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ON THE ASCENT-DESCENT SPECTRUM

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Abstract. We establish various properties as well as diverse relations of the ascent and descent spectra for bounded linear operators. We specially focus on the theory of subspectrum. Furthermore, we construct a new concept of convergence for such spectra.

Встановлюються різні властивості спектрів підйому та спуску для обмежених лінійних операторів, а також певні співвідношення між ними. Ми зосереджуємося на теорії підспектру. Крім того, ми пропонуємо нове поняття збіжності для таких спектрів.

1. Introduction and main results

Denote by $B(X)$ the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X and by $\mathcal{F}(X)$ the set of finite rank operators on X. For $T \in B(X)$, we use $N(T)$ and $R(T)$ respectively to denote the null-space and the range of T. The *ascent* of $T \in B(X)$, denoted by $\operatorname{asc}(T)$, is the smallest $n \in \mathbb{N}$ satisfying $N(T^n) = N(T^{n+1})$. If such n does not exist then $\operatorname{asc}(T) = \infty$. The *descent* of T, denoted by $\operatorname{dsc}(T)$, is the smallest $n \in \mathbb{N}$ satisfying $R(T^n) = R(T^{n+1})$. If such n does not exist then $\operatorname{dsc}(T) = \infty$. We define the generalized kernel of T by $N^\infty(T) = \bigcup_{n=1}^{\infty} N(T^n)$ and the generalized range of T by $R^\infty(T) = \bigcap_{n=1}^{\infty} R(T^n)$. Next, we denote by $\operatorname{Asc}(B(X))$ the space of bounded operators T such that $\operatorname{asc}(T)$ is finite and by $\operatorname{Dsc}(B(X))$ the space of bounded operators T such that $\operatorname{dsc}(T)$ is finite. It is worth noting that this algebraic theory was mostly developed by M. A. Kaashoek [17] and A.E. Taylor [24]. As an interesting result which characterizes ascent-descent operators is the following proposition:

Proposition 1.1. [1] *Let T be a linear operator on a vector space X and m be a positive natural number. The following assertions hold true:*
i) $\operatorname{asc}(T) \leq m < \infty$ *if and only if for every* $n \in \mathbb{N}$, $R(T^m) \cap N(T^n) = \{0\}$.
ii) $\operatorname{dsc}(T) \leq m < \infty$ *if and only if for every* $n \in \mathbb{N}$ *there exists a subspace* $Y_n \subset N(T^m)$ *satisfying* $X = Y_n \oplus R(T^n)$.

As shown in [25], see also [6], the previous proposition can be reformulated as follows:

Remark 1.2.

$$\operatorname{asc}(T) \text{ is finite } \Leftrightarrow R(T^n) \cap N(T) = \{0\}, \text{ for some } n \geq 0. \tag{1.1}$$

$$\operatorname{dsc}(T) \text{ is finite } \Leftrightarrow R(T) + N(T^n) = X, \text{ for some } n \geq 0. \tag{1.2}$$

Recently, stability problems of operators under perturbation have attracted many researchers and undergone important contributions, see for instance [2, 13, 14, 4, 6, 7, 8, 12, 15, 18, 20]. Recall that, in [10], authors have proved nice relations between the left Browder spectrum and the left invertible spectrum (respectively, between the right

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Browder spectrum and the right invertible spectrum). Motivated by these last works, we consider the following subsets:

$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin Asc(B(X))\}\text{-the ascent spectrum,}$$
$$\sigma_{dsc}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin Dsc(B(X))\}\text{-the descent spectrum.}$$

Focusing on the axiomatic theory of subspectrum which was introduced by Slodowski and Zelazko [23, 27], we first, recall the following definitions.

Definition 1.3. A subspectrum $\tilde{\sigma}$ in $B(X)$ is a mapping which assigns to every $n-$tuple $(T_1, ..., T_n)$ of mutually commuting elements of $B(X)$ a non-empty compact subset $\tilde{\sigma}(T_1, ..., T_n) \subset \mathbb{C}^n$ such that:
i) $\tilde{\sigma}(T_1, ..., T_n) \subset \sigma(T_1) \times ... \times \sigma(T_n)$,
ii) $\tilde{\sigma}(p(T_1, ..., T_n)) = p(\tilde{\sigma}(T_1, ..., T_n))$ for every commuting $T_1, ..., T_n \in B(X)$ and every polynomial mapping $p = (p_1, ..., p_m) : \mathbb{C}^n \to \mathbb{C}^m$.

The concept of Definition 1.3 is truly suitable since it comprises for example the left (right) spectrum, the left (right) approximate point spectrum, the Harte (the union of the left and right) spectrum. Despite, there are also many examples of spectrum, frequently characterized only for single elements, which are not covered by the axiomatic theory of Żelazko.

Definition 1.4. [19] Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a regularity if
i) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in R \Leftrightarrow a^n \in R$,
ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$.

Such theory was taken into account by many researchers due to its widespread concept that generalizes spectra for single elements to spectra for n-tuple elements. We recall that, in [9], the author introduced regularities and subspectra in a unital noncommutative Banach algebra and showed that there is a correspondence between them similarly to the commutative case. In [19], the authors gave the following conditions on a regularity R, to extend spectra to subspectra:

(C_1) $ab \in R \Leftrightarrow a \in R$ and $b \in R$ for all commuting elements a, $b \in \mathcal{A}$.

(C_2) "Continuity on commuting elements", i.e: If a_n, $a \in \mathcal{A}$, a_n converges to a and $a_n a = a a_n$ for every n then $\lambda \in \sigma_R(a)$ if and only if there exists a sequence $\lambda_n \in \sigma_R(a_n)$ such that λ_n converges to λ.

It is worth noting that the space of bounded linear operators $B(X)$ is a particular case of the Banach algebra $\mathcal{A}$ and $Asc(B(X))$ and $Dsc(B(X))$ are regularities on $B(X)$. For more information, see [14, 5, 20]. Throughout our work, under suitable optimal hypothesis, we will prove that conditions (C_1) and (C_2) are approved properties for the ascent and descent spectra. For the condition (C_1), we need the following assumptions:
(H_1): $ST = TS$ and $\forall p \in \mathbb{N}$, $N((TS)^p) = N(T^p) \oplus N(S^p)$.
(H_2): ST is with finite descent n_0 and $N(S^{n_0}) \subseteq R(T)$ or $N(T^{n_0}) \subseteq R(S)$.
$(H_2^{'})$: ST is with finite descent n_0 and $N(S^{n_0}) \subseteq R(T)$ and $N(T^{n_0}) \subseteq R(S)$.

Consider the following subsets:

$$\tilde{\mathcal{F}} : = \Big\{F \in B(X) \text{ such that there exists } n_0 \in \mathbb{N} \text{ for which } F^{n_0} \in \mathcal{F}(X)\Big\},$$

$$\mathcal{R}: = \Big\{\lambda \in \mathbb{C} \text{ such that if } S+T-\lambda \text{ has finite ascent then } (S-\lambda) \text{ and } (T-\lambda) \text{ do not satisfy } (H_1)\Big\},$$

$$\mathcal{M}: = \Big\{\lambda \in \mathbb{C} \text{ such that if } S+T-\lambda \text{ has finite descent then } (S-\lambda) \text{ and } (T-\lambda) \text{ do not satisfy } (H_1) \text{ or } (H_2^{'})\Big\}.$$

Our main results assuring condition (C_1) reside in:

Theorem 1.5. *Assume that S and T be two bounded linear operators satisfying (H_1) and that $ST \in \tilde{F}$. Then,*

$$\mathcal{R} \cup \sigma_{asc}(S+T) \setminus \{0\} = \mathcal{R} \cup (\sigma_{asc}(S) \cup \sigma_{asc}(T)) \setminus \{0\}.$$

In the descent case we obtained the following results:

Theorem 1.6. *Assume that S and T are two bounded linear operators, satisfying (H_1) and $(H_2^{'})$. Then,*

$$dsc(T) \leq n_0 \textit{ and } dsc(S) \leq n_0 \textit{ if and only if } dsc(TS) \leq n_0.$$

Corollary 1.7. *If $ST = TS \in \tilde{\mathcal{F}}$ then:*
i)

$$\sigma_{dsc}(S+T) \setminus \{0\} \subseteq (\sigma_{dsc}(S) \cup \sigma_{dsc}(T)) \setminus \{0\}.$$

ii)

$$\mathcal{M} \cup \sigma_{dsc}(S+T) \setminus \{0\} = \mathcal{M} \cup (\sigma_{dsc}(S) \cup \sigma_{dsc}(T)) \setminus \{0\}.$$

In [19], the authors show that if R is a regularity in a Banach algebra $\mathcal{A}$ then we have, if a, $b \in \mathcal{A}$, $ab = ba$ and $a \in Inv(\mathcal{A})$ then:

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R. \tag{1.3}$$

In our work, dealing with ascent and descent operators as a special case of a regularity, we prove in Lemma 2.1 and Theorem 1.6 that property (1.3) is satisfied without needing the invertibility of any of the bounded operators T and S.

On the other hand, inspired by the continuity concept of families of magnetic pseudo-differential operators given in [3], we create a concept of convergence of spaces to prove condition (C_2). To start off, we recall the concept of the reduced minimum modulus:

Definition 1.8. Let X be a Banach space and let $T : X \to X$ be a non zero operator. We define the reduced minimum modulus of T by

$$\gamma(T) := \inf\{\|Tx\|;\ x \in X, dist(x, N(T)) = 1\}.$$

Formally, we set $\gamma(0) = \infty$.

Our main result in this context follow on:

Theorem 1.9. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators convergent to T in the operator norm.*

(1) *If $(\lambda_n)_n$ is a sequence convergent to $\lambda \in \sigma_{asc}(T)$ when $T-\lambda$ has a closed range and for all x in $N(T-\lambda)$, $dist(x, N(T_n-\lambda_n))$ is reached from some rank, then $\lambda_n \in \sigma_{asc}(T_n)$ from some rank.*
(2) *Assume that for every $n \in \mathbb{N}$, $\lambda_n \in \sigma_{asc}(T_n)$ and $\limsup \gamma(T_n - \lambda_n) > 0$. If $(\lambda_n)_n$ converges to λ then $\lambda \in \sigma_{asc}(T)$.*

(3) *Let* $(\lambda_n)_n$ *be a sequence converge to* $\lambda \in \sigma_{dsc}(T)$. *If* $\limsup \gamma(T_n - \lambda_n) > 0$, *then* $\lambda_n \in \sigma_{dsc}(T_n)$ *from some rank.*
(4) *Assume that for every* $n \in \mathbb{N}$, $\lambda_n \in \sigma_{dsc}(T_n)$. *If* λ_n *converges to* λ, *then* $\lambda \in \sigma_{dsc}(T)$.

Theorem 1.9 is based on the fact that the limit operator T has a closed range. We do not need neither the commutativity of the operators $(T_n)_n$ nor the fact that T_n has a closed range for every $n \in \mathbb{N}$. It suffices that the range of $(T_n)_n$ is closed from some $p \in \mathbb{N}$. Note that in the case of Hilbert space, the distance hypothesis in 1) is always satisfied.

Finally, we illustrate our theoretical results by an application. It might be said that our approach throughout this paper is purely algebraic. This is different from the approaches used in [16], which is based on the use of analytic functions and the SVEP condition. Note that, The paper is structured as follows: the next section is devoted to proving our main results (Theorem 1.5, Theorem 1.6, Corollary 1.7 and Theorem 1.9), In the third section, we give some properties of corresponding spectra. In the last section, we provide an application.

2. Proofs of Main Results

2.1. Results assuring condition C_1.

2.1.1. *Theorem 1.5.* First, we demonstrate the following lemmas.

Lemma 2.1. *Let* T *and* S *be two bounded linear operators satisfying* (H_1). *Then,* T *and* S *have finite ascents if and only if* TS *has a finite ascent.*

Proof. Let T and S be respectively with finite ascents. Then, there exists $n_0 \in \mathbb{N}$ such that for all $p \geq n_0$, we have:

$$\begin{cases} T^p x = 0 & \Rightarrow T^{n_0} x = 0, \\ & \textit{and} \\ S^p x = 0 & \Rightarrow S^{n_0} x = 0. \end{cases}$$

On the other hand, we have $(TS)^p x = 0 \Rightarrow T^p(S^p x) = 0 \Rightarrow T^{n_0}(S^p x) = 0 \Rightarrow S^p(T^{n_0} x) = 0 \Rightarrow S^{n_0}(T^{n_0} x) = 0 \Rightarrow (TS)^{n_0} x = 0$. This gives that TS is with finite ascent.

Concerning the inverse assertion, we have TS is with finite ascent means that there exists $n_0 \in \mathbb{N}$ such that for all $p \geq n_0$, we have:

$$x \in N((TS)^p) \Rightarrow x \in N((TS)^{n_0}). \tag{2.4}$$

Now, without loss of generality, let $x \in N(S^p)$ such that $p \geq n_0$ and let us show that $x \in N(S^{n_0})$. In fact, $x \in N(S^p)$ implies that $x \in N((ST)^p)$. In view of (2.4), it yields that $x \in N((ST)^{n_0})$. Thus, by (H_1), there exists $x_1 \in N(S^{n_0})$ and $x_2 \in N(T^{n_0})$ such that $x = x_1 + x_2$. Since $N(S^p) \subseteq N(S^{p+1})$ and $N(T^p) \subseteq N(T^{p+1})$ for all $p \in \mathbb{N}$ then $x_1 \in N(S^p)$ and $x_2 \in N(T^p)$. Furthermore, as $N(S^p)$ is a vector subspace and $x_2 = x - x_1$, it yields that $x_2 \in N(S^p) \cap N(T^p)$. By assumption (H_1), we obtain $x_2 = 0$. Hence, $x = x_1 \in N(S^{n_0})$. This proves that S has a finite ascent. ☐

Lemma 2.2. *Let* S *and* T *be two bounded operators satisfying* $ST = TS \in \tilde{\mathcal{F}}$ *then:*

$$\sigma_{asc}(S+T) \setminus \{0\} \subseteq (\sigma_{asc}(S) \cup \sigma_{asc}(T)) \setminus \{0\}.$$

Proof. Let $\lambda \neq 0$ and $\lambda \in \rho_{asc}(S) \cap \rho_{asc}(T)$, then:

$$\begin{cases} (S-\lambda) & \in Asc(B(X)), \\ & \textit{and} \\ (T-\lambda) & \in Asc(B(X)). \end{cases}$$

Since $ST = TS$, then

$$(S-\lambda)(T-\lambda) = (T-\lambda)(S-\lambda).$$

By the direct assertion of Lemma 2.1 which is true for any two commutative operators, we obtain that

$$(S-\lambda)(T-\lambda) \in Asc(B(X)).$$

As,

$$(S-\lambda)(T-\lambda) = ST - \lambda(S+T-\lambda), \tag{2.5}$$

it yields that,

$$ST - \lambda(S+T-\lambda) \in Asc(B(X)). \tag{2.6}$$

Since,

$$ST(S+T-\lambda) = (S+T-\lambda)ST \quad \text{and} \quad ST \in \tilde{\mathcal{F}},$$

we obtain by [18, Theorem 2.2], in view of (2.6), that $(S+T-\lambda) \in Asc(B(X))$. Consequently,

$$\sigma_{asc}(S+T) \setminus \{0\} \subseteq (\sigma_{asc}(S) \cup \sigma_{asc}(T)) \setminus \{0\}. \tag{2.7}$$

□

Proof of Theorem 1.5. The direct inclusion follows from Lemma 2.2.
To prove the reciprocal inclusion, let $\lambda \in (\sigma_{asc}(S) \cup \sigma_{asc}(T)) \setminus \{0\}$ and assume that $\lambda \notin \sigma_{asc}(S+T) \setminus \{0\}$ and $\lambda \notin \mathcal{R}$. Then, $S+T-\lambda \in Asc(B(X))$ and $(S-\lambda)$ and $(T-\lambda)$ satisfy (H_1). Since, $ST = TS \in \tilde{\mathcal{F}}$, then by [18, Theorem 2.2], we obtain in view of (2.5) that:

$$(S-\lambda)(T-\lambda) = (T-\lambda)(S-\lambda) \in Asc(B(X)).$$

As $(S-\lambda)$ and $(T-\lambda)$ satisfy (H_1), it follows from Lemma 2.1 that $(S-\lambda)$ and $(T-\lambda)$ have finite ascents, which is absurd. This proves the second inclusion. □

2.1.2. *Theorem 1.6.* In order to prove Theorem 1.6, we first prove some auxiliary results.

Lemma 2.3. *Let T and S be two bounded linear operators such that $TS = ST$. Then,*

T and S have finite descents $\Rightarrow$ TS has a finite descent.

Proof. Let T and S be respectively with finite descents. Then there exists $n_0 \in \mathbb{N}$ such that for all $p \geq n_0$ we have

$$\begin{cases} T^p(X) & = T^{p+1}(X), \\ & \textit{and} \\ S^p(X) & = S^{p+1}(X). \end{cases}$$

Furthermore, we have

$$(ST)^p(X) = S^pT^p(X) = S^pT^{p+1}(X) = S^{p+1}T^{p+1}(X) = (ST)^{p+1}(X).$$

Consequently, ST has a finite descent. □

Lemma 2.4. *Let T and S be two bounded linear operators.*
1) If T and S satisfy (H_1) and (H_2), then, T or S has a finite descent.
2) If T and S satisfy (H_1) and $(H_2^{'})$, then, T and S have finite descents.

Proof. Since $dsc(TS) = n_0$ is finite, we obtain by Proposition 1.1 that for every $x \in X$ there exists $w \in R(TS)$ and $x_2 \in N((TS)^{n_0})$ satisfying:

$$x = w + x_2.$$

In other words, there exists $x_1 \in X$ such that $w = TSx_1$ and $T^{n_0}S^{n_0}x_2 = 0$. By (H_1), there exists $x_2' \in N(S^{n_0})$ and $x_2'' \in N(T^{n_0})$ satisfying:

$$x = (TS)x_1 + x_2' + x_2''.$$

1) If T and S satisfy (H_2), without loss of generality, we assume that $N(S^{n_0}) \subseteq R(T)$. Hence, there exists $y_2 \in X$ such that $x_2' = Ty_2$. Put $y_1 := Sx_1$, it follows that

$$x = T(y_1 + y_2) + x_2''.$$

The result follows from Remark 1.2, (1.2).
2) If T and S satisfy $(H_2^{'})$, then there exists $y_2 \in X$ such that $x_2' = Ty_2$. Put $y_1 := Sx_1$, it implies that

$$x = T(y_1 + y_2) + x_2''.$$

and there is also $z_2 \in X$ such that $x_2'' = Sz_2$. Set $z_1 := Tx_1$, it yields that

$$x = S(z_1 + z_2) + x_2'.$$

Then, from Remark 1.2, (1.2), S and T have finite descents. □

Proof of Theorem 1.6. The direct sense, it follows from Lemma 2.3.
The reciprocal sense, it is obtained from Lemma 2.4, 2). □

Proof of Corollary 1.7: i) Let $\lambda \neq 0$ and $\lambda \in \rho_{dsc}(S) \cap \rho_{dsc}(T)$, then:

$$\begin{cases} (S-\lambda) & \in Dsc(B(X)), \\ & and \\ (T-\lambda) & \in Dsc(B(X)). \end{cases}$$

Since $ST = TS$, then

$$(S-\lambda)(T-\lambda) = (T-\lambda)(S-\lambda).$$

Using Lemma 2.3, we obtain

$$(S-\lambda)(T-\lambda) \in Dsc(B(X)).$$

Since,

$$(S-\lambda)(T-\lambda) = ST - \lambda(S+T-\lambda), \tag{2.8}$$

we have,

$$ST - \lambda(S+T-\lambda) \in Dsc(B(X)). \tag{2.9}$$

Remark that,

$$ST(S+T-\lambda) = (S+T-\lambda)ST \quad \text{and} \quad ST \in \tilde{\mathcal{F}}.$$

It follows by [6, Theorem 3.1] and (2.9) that $(S+T-\lambda) \in Dsc(B(X))$. Hence,

$$\sigma_{dsc}(S+T) \setminus \{0\} \subseteq (\sigma_{dsc}(S) \cup \sigma_{dsc}(T)) \setminus \{0\}. \tag{2.10}$$

ii) According to i), we have:

$$\mathcal{M} \cup \sigma_{dsc}(S+T) \setminus \{0\} \subseteq \mathcal{M} \cup (\sigma_{dsc}(S) \cup \sigma_{dsc}(T)) \setminus \{0\}. \tag{2.11}$$

Now, concerning the inverse inclusion, let $\lambda \in (\sigma_{dsc}(S) \cup \sigma_{dsc}(T)) \setminus \{0\}$ and assume that $\lambda \notin \sigma_{dsc}(S+T) \setminus \{0\}$ and $\lambda \notin \mathcal{M}$. Then, $S+T-\lambda \in Dsc(B(X))$. Since $ST = TS \in \tilde{\mathcal{F}}$, then by [6, Theorem 3.1] and (2.8), we obtain

$$(S-\lambda)(T-\lambda) = (T-\lambda)(S-\lambda) \in Dsc(B(X)).$$

Using the fact that $(S+T-\lambda) \in Dsc(B(X))$ and $\lambda \notin \mathcal{M}$, it follows that $(S-\lambda)$ and $(T-\lambda)$ satisfy (H_1) and $(H_2^{'})$. Hence, by Theorem 1.6, we obtain $dsc(S-\lambda)$ and $dsc(T-\lambda)$ are finite, which is absurd. Thus,

$$\mathcal{M} \cup (\sigma_{dsc}(S) \cup \sigma_{dsc}(T)) \setminus \{0\} \subseteq \mathcal{M} \cup \sigma_{dsc}(S+T) \setminus \{0\}. \tag{2.12}$$

The result follows from (2.11) and (2.12). □

2.2. **Results assuring condition** (C_2)**.** We, first give the following concept of convergence of spaces.

Definition 2.5. Let $(E_n)_n$ be a sequence of normed subspaces of X.
i) We say that $(E_n)_n$ upper-converges to a vector space E, and we write $u-\lim_{n\to\infty} E_n = E$, if x is an adherent point of a sequence $(x_n)_n \subset X$ such that $x_n \in E_n$ from some rank implies that $x \in E$.
ii) We say that $(E_n)_n$ lower-converges to a vector space E, and we write $l-\lim_{n\to\infty} E_n = E$, if x belongs to E implies that x is an adherent point of a sequence $(x_n)_n \subset X$ such that $x_n \in E_n$ from some rank.

Let X be a Banach space and let Y, Z be subspaces of X and define

$$\delta(Y,Z) := \sup_{x\in Y, \|x\|\leq 1} dist(x,Z).$$

The gap $\widehat{\delta}(Y,Z)$ is defined by $\widehat{\delta}(Y,Z) = \max(\delta(Y,Z),\delta(Z,Y))$.

To prove Theorem 1.9, the main result of this subsection we need to prove next lemmas:

Lemma 2.6. *i) Let $(E_n)_n$ be a sequence of normed subspaces of a Banach space X, upper-convergent to a normed vector space E. Then $\delta(E_n, E)$ converges to 0.*
ii) Let $(E_n)_n$ be a sequence of normed subspaces of a Banach space X, lower-convergent to a normed vector space E. Then $\delta(E, E_n)$ converges to 0.
iii) Let $(E_n)_n$ be a sequence of Banach subspaces, upper-convergent to $\{0\}$. Then $E_n = \{0\}$ from some rank.
iv) Let $(E_n)_n$ be a sequence of Banach subspaces, lower-convergent to X. Then $E_n = X$ from some rank.

Proof. It is easy to prove i) and ii). Concerning iii) and iv), it suffices to use i), ii) as well as Theorem 17 page 102 in [21]. □

Lemma 2.7. *Let $(T_n)_n$ be a sequence of bounded linear operators convergent to T in the operator norm. Then, $(N(T_n))_n$ upper-converges to $N(T)$.*

Proof. Let $(x_n)_n \subset X$ such that $x_n \in N(T_n)$ from some rank. Next, we prove that for every adherent point x of $(x_n)_n$, we have $x \in N(T)$. Indeed, for every $n \in \mathbb{N}$

$$\begin{aligned} \|Tx\| &= \|Tx + Tx_n - Tx_n + T_nx_n - T_nx_n\| \\ &\leq \|T\|\|x - x_n\| + \|(T - T_n)x_n\| + \|T_nx_n\|. \end{aligned} \tag{2.13}$$

Since $x_n \in N(T_n)$ from some rank $n_0 \in \mathbb{N}$, then $\|T_nx_n\| = 0$, for all $n \geq n_0$. Let $\varepsilon > 0$, for all $N \in \mathbb{N}$ there exists $n \geq N$, such that $\|x_n - x\| < \varepsilon$. Besides, there is $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, we have $\|(T - T_n)x\| < \varepsilon$ for all $x \in X$. Thus, we obtain by (2.13) that for all $N_0 = \sup(n_0, n_1)$ there exists $n \geq N_0$ satisfying $\|Tx\| < (\|T\| + 1)\varepsilon$. Consequently, $x \in N(T)$. □

Lemma 2.8. *Let $(T_n)_n$ be a sequence of bounded linear operators convergent to T in the operator norm. Assume that T has a closed range and for all x in $N(T)$, $dist(x, N(T_n))$ is reached from some rank, then $(N(T_n))_n$ lower-converges to $N(T)$.*

Proof. By Lemma 2.6 i) and Lemma 2.7, $\delta(N(T_n), N(T))$ tends to 0 as $n \to \infty$. Using Theorem 17 page 102 in [21], we have that if T is with closed range then $\delta(N(T), N(T_n))$ tends to 0 as $n \to \infty$. This implies that

$$\sup_{x\in N(T),\ \|x\|\leq 1} \inf_{y\in N(T_n)} \|x - y\| \text{ tends to 0 as } n \to \infty,$$

which implies that

$$\forall x \in N(T), \|x\| \leq 1, \inf_{y \in N(T_n)} \|x - y\| \text{ tends to } 0 \text{ as } n \to \infty.$$

That is, for all $x \in N(T)$ and $\|x\| \leq 1$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is $y_n \in N(T_n)$ satisfying $\|x - y_n\| = \inf_{y \in N(T_n)} \|x - y\|$. □

Lemma 2.9. *Let $(T_n)_n$ be a sequence of bounded linear operators convergent to T in the operator norm. Assume that* $\limsup \gamma(T_n) > 0$. *Then, $(R(T_n))_n$ upper-converges to $R(T)$.*

Proof. See Corollary 19 page 103 in [21]. □

Lemma 2.10. *Let $(T_n)_n$ be a sequence of bounded linear operators convergent to T in the operator norm. Then, $(R(T_n))_n$ lower-converges to $R(T)$.*

Proof. Consider,

$$F = \{(y_n)_n \subset X \text{ such that } y_n \in R(T_n) \text{ from some rank}\}.$$

Assume that $y \in R(T)$. We will prove that y is a limit of a sequence $(y_n)_n$ belonging to F. In fact, $y \in R(T)$ means that there exists $x \in X$ satisfying $Tx = y$. Consider the sequence $y_n = T_n x$, from some rank $n_0 \in \mathbb{N}$. Since T_n converges to T in the operator norm, then for all $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that every $n \geq n_1$, $\|Tx - T_n x\| < \varepsilon$. That is $\|y - y_n\| < \varepsilon$. Hence, $(y_n)_n$ converges to y and $(y_n)_n \in F$. □

Proof of Theorem 1.9. 1) Let $\lambda \in \sigma_{asc}(T)$. By Proposition 1.1, this is equivalent to say that:

$$\text{For all } m \geq 0, \text{ there exists } d \geq m \text{ satisfying } R((T-\lambda)^d) \cap N(T-\lambda) \neq \{0\}. \quad (2.14)$$

Since $T_n - \lambda_n$ converges to $T - \lambda$ in the operator norm, then by Lemma 2.10 (respectively Lemma 2.8), $R(T-\lambda)^d$ (respectively $N(T-\lambda)$) is the $l-limit$ of the sequence $R(T_n-\lambda_n)^d$ (respectively $N(T_n - \lambda_n)$). Thus, (2.14) is equivalent to say that:

$$l - \lim_{n\to\infty} \left(N(T_n - \lambda_n) \cap R((T_n - \lambda_n)^d)\right) \neq \{0\}. \quad (2.15)$$

From (2.15) and Lemma 2.6 ii), for every $m \geq 0$ there exists $d \geq m$ satisfying:

$$N(T_n - \lambda_n) \cap R((T_n - \lambda_n)^d) \neq \{0\}, \text{ from some rank } n \in \mathbb{N}.$$

Therefore, according to Proposition 1.1, $\lambda_n \in \sigma_{asc}(T_n)$, from some rank.

2) Let $\lambda_n \in \sigma_{asc}(T_n)$. This means, in view of Proposition 1.1, that from some rank $n \in \mathbb{N}$,

$$\text{for all } m \geq 0, \text{ there exists } d \geq m \text{ satisfying } R((T_n-\lambda_n)^d) \cap N(T_n-\lambda_n) \neq \{0\}, \quad (2.16)$$

Since $(T_n - \lambda_n)_n$ converges to $(T - \lambda)$ in the operator norm, then by Lemma 2.7 and Lemma 2.9, $N(T_n - \lambda_n)$ upper-converges to $N(T - \lambda)$ and $R(T_n - \lambda_n)^d$ upper-converges to $R((T - \lambda)^d)$. Using Lemma 2.6 iii), (2.16) implies, For all $m \geq 0$ there exists $d \geq m$ satisfying:

$$R((T-\lambda)^d) \cap N(T-\lambda) \neq \{0\}.$$

Consequently, by Proposition 1.1, $asc(T - \lambda)$ is infinite. Hence, $\lambda \in \sigma_{asc}(T)$.

3) Let $\lambda \in \sigma_{dsc}(T)$. Using Proposition 1.1, we have:

$$\text{For all } m \geq 0, \text{ there exists } d \geq m \text{ satisfying } R(T-\lambda) + N((T-\lambda)^d) \subsetneq X. \quad (2.17)$$

Since $(T_n - \lambda_n)_n$ converges to $(T - \lambda)$, then by Lemma 2.7 (respectively, Lemma 2.9), $N(T - \lambda)$ (respectively, $R(T - \lambda)$) is the $u - limit$ of the sequence $(N(T_n - \lambda_n))_n$

(respectively, $(R(T_n-\lambda_n))_n$). Thus, (2.17) is equivalent to say that, for every $m \geq 0$ there exists $d \geq m$ satisfying:

$$u-\lim_{n\to\infty}\left(R(T_n-\lambda_n)+N((T_n-\lambda_n)^d)\right) \subsetneq X.$$

Using Lemma 2.6 i) it follows:
From some rank $n \in \mathbb{N}$, for every $m \geq 0$, there exists $d \geq m$ satisfying

$$R(T_n-\lambda_n)+N((T_n-\lambda_n)^d) \subsetneq X. \tag{2.18}$$

Hence, $\lambda_n \in \sigma_{dsc}(T_n)$, from some rank.

4) Just using [6, Proposition 1.1] and [19, Proposition 1.6]. □

3. Properties of corresponding spectra

Denote by $\mathcal{P}(X)$ the set of all projections $P \in B(X)$ such that $codimR(P)$ is finite. For $T \in B(X)$ and $P \in \mathcal{P}(X)$, the compression $T_P : R(P) \to R(P)$ is defined by $T_P y = PTy$, $y \in R(P)$, i.e. $T_P = PT_{|R(P)}$ where $T_{|R(P)} : R(P) \to X$ is the restriction of T. Clearly, $R(P)$ is a Banach space and $T_P \in B(R(P)))$.

Lemma 3.1. *Let $T \in B(X)$ and X be a direct sum of closed subspaces X_1 and X_2 which are $T-$invariant. If $T_1 = T_{|X_1} : X_1 \to X_1$ and $T_2 = T_{|X_2} : X_2 \to X_2$ then the following statements hold true:*
i) T has a finite ascent if and only if T_1 and T_2 have respectively finite ascents.
ii) T has a finite descent if and only if T_1 and T_2 have respectively finite descents.

We prove the next result analogously as in [10]:

Proposition 3.2. *For $T \in B(X)$ and $P \in \mathcal{P}(X)$, the following assertions hold:*
i) If $TP = PT$ then $T \in Asc(B(X))$ if and only if $T_P \in Asc(B(X))$.
ii) If $TP = PT$ then $T \in Dsc(B(X))$ if and only if $T_P \in Dsc(B(X))$.

Proof. i) Assume that $T \in B(X)$, $P \in \mathcal{P}(X)$ and $TP = PT$. Then, $X = R(P) \oplus N(P)$ and the subspaces $R(P)$ and $N(P)$ are invariant by $PTP \in B(X)$. The operator PTP has the following matrix form:

$$PTP = \begin{pmatrix} T_P & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(P) \\ N(P) \end{pmatrix} \to \begin{pmatrix} R(P) \\ N(P) \end{pmatrix}.$$

From Lemma 3.1 i) (respectively, ii)), it yields that PTP is with finite ascent (respectively, descent) if and only if T_P is with finite ascent (respectively, descent). Since,

$$T = PT + (I-P)T = PTP + PT(I-P) + (I-P)T,$$

and $PT(I-P)+(I-P)T$ is a finite rank operator commuting with PTP it yields by [18, Theorem 2.2] (respectively, [6, Theorem 3.1]), that PTP is with finite ascent (respectively. descent) if and only if T is with finite ascent (respectively. descent). □

We are now ready to express our first main result of this section.

Theorem 3.3. *Let S and T be two bounded linear operators satisfying (H_1) and (H_2). Assume $codimR(S^{n_0})$ and $codimR(T^{n_0})$, for some $n_0 \in \mathbb{N}^*$ are finite and let $M_{S^{n_0}}$ (respectively. $M_{T^{n_0}}$) be the subspace of X satisfying $X = R(S^{n_0}) \oplus M_{S^{n_0}}$ (respectively. $X = R(T^{n_0}) \oplus M_{T^{n_0}}$). If we suppose that $M_{S^{n_0}}$ is T-invariant and $M_{T^{n_0}}$ is S-invariant, then*

$$dsc(T) \leq n_0 \text{ and } dsc(S) \leq n_0 \text{ if and only if } dsc(TS) \leq n_0.$$

Remark 3.4. The existence of $M_{S^{n_0}}$ and $M_{T^{n_0}}$ is an immediate consequence of [21, Lemma 2, p.156] and [22, Lemma 5.3]. The [22, Lemma 5.3] proved also that $M_{S^{n_0}}$ and $M_{T^{n_0}}$ have finite dimensions.

Proof. Concerning the direct sense, it follows from Lemma 2.3.
Now, concerning the reciprocal sense, we obtain from Lemma 2.4 1) that $dsc(T) \leq n_0$ or $dsc(S) \leq n_0$. Without loss of generality, we assume that $dsc(S) \leq n_0$. Using the fact that TS has a finite descent, and $TS = ST$, we obtain that for all $n \geq n_0$,

$$(TS)^n X = (TS)^{n+1} X, \quad S^n X = S^{n+1} X = S^{n_0} X.$$

This means that for all $n \geq n_0$,

$$T^n S^{n_0} X = T^{n+1} S^{n_0} X. \tag{3.19}$$

Using Remark 3.4, $M_{S^{n_0}}$ has a finite dimension, the fact that $M_{S^{n_0}}$ is T-invariant and from [26, Proposition 1.1], we infer that $T_{|M_{S^{n_0}}} : M_{S^{n_0}} \to M_{S^{n_0}}$ has a finite descent. Let $T_{|R(S^{n_0})} : R(S^{n_0}) \to R(S^{n_0})$. In view of 3.19, we have that for all $n \geq n_0$,

$$T^n_{|R(S^{n_0})}(R(S^{n_0})) = (TS^{n_0})^n X = T^{n+1} S^{n_0} X = (TS^{n_0})^{n+1} X = T^{n+1}_{|R(S^{n_0})}(R(S^{n_0})),$$

which means that $T_{|R(S^{n_0})}$ has a finite descent. Hence, using Lemma 3.1, $dsc(T)$ is finite. □

Before given our second main result of this section, we start by recalling an interesting result that is useful in its proof.

Proposition 3.5. [6, Proposition 1.1] *Let $T \in B(X)$ be an operator with finite descent $d := d(T)$, then there exists $\delta > 0$ such that for every $0 <| \lambda |< \delta$:*

(1) $d(T - \lambda) = 0$*;*
(2) $dimN(T - \lambda) = dim(N(T) \cap R(T^d))$*.*

Proposition 3.6. [11, Proposition 2.2] *Let X be a Banach space and $T \in B(X)$. We denote by T^* the adjoint of T. If $codimR(T)$ is finite, then*

(1) $a(T^*) = d(T)$*;*
(2) $a(T) = d(T^*)$*.*

Proposition 3.7. *Let X be a Banach reflexive space. Let T be a bounded operator on X with finite ascent a. We assume that one of the following properties is satisfied:*

(1) $\dim(N(T))$ *is finite,*
(2) $co\dim(N(T^a))$ *is finite.*

Then, there exists $\delta > 0$ such that for every λ with $0 <| \lambda |< \delta$, we have $T - \lambda$ is injective.

Proof. 1) The hypothesis, $T \in B(X)$, gives that $T^* \in B(X^*)$ and $(T^*)^* = T$. Since $dimN(T) < +\infty$, then $dimR(T^*)^\perp < \infty$, in other words $codimR(T^*)$ is finite. Thanks to the Proposition 3.6 (1), and by using the fact that T has a finite ascent, we obtain that $d(T^*) = a((T^*)^*) = a(T) = a < +\infty$. Now, according to Proposition 3.5, there is $\delta > 0$ such that for all $0 <| \lambda |< \delta$, $T^* - \lambda$ is surjective. Consequently, by applying again Proposition 3.6, we deduce that $T - \overline{\lambda}$ is injective.
2) We recall that $R(T^{a*}) \subset N(T^a)^\perp$. By using [22, Lemma 5.3], this implies that $dimR(T^{a*}) < +\infty$, therefore $R(T^{a*})$ is closed and T^{a*} has a finite descent. Subsequently, we infer that T^* has also a finite descent. Applying Proposition 3.5, there is $\delta > 0$ such that $T^* - \lambda$ is surjective for all $0 < \lambda < \delta$, we deduce that $T - \overline{\lambda}$ is injective for all $0 < \lambda < \delta$. □

4. Application

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. We consider the two 2×2 block operator matrices defined on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$M = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \in B(\mathcal{H}_1 \times \mathcal{H}_2) \text{ and } M_C = \begin{pmatrix} T & C \\ 0 & S \end{pmatrix} \in B(\mathcal{H}_1 \times \mathcal{H}_2),$$

where, $T \in B(\mathcal{H}_1)$, $S \in B(\mathcal{H}_2)$ and $C \in B(\mathcal{H}_2, \mathcal{H}_1)$. Observe that

$$M_{\frac{1}{k}C} = \begin{pmatrix} I & 0 \\ 0 & kI \end{pmatrix} \begin{pmatrix} T & C \\ 0 & S \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{pmatrix}, \text{ for } k \in \mathbb{N}^*.$$

We assume that S and T have closed ranges. Since M_C and $M_{\frac{1}{k}C}$ are similar, it follows that $\sigma_i(M_{\frac{1}{k}C}) = \sigma_i(M_C)$, $i \in \{asc, dsc\}$.

Let $\mathcal{T}$ and $\mathcal{S}$ be two 2×2 block operator matrices defined by

$$\mathcal{T} = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H}_1 \times \mathcal{H}_2) \text{ and } \mathcal{S} = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \in B(\mathcal{H}_1 \times \mathcal{H}_2).$$

It is easy to verify that hypothesis of Corollary 1.7 and Lemma 2.2 are satisfied. Consequently, $\sigma_i(M)\backslash\{0\} \subset (\sigma_i(\mathcal{T}) \cup \sigma_i(\mathcal{S}))\backslash\{0\}$. Using Lemma 3.1, $\sigma_i(M) = \sigma_i(T) \cup \sigma_i(S) = \sigma_i(\mathcal{T}) \cup \sigma_i(\mathcal{S})$. By Theorem 1.9, $\sigma_{\text{desc}}(M_C) \subset \sigma_{\text{desc}}(T) \cup \sigma_{\text{desc}}(S)$ and by previous proposition and [21, Proposition 1.6] we obtain $\sigma_{\text{asc}}(M_C) \subset \sigma_{\text{asc}}(T) \cup \sigma_{\text{asc}}(S)$.

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