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ON SOME NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

MAHDI GHASVAREH AND MOHSEN ERFANIAN OMIDVAR

Abstract. This article is devoted to studying some new numerical radius inequalities for Hilbert space operators. Our analysis enables us to improve an earlier bound for numerical radius due to Kittaneh. It is shown, among other, that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{8}\left(\|A+A^*\|^2+\|A-A^*\|^2\right)\le \omega^2(A)\le\left\|\frac{|A|^2+|A^*|^2}{2}\right\|-m\left(\left(\frac{|A|-|A^*|}{2}\right)^2\right).$$

Отримані нові нерівності для числового радіуса операторів у гільбертовім просторі. Зокрема, покращено попередній результат Кіттане. Показано, що для $A \in B(H)$,

$$\frac{1}{8}\left(\|A+A^*\|^2+\|A-A^*\|^2\right)\le \omega^2(A)\le\left\|\frac{|A|^2+|A^*|^2}{2}\right\|-m\left(\left(\frac{|A|-|A^*|}{2}\right)^2\right).$$

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. For $A \in \mathcal{B}(\mathcal{H})$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the operator norm of A, respectively. Recall that $\omega(A)=\sup_{\|x\|=1}|\langle Ax,x\rangle|$. It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\|\le\omega(A)\le\|A\|. \tag{1.1}$$

Also, it is a basic fact that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$ which satisfies the power inequality

$$\omega(A^n)\le\omega^n(A)$$

for all $n=1,2,\ldots$.

In [2], Kittaneh gave the following estimate of the numerical radius which refines the second inequality in (1.1): For every A,

$$\omega(A)\le\frac{1}{2}\||A|+|A^*|\|. \tag{1.2}$$

The following estimate of the numerical radius has been given in [4]:

$$\frac{1}{4}\left\||A|^2+|A^*|^2\right\|\le\omega^2(A)\le\frac{1}{2}\left\||A|^2+|A^*|^2\right\|. \tag{1.3}$$

The first inequality in (1.3) also refines the first inequality in (1.1). This can be seen by using the fact that for any positive operator $A, B \in \mathcal{B}(\mathcal{H})$,

$$\max(\|A\|,\|B\|)\le\|A+B\|.$$

Actually,

$$\frac{1}{4}\|A\|^2=\frac{1}{4}\max\left(\left\||A|^2\right\|,\left\||A^*|^2\right\|\right)\le\frac{1}{4}\left\||A|^2+|A^*|^2\right\|.$$

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For other properties of the numerical radius and related inequalities, the reader may consult [6, 7, 5]. In this article, we give several refinements of numerical radius inequalities. Our results mainly improve the inequalities in [4].

2. Main Results

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\left\|A \pm A^*\right\| \leq \omega\left(A\right).$$

Proof. Since $A + A^*$ is normal, we have

$$\begin{aligned}\left\|A + A^*\right\| &= \omega\left(A + A^*\right)\\ &\leq \omega\left(A\right) + \omega\left(A^*\right)\\ &= 2\omega\left(A\right).\end{aligned}$$

Therefore,

$$\frac{1}{2}\left\|A + A^*\right\| \leq \omega\left(A\right). \tag{2.4}$$

Now, by replacing A by iA in (2.4), we reach the desired result. □

Theorem 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\left\|\left|A\right|^2 + \left|A^*\right|^2\right\| \leq \frac{1}{8}\left(\left\|A + A^*\right\|^2 + \left\|A - A^*\right\|^2\right) \leq \omega^2\left(A\right).$$

Proof. For any $A, B \in \mathcal{B}(\mathcal{H})$, we have the following parallelogramm law

$$\left|A + B\right|^2 + \left|A - B\right|^2 = 2\left(\left|A\right|^2 + \left|B\right|^2\right),$$

equivalently

$$\left|\frac{A + B}{2}\right|^2 + \left|\frac{A - B}{2}\right|^2 = \frac{\left|A\right|^2 + \left|B\right|^2}{2}.$$

Therefore, by the triangle inequality for the usual operator norm and Lemma 2.1, we have

$$\begin{aligned}\frac{1}{4}\left\|\left|A\right|^2 + \left|A^*\right|^2\right\| &= \frac{1}{2}\left\|\frac{\left|A\right|^2 + \left|A^*\right|^2}{2}\right\|\\ &= \frac{1}{2}\left\|\left|\frac{A + A^*}{2}\right|^2 + \left|\frac{A - A^*}{2}\right|^2\right\|\\ &\leq \frac{1}{2}\left\|\left|\frac{A + A^*}{2}\right|^2\right\| + \frac{1}{2}\left\|\left|\frac{A - A^*}{2}\right|^2\right\|\\ &= \frac{1}{2}\left\|\frac{A + A^*}{2}\right\|^2 + \frac{1}{2}\left\|\frac{A - A^*}{2}\right\|^2\\ &\leq \omega^2\left(A\right).\end{aligned}$$

We remark here that if $T \in \mathcal{B}(\mathcal{H})$, and if f is a non-negative increasing function on $[0, \infty)$, then $\left\|f\left(\left|T\right|\right)\right\| = f\left(\left\|T\right\|\right)$. In particular, $\left\|\left|T\right|^r\right\| = \left\|T\right\|^r$ for every $r > 0$. This completes the proof of the theorem. □

We present a refinement of the first inequality from (1.3). To this end, we need the following lemma, which can found in [1].

Lemma 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A + B\| \leq \sqrt{\|A^*A + B^*B\| + 2\omega(B^*A)}.$$

By the above lemma, we can improve the first inequality in (1.3).

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\left\||A|^2+|A^*|^2\right\| \le \frac{1}{2}\sqrt{2\omega^4(A)+\frac{1}{8}\omega\left((A^*-A)^2(A^*+A)^2\right)} \le \omega^2(A). \tag{2.5}$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A. Then B and C are self-adjoint operators. One can easily check that

$$\frac{|A|^2+|A^*|^2}{4} = \frac{B^2+C^2}{2}, \tag{2.6}$$

and

$$|\langle Ax, x\rangle|^2 = \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 \tag{2.7}$$

for any unit vector $x \in \mathcal{H}$. Of course, the relation (2.7) implies

$$\langle Bx, x\rangle^2 \left(\text{resp. } \langle Cx, x\rangle^2\right) \le |\langle Ax, x\rangle|^2.$$

Now, by taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|B\|^2 \left(\text{resp. } \|C\|^2\right) \le \omega^2(A). \tag{2.8}$$

Whence,

$$\begin{aligned}
\frac{1}{4}\left\||A|^2+|A^*|^2\right\| &= \frac{1}{2}\left\|B^2+C^2\right\| \quad (\text{by } (2.6)) \\
&\le \frac{1}{2}\sqrt{\|B^4+C^4\|+2\omega\left(C^2B^2\right)} \quad (\text{by Lemma } 2.3) \\
&\le \frac{1}{2}\sqrt{\|B\|^4+\|C\|^4+2\omega\left(C^2B^2\right)} \\
&\le \frac{1}{2}\sqrt{2\omega^4(A)+2\omega\left(C^2B^2\right)} \quad (\text{by } (2.8)) \\
&\le \frac{1}{2}\sqrt{2\omega^4(A)+2\left\|C^2B^2\right\|} \quad (\text{by the second inequality in } (1.1)) \\
&\le \frac{1}{2}\sqrt{2\omega^4(A)+2\|B\|^2\|C\|^2} \\
&\qquad (\text{by the submultiplicativity of the usual operator norm}) \\
&\le \omega^2(A) \quad (\text{by } (2.8))
\end{aligned}$$

i.e.,

$$\frac{1}{4}\left\||A|^2+|A^*|^2\right\| \le \frac{1}{2}\sqrt{2\omega^4(A)+2\omega\left(C^2B^2\right)} \le \omega^2(A).$$

Since

$$\omega\left(C^2B^2\right) = \frac{1}{16}\omega\left((A^*-A)^2(A^*+A)^2\right)$$

we get the desired result (2.5). □

Remark 2.5. Notice that if A is a self-adjoint operator, then Theorem 2.2 implies

$$\frac{1}{2}\|A\|^2 \le \|A\|^2$$

while from Theorem 2.4 we infer that

$$\frac{1}{2}\|A\|^2 \le \frac{\sqrt{2}}{2}\|A\|^2 \le \|A\|^2.$$

Hence, in this case, Theorem 2.4 is better than Theorem 2.2.

The next lemma can be found in [3].

Lemma 2.6. *If A and B are positive operators in $\mathcal{B}(\mathcal{H})$, then*

$$\|A-B\| \le \max\{\|A\|,\|B\|\} - \min\{m(A), m(B)\},$$

where $m(A)=\inf\{\langle Ax,x\rangle : \ x\in\mathcal{H}, \|x\|=1\}$.

Theorem 2.7. *Let $A\in\mathcal{B}(\mathcal{H})$. Then*

$$\omega^2(A) \le \frac{1}{2}\left[\left\||A|^2+|A^*|^2\right\| - m\left((|A|-|A^*|)^2\right)\right].$$

Proof. We can write

$$\begin{aligned}
\left\|\left(\frac{|A|+|A^*|}{2}\right)^2\right\| &= \left\|\left(\frac{|A|-|A^*|}{2}\right)^2 - \frac{|A|^2+|A^*|^2}{2}\right\| \\
&\le \max\left(\left\|\left(\frac{|A|-|A^*|}{2}\right)^2\right\|, \left\|\frac{|A|^2+|A^*|^2}{2}\right\|\right) \\
&\quad - \min\left(m\left(\left(\frac{|A|-|A^*|}{2}\right)^2\right), m\left(\frac{|A|^2+|A^*|^2}{2}\right)\right) \\
&\le \left\|\frac{|A|^2+|A^*|^2}{2}\right\| - m\left(\left(\frac{|A|-|A^*|}{2}\right)^2\right).
\end{aligned}$$

On the other hand, since

$$\omega(A) \le \frac{1}{2}\||A|+|A^*|\|,$$

we have

$$\omega^2(A) \le \left\|\left(\frac{|A|+|A^*|}{2}\right)\right\|^2 = \left\|\left(\frac{|A|+|A^*|}{2}\right)^2\right\|.$$

Consequently,

$$\omega^2(A) \le \left\|\frac{|A|^2+|A^*|^2}{2}\right\| - m\left(\left(\frac{|A|-|A^*|}{2}\right)^2\right),$$

as desired. □

The following lemma, which can be found in [9, Lemma 2.3 and Proposition 2.2] is needed to prove our last estimate for $\omega(A)$.

Lemma 2.8. *Let f be a non-negative increasing convex function on $[0,\infty)$ and let $A, B\in\mathcal{B}(\mathcal{H})$ be positive operators. Then for any $0\le v\le 1$,*

$$\|f((1-v)A+vB)\| \le \|(1-v)f(A)+vf(B)\|.$$

In particular,

$$\left\|f\left(\frac{A+B}{2}\right)\right\| \le \left\|\frac{f(A)+f(B)}{2}\right\|$$

Using some ideas of [8], we prove our last result.

Theorem 2.9. *Let $A\in\mathcal{B}(\mathcal{H})$ and let f be a continuous function on the interval $[0,\infty)$ and let g be increasing and concave on $[0,\infty)$, such that gof is increasing and convex on $[0,\infty)$. Then*

$$f(\omega(A)) \le \left\|g^{-1}\left(\frac{gof(|A|)+gof(|A^*|)}{2}\right)\right\| \le \frac{1}{2}\|f(|A|)+f(|A^*|)\|.$$

Proof. As mentioned above, gof is increasing and convex on $[0,\infty)$, therefore, from the inequality (1.2),

$$\begin{aligned} gof\left(\omega\left(A\right)\right) &\leq gof\left(\left\|\frac{|A|+|A^*|}{2}\right\|\right) \\ &= \left\|gof\left(\frac{|A|+|A^*|}{2}\right)\right\| \\ &\leq \left\|\frac{gof\left(|A|\right)+gof\left(|A^*|\right)}{2}\right\| \quad \text{(by Lemma 2.8).} \end{aligned}$$

Therefore,

$$gof\left(\omega\left(A\right)\right) \leq \left\|\frac{gof\left(|A|\right)+gof\left(|A^*|\right)}{2}\right\|.$$

Now, since g^{-1} is increasing and convex, we then have

$$\begin{aligned} f\left(\omega\left(A\right)\right) &= g^{-1}\left(gof\left(\omega\left(A\right)\right)\right) \\ &\leq g^{-1}\left(\left\|\frac{gof\left(|A|\right)+gof\left(|A^*|\right)}{2}\right\|\right) \\ &= \left\|g^{-1}\left(\frac{gof\left(|A|\right)+gof\left(|A^*|\right)}{2}\right)\right\| \\ &\leq \left\|\frac{f\left(|A|\right)+f\left(|A^*|\right)}{2}\right\| \quad \text{(by Lemma 2.8)} \\ &= \frac{1}{2}\left\|f\left(|A|\right)+f\left(|A^*|\right)\right\|. \end{aligned}$$

Thus,

$$f\left(\omega\left(A\right)\right) \leq \left\|g^{-1}\left(\frac{gof\left(|A|\right)+gof\left(|A^*|\right)}{2}\right)\right\| \leq \frac{1}{2}\left\|f\left(|A|\right)+f\left(|A^*|\right)\right\|.$$

□

Corollary 2.10. *Let $A \in \mathcal{B}(\mathcal{H})$. Then for any $r \geq 2$,*

$$\begin{aligned} &\omega^r\left(A\right) \\ &\leq \frac{1}{2}\left\||A|^r+|A^*|^r+|A|^{\frac{r}{2}}+|A^*|^{\frac{r}{2}}+I-\sqrt{2\left(|A|^r+|A^*|^r+|A|^{\frac{r}{2}}+|A^*|^{\frac{r}{2}}\right)+I}\right\| \\ &\leq \frac{1}{2}\left\||A|^r+|A^*|^r\right\|. \end{aligned}$$

Proof. Define

$$g\left(x\right)=x+\sqrt{x} \qquad \& \qquad f\left(x\right)=x^r, \ \ (r\geq 2)$$

on $[0,\infty)$. Thus,

$$gof\left(x\right)=x^r+x^{\frac{r}{2}}.$$

One can quickly check that f, g, and gof satisfy all the assumptions in Theorem 2.7. Since

$$g^{-1}\left(x\right)=\frac{2x+1-\sqrt{4x+1}}{2},$$

we get the desired result. □

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Mahdi Ghasvareh: mghasvareh@gmail.com
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran

Mohsen Erfanian Omidvar: math.erfanian@gmail.com
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran

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