \big[|f_2(s,0)| + c|x^*(\varphi(\theta))|\big] \\
&\quad [d_\theta(\bigvee_{p=0}^{\theta} [g_2(s,p) - g_2^*(s,p)]\Big)] d_s(\bigvee_{p=0}^{s} g_1(t,p))
\end{aligned}
$$

$$+ K\int_0^1 \Big(|f_1^*(s)| + k\int_0^s \big[|f_2(s,0)| + c|x^*(\varphi(\theta))|\big]$$

$$d_\theta(\bigvee_{p=0}^{\theta} g_2^*(s,p)\Big) d_s(\bigvee_{p=0}^{s}[g_1(t,p) - g_1^*(t,p)])$$

$$\leq Kkc\|x - x^*\|H\mu + Kk\big[f_2^* + cr\big]\bigvee_{\theta=0}^{s}[g_2(s,\theta) - g_2^*(s,\theta)][g_1(t,1) - g_1(t,0)]$$

$$+ K\big(f_1^* + k[\big(f_2^* + cr\big)\bigvee_{\theta=0}^{s} g_2^*(s,\theta)]\big)\bigvee_{s=0}^{1}[g_1(t,s) - g_1^*(t,s)]$$

$$\leq ckKH\mu\|x - x^*\| + kK\big[f_2^* + cr\big]([g_2(s,s) - g_2^*(s,s)]$$
$$+ [g_2(s,0) - g_2^*(s,0)])\mu + K\big(f_1^* + k[\big(f_2^* + cr\big)H]\big)([g_1(t,1) - g_1^*(t,1)]$$
$$+ [g_1(t,0) - g_1^*(t,0)]).$$

Taking the supremum over $t \in I$, we get

$$\|x - x^*\| \leq kcKH\mu\|x - x^*\| + kKH\big[f_2^* + cr\big]2\delta$$
$$+ K\big(f_1^* + k[\big(f_2^* + cr\big)\mu]\big)2\delta + k[r + kr + m]2\mu\delta.$$

Thus

$$\|x - x^*\| \leq \frac{2kKK_1\big[f_2^* + cr\big]\delta + 2K\big(f_1^* + k[\big(f_2^* + cr\big)H]\big)\delta}{1 - (kcKH\mu)} = \epsilon.$$

This completes the proof. □

4. Existence results for the Set-valued problem

Consider the set-valued functional integral equation of Hammerstein-Stieltjes type (1.2),

$$x(t) \in p(t) + \int_0^1 k(t,s)F_1(s, \int_0^s f_2(\theta, x(\varphi(\theta)))d_\theta g_2(s,\theta))d_s g_1(t,s), \qquad t,s \in [0,1],$$

which is considered under the following assumptions:

(i) $p : I \to I$, is a continuous function, where $p^* = \sup_{t\in I}|p(t)|$.

(ii) $\varphi : I \to I$, is a continuous function.

(iii^*) The set-valued map $F_1 : I \times \mathbb{R} \to 2^{\mathbb{R}}$, is a Lipschitzian set-valued map with a nonempty compact convex subset of $2^{\mathbb{R}}$, and a Lipschitz constant $k > 0$,

$$\|F_1(t,x) - F_1(t,y)\| \leq k\|x - y\|.$$

We remark that from this assumption and Theorem 2.3, we can deduce the set of Lipschitz selections of F_1 is not empty and there exists $f_1 \in F_1$ such that

$$|f_1(t,x) - f_1(t,y)| \leq k|x - y|.$$

(iv) $f_2 : I \times \mathbb{R} \to \mathbb{R}$, is a continuous function and there exist two constants a and b, such that

$$|f_2(t,x)| \leq a + b|x|, \qquad \forall\, t \in [0,T] \quad \text{and} \quad x \in \mathbb{R}.$$

(v) $k(t,s)$ is a continuous function, such that

$$K = \sup_{t\in I}|k(t,s)|, \qquad \text{and } K \text{ is a positive constant.}$$

(vi) The functions g_i are continuous on the triangle $\triangle_i$, for $i = 1, 2$, where

$$\triangle_1 = \{(t, s) : 0 \le s \le t \le T\},$$

$$\triangle_2 = \{(s, \theta) : 0 \le \theta \le s \le T\}.$$

(vii) The functions $s \to g_i(t, s)$ are of bounded variation on $[0, t]$ for each $t \in I$, i=1,2.

(viii) For any $\epsilon > 0$ there exists $\delta > 0$ such that, for all $t_1; t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \le \delta$ the following inequality holds:

$$\bigvee_0^{t_1}[g_i(t_2, s) - g_i(t_1, s)] \le \epsilon$$

for $i = 1, 2$.

(ix) $g_i(t, 0) = 0$ for any $t \in I$, $i = 1, 2$.

4.1. **Existence of solution.**

Theorem 4.1. *Let the assumptions (i)-(ii)-(iii)*** and (iv)-(ix) be satisfied. Then the Hammerstein-Stieltjes functional integral inclusion (1.2) has at least one solution $x \in C(I, \mathbb{R})$.*

Proof. It is clear that from Theorem 2.3 and assumption $(iii)^{***}$ that the set of Lipschitz selection of F_1 is not empty. So, a solution of the single-valued integral equation (1.1) where $f_1 \in S_{F_1}$, is a solution to the inclusion (1.2).

It must be noted that the Lipschitz selection $f_1 : I \times R \to R$, satisfies the Lipschitz condition

$$|f_1(t, x_1) - f_1(t, x_2)| \le k|x_1 - x_2|.$$

From this condition with $f_1^* = \sup_{t \in I} |f_1(t, 0)|$, we have

$$|f_1(t, x(s))| - |f_1(t, 0)| \le |f_1(t, x(s)) - f_1(t, 0)| \le k|x|,$$

then

$$|f_1(t, x(s))| \le k|x| + f_1(t, 0)|$$

and

$$|f_1(t, x(s))| \le k|x| + f_1^*,$$

i.e., assumption (iii) of Theorem 3.5 is satisfied. So, all the conditions of Theorem 3.5 hold.

Observe that if $x \in C(I, \mathbb{R})$ is a solution of the functional integral equation (1.1), then x is a solution to the functional integral inclusion (1.2). This completes the proof. □

4.2. **Continuous dependence on the set of selections S_{F_1}.**

Definition 4.2. The solutions of the Hammerstein-Stieltjes functional integral inclusion (1.2) depends continuously on the set S_{F_1} of selections of the set-valued function F_1, if $\forall \epsilon > 0$ $\exists \delta > 0$ such that

$$|f_1(t, x(t)) - f_1^*(t, x(t))| < \delta, \qquad f_1, f_1^* \in S_{F_1},\ t \in I,$$

then

$$\|x - x^*\| < \epsilon.$$

Now, we have the following theorem

Theorem 4.3. *The solution of the Hammerstein-Stieltjes inclusion (1.2) depends continuously on the S_{F_1} of all Lipschitzian selection of F_1.*

Proof. For two solutions $x(t)$ and $x^*(t)$ of (1.2) corresponding to the two selections $f_1, f_1^* \in S_{F_1}$, we have

$$x(t) - x^*(t) = p(t) + \int_0^1 k(t,s) f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) d_s g_1(t,s)$$
$$- p(t) - \int_0^1 k(t,s) f_1^*(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta)) d_s g_1(t,s),$$

hence

$$\begin{aligned}
|x(t) - x^*(t)| &\leq | \int_0^1 k(t,s)[f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) \\
&\quad - f_1^*(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta))] d_s g_1(t,s)| \\
&\leq \int_0^1 |k(t,s)||f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) \\
&\quad - f_1^*(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_s g_2(s,\theta))||d_s g_1(t,s)| \\
&\leq \int_0^1 |k(t,s)||f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) \\
&\quad - f_1(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta))||d_s g_1(t,s)| \\
&\quad + \int_0^1 |k(t,s)||f_1(s, \int_0^s f_2(s, x^*(\varphi(s))) d_s g(t,s)) \\
&\quad - f_1^*(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta))||d_s g_1(t,s)| \\
&\leq K \int_0^1 |f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) \\
&\quad - f_1(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta))||d_s g_1(t,s)| \\
&\quad + K \int_0^1 |f_1(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta)) \\
&\quad - f_1^*(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta))||d_s g_1(t,s)| \\
&\leq K \int_0^1 |f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) d_\theta g_2(s,\theta)) \\
&\quad - f_1(s, \int_0^s f_2(\theta, x^*(\varphi(\theta))) d_\theta g_2(s,\theta))||d_s g_1(t,s)| + K\delta \int_0^1 |d_s g_1(t,s)| \\
&\leq kK \int_0^1 \int_0^s |f_2(\theta, x(\varphi(\theta))) \\
&\quad - f_2(\theta, x^*(\varphi(\theta)))| d_\theta |g_2(s,\theta))||d_s g_1(t,s)| + K\delta \int_0^1 d_s (\bigvee_{p=0}^{s} g_1(t,p)) \\
&\leq ckK \int_0^1 \int_0^s |x(\varphi(\theta)) - x^*(\varphi(\theta))||d_\theta g_2(s,\theta))||d_s g_1(t,s)| \\
&\quad + K\delta \int_0^1 d_s g_1(t,s)
\end{aligned}$$

$$\leq ckK\|x-x^*\|\int_0^1 d_s \bigvee_{p=0}^{s} g_1(t,p)\int_0^s d_\theta \bigvee_{q=0}^{\theta} g_2(s,q) + K\delta[g_1(t,1)-g_1(t,0)]$$

$$\leq ckK\|x-x^*\|\int_0^1 d_s g_1(t,p)\int_0^s d_\theta \bigvee_{q=0}^{\theta} g_2(s,q) + K\delta[|g_1(t,1)|+|g_1(t,0)|]$$

$$\leq ckK\|x-x^*\|[g_1(t,1)-g_1(t,0)]\bigvee_{\theta=0}^{s} g_2(s,\theta) + K\delta[|g_1(t,1)|+|g_1(t,0)|]$$

$$\leq ckK\|x-x^*\|[g_1(t,1)+g_1(t,0)]\bigvee_{\theta=0}^{s} g_2(s,\theta) + K\delta[|g_1(t,1)|+|g_1(t,0)|]$$

$$\leq K[|g_1(t,1)|+|g_1(t,0)|](kc\|x-x^*\|\sup_{s\in[0,T]}\bigvee_{\theta=0}^{s} g_2(s,\theta)+\delta).$$

Taking the supremum over $t \in I$, we get

$$\|x-x^*\| \leq \mu K(ckH\|x-x^*\|+\delta).$$

Hence

$$\|x-x^*\| \leq \mu K\delta(1-\mu kcKH)^{-1} = \epsilon.$$

which proves the continuous dependence of the solutions on the set S_{F_1} of all Lipschitzian selection of F_1. This completes the proof. □

4.3. **Set-valued Chandrasekhar nonlinear functional integral equation.** Now, as an application of the nonlinear a set-valued functional integral equation of the Hammerstein-Stieltjes type (1.2), we have the following.

Let the functions g_i, $i = 1,2$, be defined by

$$g_1(t,s) = \begin{cases} t\ln\frac{t+s}{t}, & \text{for } t\in(0,1],\ s\in I,\\ 0, & \text{for } t=0,\ s\in I,\end{cases}$$

and

$$g_2(s,\theta) = \begin{cases} s\ln\frac{s+\theta}{s}, & \text{for } s\in(0,1],\ \theta\in I,\\ 0, & \text{for } s=0,\ \theta\in I.\end{cases}$$

Let $f_2(t,x(t)) = b_1(t)x(t)$, $k(t,s)=1$, and $\varphi(t)=t$ in (1.2).

Further, using the fact that functions g_i, $i=1,2$, satisfy assumptions (vi)-(ix) in Theorem 4.1 (see [5]), we obtain the set-valued Chandrasekhar nonlinear functional integral inclusion (1.3)

$$x(t) \in a(t) + \int_0^1 \frac{t}{t+s} F_1\Big(s, \int_0^s \frac{s}{s+\theta} b_1(\theta)x(\theta)d\theta\Big)dst, \in I.$$

Now, we can formulate the following existence result for the nonlinear Chandrasekhar functional integral inclusions (1.3).

Theorem 4.4. *Under the assumptions of Theorem 4.1, the functional integral inclusions (1.3) has at least one continuous solution $x \in C(I,\mathbb{R})$.*

4.4. **Set-valued Hammerstein nonlinear functional integral equation of fractional order.** In this section, we will consider the fractional integral inclusion (1.4)

$$x(t) \in p(t) + \int_0^1 k(t,s)F_1(s, \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} f_2(\theta, x(\varphi(\theta)))d\theta)ds, \qquad s,\theta\in I,$$

where $\Gamma(\alpha)$ denotes the gamma function and $\alpha \in (0,1)$.

Let us mention that (1.4) represents the so-called nonlinear Hammerstein integral inclusion of fractional orders. Recently, the inclusion of such a type was intensively investigated in some papers, see [13, 14, 15].

Now, we show that the functional integral inclusion of fractional orders (1.4) can be treated as a particular case of the set-valued functional integral equation of Hammerstein-Stieltjes (1.2) studied in Section 3.

Indeed, we can consider functions $g_i(w,z) = g_i : \triangle_i \to \mathbb{R}$, $i = 1, 2$, defined by the formula

$$g_1(t,s) = s, \qquad s \in I,$$
$$g_2(s,\theta) = \frac{s^\alpha - (s-\theta)^\alpha}{\Gamma(\alpha+1)}, \qquad s,\theta \in I.$$

Note that the functions g_1 and g_2 satisfy assumptions (vi)-(ix) in Theorem 3.5, see [5, 16].

Now, we can formulate the following existence results concerning the fractional integral inclusion of fractional order (1.4).

Theorem 4.5. *Under the assumptions (i)-(iv) of Theorem 4.1, the fractional integral inclusion (1.4) has at least one continuous solution $x \in C(I,\mathbb{R})$.*

4.4.1. *Differential inclusion.* Consider now the initial value problem of the differential inclusion (1.5) with the initial data (1.6).

Theorem 4.6. *Let the assumption of Theorem 4.5 be satisfied. Then the initial value problem (1.5)-(1.6) has at least one positive solution $x \in C(I,\mathbb{R})$*

Proof. Let $y(t) = \frac{dx(t)}{dt}$. Then the inclusion (1.5), will be

$$y(t) \in \int_0^1 F_1(s, I^{1-\tau} y(s))ds. \tag{4.10}$$

By letting $f_2(t,x) = x(t)$, $\varphi(t) = t$, $k(t,s) = 1$, and $\alpha = 1-\tau$, and applying Theorem 4.5 to the functional inclusion (1.4), we deduce that there exists a continuous solution $y \in C(I,\mathbb{R})$ of the functional inclusion (1.4) and this solution depends on the set S_{F_1}.

This implies existence of a solution $x \in C(I,\mathbb{R})$,

$$x(t) = x_\circ + \int_0^t y(s)ds,$$

of the initial-value problem (1.5)-(1.6). □

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