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COMPACTNESS PROPERTIES OF LIMITED OPERATORS

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Abstract. The aim of this paper is to investigate the relationship between limited operators and weakly compact (resp. compact) operators. Mainly, it is proved that if every limited operator $T : E \to X$ from a Banach lattice E into Banach space X is weakly compact (resp. compact) then the norm of E' is order continuous or X has the (BD) property (resp. GP property). Also, it is proved that if every weakly compact operator $T : E \to X$ is limited then the norm of E' is order continuous or X has the DP* property.

Метою цієї роботи є дослідження зв'язку між обмежувальними операторами та слабо компактними (відповідно компактними) операторами. Доведено, що якщо кожен обмежувальний оператор $T : E \to X$ з банахової ґратки E в банаховий простір X є слабо компактним (відповідно компактним), то норма в E' є порядково неперервною або X має (BD)–властивість (відповідно GP–властивість). Також доведено, що якщо кожний слабо компактний оператор $T : E \to X$ обмежений, то норма в E' є порядково неперервною або X має DP*–властивість.

1. Introduction

Throughout this paper X, Y will denote real Banach spaces and E, F will denote real Banach lattices. The unit closed ball of a Banach space X (resp. Banach lattice E) will be denoted by B_X (resp. B_E). We refer to [1, 9, 11] for unexplained terminology of the Banach lattice theory and positive operators.

A bounded subset A of a Banach space X is called a *limited* subset of X if for each weak* null sequence (x'_n) in X',

$$\lim_{n\to\infty} \sup\{|x'_n(x)| : x \in A\} = 0.$$

A sequence (x_n) in X is called *limited* if $\{x_n : n \in \mathbb{N}\}$ is a limited subset of X. We know that every relatively (norm) compact subset of a Banach space X is limited, but the converse of that is false in general. For example, if c_0 is the Banach space of convergent to zero real sequences with the supremum norm, then its closed unit ball B_{c_0} is a limited subset of ℓ^∞, but it is not relatively compact. If all limited subsets of X are relatively compact, then X is said to have the *Gelfand-Phillips property* (abb. *GP property*). Alternatively, a Banach space X has the GP property if and only if every weak null limited sequence (x_n) in X is norm null [5]. The following spaces have the Gelfand-Phillips property: Schur spaces, separable spaces, reflexive spaces, duals of spaces containing no copy of ℓ^1. Some useful and additional properties of limited sets and Banach spaces with the GP property can be found in [3, 5, 6].

We shall say that X has the *(BD) property* if any limited set in X is relatively weakly compact [8]. By using the Eberlein–Šmulian Theorem, we see that X has the (BD) property if and only if every limited sequence (x_n) in X has a weak convergent subsequence. Gelfand-Phillips spaces and spaces not containing ℓ^1 have the (BD) property [3].

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Recall from [2] that a Banach space X has the *Dunford-Pettis* * *property* (abb. *DP* * *property*) if every relatively weakly compact subset of X is limited. It turns out that a Banach space X has the DP* property if and only if $\lim f_n(x_n) = 0$ for every weakly null sequence (x_n) in X and every weak* null sequence (f_n) in X'. Alternatively, a Banach space X has the DP* property if and only if every weakly compact operator from ℓ^1 to X is limited [7, Corollary 3.3]. As examples, the classical Banach spaces ℓ^1, ℓ^∞ have the DP* property (see [4]).

An operator $T : X \longrightarrow Y$ from a Banach space X into a Banach space Y is called *limited* (resp. *weakly compact*) if T maps the closed unit ball of X to limited (resp. relatively weakly compact) subset of Y. Clearly, every compact operator is limited but there exists a limited operator which is not compact (weakly compact). Indeed, the canonical injection $i : c_0 \hookrightarrow \ell^\infty$ is a limited operator (see [1, Theorem 4.67]), but it is not compact (weakly compact). Also, there exists a weakly compact operator which is not limited (for example, the identity operator $Id_{\ell^2} : \ell^2 \longrightarrow \ell^2$ is weakly compact but it is not limited).

In this paper, we investigate the relationship between limited operators and weakly compact (resp. compact) operators. More precisely, we prove that if every limited operator $T : E \to X$ from a Banach lattice E into Banach space X is weakly compact (resp. compact) then the norm of E' is order continuous or X has the (BD) property (resp. GP property). Also, it is proved that if every weakly compact operator $T : E \to X$ is limited then the norm of E' is order continuous or X has the DP* property.

2. Main Results

We start the section with a characterization of (BD) property (resp. GP property).

Proposition 2.1. *Let X be a Banach space.*

(1) *A sequence (x_n) in X is limited if and only if the operator $T : \ell^1 \to X$ defined by $T((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n$ is limited;*

(2) *X has the (BD) property if and only if every limited operator $T : \ell^1 \to X$ is weakly compact;*

(3) *X has the GP property if and only if every limited operator $T : \ell^1 \to X$ is compact.*

Proof. (1) Assume that (x_n) is a limited sequence in X. Then (x_n) is norm bounded and so the operator $T : \ell^1 \to X$ defined by

$$T((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n$$

is well defined. If $T' : X' \to \ell^\infty$ denotes the adjoint of T, then for each weak* null sequence (f_n) in X' we have

$$\|T'(f_n)\|_\infty = \sup_{k \geq 1} |f_n(x_k)| \longrightarrow 0,$$

as (x_n) is a limited sequence in X. Thus, T is limited.

For the converse, assume that the operator $T : \ell^1 \to X$ defined by

$$T((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n$$

is limited, that is $T(B_{\ell^1})$ is a limited subset of X. Now let (e_n) denote the sequence of basis vectors of ℓ^1. From $x_n = T(e_n) \in T(B_{\ell^1})$ we conclude that (x_n) is a limited sequence in X.

(2) The "only if" part is trivial. The "if" part needs proof. So, assume that every limited operator from ℓ^1 to X is weakly compact. Let (x_n) be a limited sequence in X. Then by (1) the operator $T : \ell^1 \to X$ defined by

$$T((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n$$

is limited, and so by our hypothesis T is weakly compact. From $T(e_n) = x_n$ we conclude that (x_n) has a weakly convergent subsequence, and this proves that X has the (BD) property.

(3) As in (2) the "only if" part is trivial. The "if" part needs proof. So, assume that every limited operator from ℓ^1 to X is compact. Let (x_n) be a weak null limited sequence in X. Then by (1) the operator $T : \ell^1 \to X$ defined by

$$T((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n,$$

is limited, and so by our hypothesis T is compact. Clearly, $T(e_n) = x_n \overset{w}{\to} 0$ holds. On the other hand, since T is compact, we see that every subsequence of (x_n) has a subsequence converging in norm to zero. Therefore, $\|x_n\| \to 0$ holds, and this proves that X has the GP property. □

The following result tells us what happens when every limited operator $T : E \to X$ is always weakly compact.

Theorem 2.2. *Let E be a Banach lattice and X be a Banach space. If every limited operator $T : E \to X$ is weakly compact then one of the following conditions is valid:*

(1) *the norm of E' is order continuous;*

(2) *X has the (BD) property.*

Proof. Assume by way of contradiction that the norm of E' is not order continuous and X does not have the (BD) property. To finish the proof, we have to construct a limited operator from E into X which is not weakly compact.

Since the norm of E' is not order continuous, by Theorem 116.1 of [11], there exists a norm bounded disjoint sequence (u_n) of positive elements in E which does not converge weakly to zero. Hence, we may assume that $||u_n|| \leqslant 1$ for all n and also satisfies $\phi(u_n) = 1$ for some $0 \leqslant \phi \in E'$ and all n. Consider each $u_n \in E$ as an element in E'' and let

$$N_{u_n} := \{f \in E' : |f|(|u_n|) = 0\}$$

and

$$C_{u_n} := N_{u_n}^d = \{f \in E' : f \perp N_{u_n}\}.$$

Let P_n be the order projection of $E' = N_{u_n} \oplus C_{u_n}$ onto the carrier C_{u_n} of u_n, and let $\phi_n := P_n(\phi)$ be the component of ϕ in C_{u_n}.

By Theorem 116.3 of [11] the components ϕ_n of ϕ in C_{u_n} form an order bounded disjoint sequence in $(E')^+$ such that $\phi_n(u_n) = \phi(u_n) = 1$ for all n and $\phi_n(u_m) = 0$ if $n \neq m$. Define the operator $P : E \to \ell^1$ by

$$P(x) = (\phi_n(x))_{n=1}^{\infty} \quad \text{for all } x \in E.$$

Note that in view of the inequality

$$\sum_{n=1}^{\infty} |\phi_n(x)| \leqslant \sum_{n=1}^{\infty} \phi_n(|x|) \leqslant \phi(|x|),$$

we see that $P(x) \in \ell^1$ and $\|P(x)\|_1 \leq \|\phi\| \, \|x\|$ holds for each $x \in E$. Then P is a bounded linear operator.

On the other hand, since X does not have the (BD) property there exists a limited sequence (y_n) in X which does not have any weakly convergent subsequence. Define the positive operator $S : \ell^1 \longrightarrow X$ by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for all } (\lambda_n) \in \ell^1.$$

Clearly S is well defined and is limited (Proposition 2.1). Next, we consider the product operator $T = S \circ P : E \to \ell^1 \to X$ defined by

$$T(x) = \sum_{n=1}^{\infty} \phi_n(x) y_n \qquad \text{for all } x \in E.$$

Since S is limited then $T = S \circ P$ is limited. Indeed, if (f_n) is a weak* null sequence in X', then $\|S'(f_n)\|_\infty \to 0$, and hence $\|T'(f_n)\| = \|P'(S'(f_n))\| \to 0$. Thus T is limited. If T is a weakly compact operator, then the sequence $(T(u_n)) = (y_n)$ contains a weakly convergent subsequence which contradicts the choice of (y_n), and the proof is finished. □

Corollary 2.3. *Let E be a Banach lattice and X be a Banach space. If E is KB-space then the following statements are equivalent:*

(1) *Every limited operator $T : E \longrightarrow X$ is weakly compact;*
(2) *One of the conditions is valid:*
  (a) *E is reflexive;*
  (b) *X has the (BD) Property.*

Proof. (1) ⇒ (2) Follows from Theorem 2.2 by noting that E is reflexive if and only if E and E' are KB-spaces.

(2.a) ⇒ (1) In this case, every operator $T : E \longrightarrow X$ is weakly compact.

(2.b) ⇒ (1) Obvious. □

The next result tells us what happens when every weakly compact operator $T : E \to X$ is always limited.

Theorem 2.4. *Let E be a Banach lattice and X be a Banach space. If every weakly compact operator $T : E \longrightarrow X$ is limited then one of the following conditions is valid:*

(1) *the norm of E' is order continuous;*
(2) *X has the DP* property.*

Proof. Assume that the norm of E' is not order continuous. We have to show that X has the DP* property. To this end, let (y_n) be a weak null sequence in X and (g_n) be a weak* null sequence in X'. Consider the operator $S : \ell^1 \longrightarrow X$ defined by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for all } (\lambda_n) \in \ell^1.$$

By Theorem 5.26 of [1], the operator S is weakly compact. Since the norm of E' is not order continuous, by Theorem 2.4.2 of [9], there exists a positive order bounded disjoint sequence $(x'_n) \subset E'$ satisfying $||x'_n|| = 1$ for all n. Let $x' \in (E')^+$ such that $0 \leqslant x'_n \leqslant x'$ for all n. Defined the operator $P : E \longrightarrow \ell^1$ by

$$P(x) = (x'_n(x))_{n=1}^{\infty} \quad \text{for all } x \in E.$$

Since $\sum_{k=1}^{n} |x'_k(x)| \leq \sum_{k=1}^{n} x'_k(|x|) \leq x'(|x|)$, the operator P is well defined. Now, consider the composed operator $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow X$ such that

$$T(x) = \sum_{n=1}^{\infty} x'_n(x) y_n \quad \text{for all } x \in E.$$

Clearly, T is a weakly compact operator. So, by our hypothesis T is limited. Its adjoint $T' : X' \longrightarrow E'$ defined by

$$T'(y') = \sum_{n=1}^{\infty} y'(y_n)x'_n \quad \text{for all } y' \in X'.$$

Since (g_n) is a weak* null sequence in X', $||T'(g_n)|| \longrightarrow 0$ as $n \longrightarrow \infty$. Now from the inequality,

$$|T'(g_n)| = \sum_{i=1}^{\infty} |g_n(y_i)|x'_i \geqslant |g_n(y_n)|x'_n \geqslant 0,$$

we conclude that

$$|g_n(y_n)| = |g_n(y_n)| \ \ ||x'_n|| \leqslant ||T'(g_n)|| \longrightarrow 0,$$

and hence $g_n(y_n) \longrightarrow 0$, and the proof is finished. □

Corollary 2.5. *If E is an infinite dimensional AL-space, then the following conditions are equivalent:*

(1) *Every weakly compact operator $T : E \longrightarrow X$ is limited;*
(2) *X has DP* property.*

Remark 2.6. The first necessary condition in the previous theorem is not sufficient. In fact, if we take $E = \ell^\infty$ and $X = c_0$, it is known that every operator from ℓ^∞ into c_0 is weakly compact (because ℓ^∞ has the Grothendieck property). On the other hand, it follows from a result of Wnuk [10, p. 198] that there exists a non regular operator $T : \ell^\infty \to c_0$ which is necessarily not compact (and hence not limited).

However, if E' has the Schur property then every weakly compact operator $T : E \longrightarrow X$ is compact (and hence limited).

Finally we obtain the following result that tells us what happens when every limited operator $T : E \to X$ is always compact.

Theorem 2.7. *Let E be a Banach lattice and X be a Banach space. If every limited operator $T : E \to X$ is compact then one of the following conditions is valid:*

(1) *the norm of E' is order continuous;*
(2) *X has Gelfand-Phillips property.*

Proof. The proof is very similar to the proof of Theorem 2.2. Assume by way of contradiction that the norm of E' is not order continuous and X does not have the GP property. To finish the proof, we have to construct a limited operator from E into X which is not compact.

We consider the same operators $P : E \to \ell^1$ and $S : \ell^1 \longrightarrow X$ defined as in the proof of Theorem 2.2, but now we assume that (y_n) is a weakly null limited sequence in X such that $\|y_n\| \nrightarrow 0$. By Proposition 2.1, the operator S is limited and hence $T = S \circ P$ is limited. But T is not compact. To see this, note that $T(u_n) = y_n \xrightarrow{w} 0$. If T is a compact operator, every subsequence of (y_n) has a subsequence converging in norm to zero. Therefore, $\|y_n\| \to 0$ holds, which contradicts the choice of (y_n), and the proof is finished. □

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