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ON THE SECOND LARGEST MULTIPLICITY OF EIGENVALUES FOR THE STIELTJES STRING SPECTRAL PROBLEM ON TREES

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Abstract. The largest possible multiplicity of an eigenvalue of a spectral problem generated by the Stieltjes string equations on a metric tree is $p_{pen} - 1$, where p_{pen} is the number of pendant vertices. We propose how to find the second largest possible multiplicity of an eigenvalue of such a problem. This multiplicity depends on the numbers of point masses on the edges of the trees.

Максимально можлива кратність власного значення спектральної задачі, породженої рівняннями струни Стілтьєса на метричному дереві, дорівнює $p_{pen} -$ 1, де p_{pen} — кількість висячих вершин. Ми пропонуємо, як знайти другу за величиною кратність власного значення такої задачі. Ця кратність залежить від кількості точкових мас на ребрах дерев.

1. Introduction

Second order difference equations appear in different fields of physics (synthesis of electrical circuits [6, p.129], transverse vibrations of the so-called Stieltjes strings (massless elastic threads bearing point masses) [16, 3], and longitudinal vibrations of point masses connected by springs [18].

It is known, see [3], that the eigenvalues of the Dirichlet problem (the spectral problem with the Dirichlet boundary conditions at both ends) generated by the Stieltjes string equations on an interval are simple and for any sequence of distinct positive numbers $\{z_k\}_{k=1}^n$ there exists a (not unique) pair of sequences $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$ of positive numbers such that $\{z_k\}_{k=1}^n$ is the spectrum of the corresponding Dirichlet problem while m_k-s are the values of the point masses and l_k-s are the lengths of the subintervals into which the string is divided by the masses. Also it is known from [3] that the data necessary and sufficient to solve the inverse problem of recovering the sequences $\{m_k\}_{k=1}^n$, $\{l_k\}_{k=0}^n$ consist of two spectra: the spectrum of the Dirichlet spectral problem, and the spectrum of the Dirichlet-Neumann spectral problem (the problem with the Dirichlet boundary condition at one end and the Neumann boundary condition at the other end) and the total length of the string.

Natural generalizations of such problems are the problems generated by Stieltjes string equations on metric trees [4, 5, 7]. For applications, see [8]. In the case of a graph domain, the problem can have multiple eigenvalues. Since we consider selfadjoint problems there is no ambiguity between algebraic and geometric multiplicity.

For trees, the maximal multiplicity equals $p_{pen}-1$ where p_{pen} is the number of pendant vertices in the tree. This result, well known in quantum graph theory [17, 15], was proved for the finite dimensional case in [1]. It should also be mentioned that related results for the so-called tree-patterned matrices were obtained in [9] and [11]

Unfortunately, any general answer about restrictions on eigenvalue multiplicities for a spectral problem on an arbitrary tree as well as for an arbitrary tree patterned matrices is not known in spite of many particular results for tree-patterned matrices in [10, 11, 13].

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Such restrictions are known for tree-patterned matrices whose graphs are generalized star and generalized double star [11] and for spectral problems generated by the Stieltjes string equations on a star graphs [21, 22] and on the so-called snowflake graphs in [23].

In this paper we give an answer to the following question: if the spectral problem generated by the Stieltjes string equations on a tree has an eigenvalue of maximal multiplicity $p_{pen}-1$, what is the second largest possible multiplicity of an eigenvalue of this problem?

In Section 2 we give the corresponding definitions and describe the spectral problem generated by the Stieltjes string recurrence relations on a connected graph. In Section 3 we recall the result on maximal possible multiplicity of an eigenvalue for spectral problem on a tree and prove an auxiliary theorem. In Section 4 we present an algorithm for finding the second largest possible multiplicity and consider examples.

2. Formulation of the problems

For a tree T we denote its vertices by v_i, $i=1,2,...,p$, where p is the number of the vertices of T, its edges by e_j, $j=1,2,...,g$, where g is the number of edges of T. For each i denote by $d(v_i)$ the degree of the vertex v_i and for each j we denote by l_j the length of the edge e_j and by n_j the number of point masses on this edge.

We choose a pendant vertex $\mathbf{v}$ as the root and direct the edges of T away from the root to obtain an oriented graph. Then in addition to the degree $d(v_i)$ of a vertex v_i we introduce $d^+(v_i)$, the *indegree*, the number of incident edges directed towards the vertex and $d^-(v_i)$, the *outdegree*, the number of incident edges directed away from the vertex v_i. For each vertex except of the root $d^+(v_i)=1$ and for the root $d^+(\mathbf{v})=0$). For each pendant vertex except of the root $d^-(v_j)=0$.

The local coordinate identifies a directed edge e_j $(j=1,2,...,g)$ of T with the interval $[0,l_j]$ and the coordinate x_j increases in the direction of the edge.

Each edge e_j is divided into n_j+1 subintervals of the lengths $l_0^{(j)}$, $l_1^{(j)}$,..., $l_{n_j}^{(j)}$ by point masses $m_1^{(j)}$, $m_2^{(j)}$, ..., $m_{n_j}^{(j)}$ $(l_k^{(j)}>0,\ m_k^{(j)}>0,\ l_j=\sum\limits_{k=0}^{n_j} l_k^{(j)})$. An interior vertex v_i has outgoing edges e_j starting with subintervals of lengths $l_0^{(j)}$, while the incoming edge e_r ends at v_i with an interval of length $l_{n_r}^{(r)}$. It is assumed that the graph is stretched and the pendant vertices are fixed. The graph can vibrate in the direction orthogonal to the plane of equilibrium position of the tree. We denote by $v_k^{(j)}(t)$ the transverse displacement of the mass $m_k^{(j)}$. If an edge e_j is incoming to an interior vertex v_i then the displacement of the incoming end of the edge is denoted by $v_{n_j+1}^{(j)}(t)$, while if an edge e_r is outgoing from a vertex v_i then the displacement of the outgoing end of the edge is denoted $v_0^{(r)}(t)$. Using such notation, transverse vibrations of the graph can be described by the system of equations

$$\frac{v_k^{(j)}(t)-v_{k+1}^{(j)}(t)}{l_k^{(j)}}+\frac{v_k^{(j)}(t)-v_{k-1}^{(j)}(t)}{l_{k-1}^{(j)}}+m_k^{(j)}\frac{\partial^2 v_k^{(j)}}{\partial t^2}(t)=0 \tag{2.1}$$

$$(k=1,2,\ldots,n_j,\ n_j\geq 1,\ j=1,2,...,g).$$

Let W_i^- be the set of indices of the edge outgoing away from the vertex v_i. For pendant vertices except the root $W_i^-=\emptyset$.

For each interior vertex with the incoming edge e_j and outgoing edges e_r $(r\in W_i^-)$ we impose the continuity conditions

$$v_0^{(r)}(t)=v_{n_j+1}^{(j)}(t). \tag{2.2}$$

The balance of forces at such a vertex implies

$$\sum_{r\in W_i^-}\frac{v_1^{(r)}(t)-v_0^{(r)}(t)}{l_0^{(r)}}=\frac{v_{n_j+1}^{(j)}(t)-v_{n_j}^{(j)}(t)}{l_{n_j}^{(j)}}. \tag{2.3}$$

For an edge e_j incident with a pendant vertex except of the root we impose the Dirichlet boundary condition

$$v_{n_j+1}^{(j)}(t)=0. \tag{2.4}$$

At the root we impose the Dirichlet condition for the problem which we call Dirichlet problem on the tree T

$$v_0^{(j)}(t)=0. \tag{2.5}$$

and

$$v_1^{(j)}(t)=v_0^{(j)}(t). \tag{2.6}$$

for the problem which we call Neumann problem on the tree T.

Using the ansatz $v_k^{(j)}(t)=e^{i\lambda t}u_k^{(j)}$, $z=\lambda^2$ we obtain from (2.1)–(2.6):

$$\frac{u_k^{(j)}-u_{k+1}^{(j)}}{l_k^{(j)}}+\frac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}-m_k^{(j)}zu_k^{(j)}=0 \tag{2.7}$$

$$(k=1,2,\dots,n_j,\ \ j=1,2,...,g).$$

For each interior vertex with incoming edge e_j and outgoing edges e_r $(r\in W_i^-)$ we have

$$u_0^{(r)}=u_{n_j+1}^{(j)}. \tag{2.8}$$

$$\sum_{r\in W_i^-}\frac{u_1^{(r)}-u_0^{(r)}}{l_0^{(r)}}=\frac{u_{n_j+1}^{(j)}-u_{n_j}^{(j)}}{l_{n_j}^{(j)}}. \tag{2.9}$$

For each edge e_j incident with a pendant vertex except of the root we have

$$u_{n_j+1}^{(j)}=0. \tag{2.10}$$

At the root we obtain

$$u_0^{(j)}=0. \tag{2.11}$$

for the Dirichlet problem and

$$u_1^{(j)}=u_0^{(j)} \tag{2.12}$$

for the Neumann problem on the tree T.

Assumption 2.1. Since presence of vertices of degree two does not change the multiplicities of eigenvalues we assume in the sequel that each interior vertex is of degree higher than two.

The fundamental system of two linearly independent solutions to (2.7) can be composed by the polynomials $R_{2k-2}^{(j)}(z,0)$ and $R_{2k-2}^{(j)}(z,1)$ which satisfy (see e.g. [3], Addition II) the initial conditions $R_0^{(j)}(z,0)=1$, $R_{-1}^{(j)}(z,0)=\frac{1}{l_0^{(j)}}$, $R_0^{(j)}(z,1)=1$, $R_{-1}^{(j)}(z,1)=0$ and the recurrence relations

$$\begin{aligned}
R_{2k-1}^{(j)}(z,0)&=-zm_k^{(j)}R_{2k-2}^{(j)}(z,0)+R_{2k-3}^{(j)}(z,0),\\
R_{2k-1}^{(j)}(z,1)&=-zm_k^{(j)}R_{2k-2}^{(j)}(z,1)+R_{2k-3}^{(j)}(z,1),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
R_{2k}^{(j)}(z,0)&=l_k^{(j)}R_{2k-1}^{(j)}(z,0)+R_{2k-2}^{(j)}(z,0)\quad(k=1,2,\dots,n_j),\\
R_{2k}^{(j)}(z,1)&=l_k^{(j)}R_{2k-1}^{(j)}(z,1)+R_{2k-2}^{(j)}(z,1)\quad(k=1,2,\dots,n_j),
\end{aligned} \tag{2.14}$$

The general solution to (2.7) can be given in the form

$$u_k^{(j)} = R_{2k-2}^{(j)}(z,0)q_1^{(j)} + R_{2k-2}^{(j)}(z,1)h_1^{(j)}$$

on the edge e_j with constants $q_1^{(j)}$ and $h_1^{(j)}$.

With this notations we obtain using (2.8)–(2.12):

$$h_1^{(r)} = R_{2n_j}^{(j)}(z,0)q_1^{(j)} + R_{2n_j}^{(j)}(z,1)h_1^{(j)}, \tag{2.15}$$

$$\sum_{j\in W_r^-} \frac{q_1^{(r)}}{l_0^{(r)}} = R_{2n_j-1}^{(j)}(z,0)q_1^{(j)} + R_{2n_j-1}^{(j)}(z,1)h_1^{(j)}. \tag{2.16}$$

for each interior vertex with incoming edges e_j and outgoing edge e_r: and

$$R_{2n_j}^{(j)}(z,0)q_1^{(j)} + R_{2n_j}^{(j)}(z,1)h_1^{(j)} = 0 \tag{2.17}$$

for each edge e_j incident with a pendant vertex.

At the root we have

$$h_1^{(j)} = 0. \tag{2.18}$$

for the Dirichlet problem and

$$q_1^{(j)} = 0. \tag{2.19}$$

for the Neumann problem.

Then the *characteristic polynomial* of problem (2.7)–(2.11), i.e. a polynomial whose set of zeros coincides with the spectrum of the problem can be expressed by $l_0^{(j)}R_{2n_j}^{(j)}(z,0)$, $l_0^{(j)}R_{2n_j-1}^{(j)}(z,0)$, $R_{2n_j}^{(j)}(z,1)$ and $R_{2n_j-1}^{(j)}(z,1)$. To do it we introduce the following system of vectors

$$\psi_j(z) = \mathrm{col}\{\underbrace{0,0,...0,l_0^{(j)}R_{-2}^{(j)}(z,0),l_0^{(j)}R_0^{(j)}(z,0),...,l_0^{(j)}R_{2n_j}^{(j)}(z,0),0,0,...,0}_{n+2g},\underbrace{0,0,...,0}_{n+2g}\},$$

$$\psi_{j+g}(z) = \mathrm{col}\{\underbrace{0,0,...,0}_{n+2g},\underbrace{0,0,...,0,R_{-2}^{(j)}(z,1),R_0^{(j)}(z,1),...,R_{2n_j}^{(j)}(z,1),0,0,...,0}_{n+2g}\}$$

for $j=1,2,...,g$, where g is the number of edges in G, $n=\sum_{j=1}^{g} n_j$. We denote by L_j $(j=1,2,...,2g)$ the linear functionals $C^{2n+4g}\to C$ generated by (2.7)–(2.11). Then $\Phi(z)=\{L_j(\psi_p(z)\}_{j,p}^{2g}$ is the characteristic matrix which represents the system of linear equations describing the boundary conditions at pendant vertices and continuity and balance of forces conditions for the interior vertices. We call

$$\phi_D(z) := \det(\Phi(z))$$

the *characteristic polynomial* of problem (2.7)–(2.11). In the same way we construct $\phi_N(z)$.

Let T be the above described tree with $n_j\ge 1$ edges masses on the edge e_1 $(j=1,2,...,g)$. Changing the masses $m_k^{(j)}$ $(k=1,2,...,n_j, j=1,2,...,g)$ and the intervals $l_k^{(j)}$ $(k=0,1,...,n_j, j=1,2,...,g)$ we change the ϕ_D and ϕ_N and therefore the sets of their zeros, i.e. the spectra of the corresponding operators $\mathcal{L}(D)$ and $\mathcal{L}(N)$ and the multiplicities of their eigenvalues too.

We denote the set of all obtained operators by $\mathcal{L}_T(D)$ and $\mathcal{L}_T(N)$. In the next section we describe the maximal possible value of an eigenvalue multiplicity of the operators $\mathcal{L}(D)\in\mathcal{L}_T(D)$ and $\mathcal{L}(N)\in\mathcal{L}_T(N)$ for a graph T of a tree of a given form and given numbers n_j.

3. Maximal multiplicity and auxiliary results

The maximum of eigenvalue multiplicity of the operator $\mathcal{L}(D)$ for a tree is given by the following theorem (see [1], Theorem 4.3).

Theorem 3.1. *Let $n_j \geq 1$ for all j. Then the maximal multiplicity of an eigenvalue of the operator $\mathcal{L}(D)$ defined on a tree with $g \geq 1$ is $\omega := p_{pen} - 1$, where p_{pen} is the number of pendant vertices. Equivalently, $\omega = g - p_{in}$ where p_{in} is the number of interior vertices.*

Definition 3.2. (see, e.g. [19], Definitions 5.1.20, 5.1.24). A function $f : z \mapsto f(z)$ of a complex variable z (or simply $f(z)$ by abuse of notation) is called *Nevanlinna function* (*R-function* in terms of [14]) if

1) f is analytic for z in the half-planes $\mathrm{Im} z > 0$ and $\mathrm{Im} z < 0$,
2) $f(\overline{z}) = \overline{f(z)}$ for $\mathrm{Im} z \neq 0$,
3) $\mathrm{Im} z \cdot \mathrm{Im} f(z) \geq 0$ for $\mathrm{Im} z \neq 0$,

and it is called an *S-function* if, in addition,

4) f is analytic for $z \notin [0, \infty)$,
5) $f(z) > 0$ for $z \in (-\infty, 0)$;

an S-function $f(z)$ is called an S_0*-function* if

6) 0 is not a pole of f.

Theorem 3.3. *Let T be a tree of g edges with the numbers of masses on the edges $n_j \geq 1$ for all $j = 1, 2, ..., g$, $n = \sum_{j=1}^{g} n_j$ and let $0 < \nu_0 < \tilde{\nu}$. Then there exists a collection $\{\{m_k^{(j)}\}_{k=1}^{n_j}, \{l_k^{(j)}\}_{k=0}^{n_j}, j = 1.2., , , .g\}$ such that ν_0 is the lowest (simple) eigenvalue of problem (2.7)–(2.11) and $\tilde{\nu}$ is an eigenvalue of multiplicity $p_{pen} - 1$.*

Proof. It is known (see [24]) that the number of distinct eigenvalues of problem (2.7)–(2.11) is not less than the maximal length n_0 of the paths in the tree measured in numbers of masses on the edges of the path.

Choose any numbers $\{\nu_k\}_{k=1}^{n_0-2}$ and $\{\mu_k\}_{k=0}^{n_0}$ such that

$$0 < \mu_0 < \nu_0 < \mu_1 < \nu_1 < ... < \nu_{n_0-2} < \mu_{n_0-1} < \tilde{\nu}. \tag{3.1}$$

Then

$$F(z) = l \frac{\prod_{s=0}^{n_0-2} \left(1 - \frac{z}{\nu_s}\right) \left(1 - \frac{z}{\tilde{\nu}}\right)}{\prod_{s=0}^{n_0-1} \left(1 - \frac{z}{\mu_s}\right)}$$

where arbitrary $l > 0$ is an S_0-function (see Lemma 2.2 in [21]). Being such it can be expanded in the continued fraction:

$$F(z) = a_0^{(1)} + \cfrac{1}{-b_1^{(1)} z + \cfrac{1}{a_1^{(1)} + \cfrac{1}{-b_2^{(1)} z + \ldots + \cfrac{1}{a_{n_0-1}^{(1)} + \cfrac{1}{-b_{n_0}^{(1)} z + \cfrac{1}{a_{n_0}^{(1)}}}}}}}, \tag{3.2}$$

where $a_s^{(1)} > 0$ $(s = 0, 1, ..., n_0)$, $b_s^{(1)} > 0$ $(s = 1, 2, ..., n_0)$.

Due to (3.1) the sequence $\{\nu_k\}_{k=0}^{n_0-1} \cup \{\tilde{\nu}\}$ is the spectrum of the Dirichlet-Dirichlet problem (see, e.g. [3])

$$\frac{u_k^{(1)} - u_{k+1}^{(1)}}{a_k^{(1)}} + \frac{u_k^{(1)} - u_{k-1}^{(1)}}{a_{k-1}^{(1j)}} - b_k^{(1)} z u_k^{(1)} = 0 \tag{3.3}$$

$$(k = 1, 2, \ldots, n_{n_0})$$

$$u_0^{(1)} = u_{n_0+1}^{(1)} = 0 \tag{3.4}$$

on an interval of the length l and $\{\mu_k\}_{k=0}^{n_0}$ is the spectrum of the corresponding Neumann-Dirichlet problem which consists of equation (3.2) and the boundary conditions

$$u_0^{(1)} - u_1^{(1)} = u_{n_0+1}^{(1)} = 0. \tag{3.5}$$

We choose the pendant vertex $\mathbf{v}$ which is the initial vertex of the path of maximal length as the root of the tree T and denote by e_1 the edge incident with the root. The vertex v_1 at the other end of e_1 has the outdegree $d^-(v_1)$. Denote by e_2, e_3, ..., $e_{d^-(v_1)+1}$ the edges outgoing from v_1 and by T_j $(j = 2, 3, ..., d^-(v_1) + 1)$ the subtrees rooted at v_1. Denote by N_j the number of the masses in T_j $(j = 2, 3...., d^-(v_1) + 1)$.

Let n_1 be the number of masses on e_1. Since each edge bears at least one mass we have $n_1 < n_0$. Then we identify the coefficients $\{b_k^{(1)}\}_{k=1}^{n_1}$ with the masses and $\{a_k^{(1)}\}_{k=0}^{n_1-1} \cup \{\tilde{a}_{n_1}^{(1)}\}$ with the subintervals on e_1. Since $\tilde{\nu}$ is the largest eigenvalue of Dirichlet-Dirichlet problem (3.3)–(3.4) the corresponding eigenvector has exactly one zero between each two masses (see [2]). Therefore, a value $\tilde{a}_{n_1}^{(1)} \in (0, a_{n_1}^{(1)})$ can be chosen such that

$$a_0^{(1)} + \cfrac{1}{-b_1^{(1)}\tilde{\nu} + \cfrac{1}{a_1^{(1)} + \cfrac{1}{-b_2^{(1)}\tilde{\nu}..+\cfrac{1}{a_{n_1-1}^{(1)} + \cfrac{1}{-b_{n_1}^{(1)}\tilde{\nu} + \cfrac{1}{\tilde{a}_{n_1}^{(1)}}}}}}} = 0 \tag{3.6}$$

and $f_1(\tilde{\nu}) = 0$ where

$$f_1(z) = a_{n_1}^{(1)} - \tilde{a}_{n_1}^{(1)} + \cfrac{1}{-b_{n_1+1}^{(1)}z + \cfrac{1}{a_{(n_1+1)}^{(1)} + \cfrac{1}{-b_{n_1+1}^{(1)}z+\ldots+\cfrac{1}{a_{n_0-1}^{(1)} + \cfrac{1}{-b_{n_0}^{(1)}z + \cfrac{1}{a_{n_0}^{(1)}}}}}}}. \tag{3.7}$$

Since $f_1(\tilde{\nu}) = 0$ and $f_1(z)$ is an S_0-function, it can be presented in the form

$$f_1(z) = \left(B + \frac{A}{z - \tilde{\nu}} + \sum_{s=1}^{n_0-n_1-1} \frac{A_s}{z - \tau_s} \right)^{-1} \tag{3.8}$$

where $0 < \tau_1 < \tau_2 < ... < \tau_{n_0-n_1-1} < \tilde{\nu}$ and $B > 0$, $A > 0$, $A_s > 0$ $(s = 1, 2, ..., n_0-n_1-1)$. The inequality $\tau_k < \tilde{\nu}$ is a consequence of the fact that the eigenvector of the Dirichlet-Dirichlet problem (3.3)–(3.4) corresponding to the (largest) eigenvalue $\tilde{\nu}$ has $n_0 - 1$ nodes (points of zero amplitude of vibration) and therefore the projection of this vector onto the interval $(\tilde{l}, l)$ having $n_0 - n_1 - 1$ nodes is the eigenvector of problem

$$\frac{u_k^{(1)} - u_{k+1}^{(1)}}{a_k^{(1)}} + \frac{u_k^{(1)} - u_{k-1}^{(1)}}{a_{k-1}^{(1j)}} - b_k^{(1)} z u_k^{(1)} = 0 \tag{3.9}$$

$$(k = n_1 + 1, \ldots, n_{n_0})$$

$$\tilde{u_0}^{(1)} = u_{n_0+1}^{(1)} = 0 \tag{3.10}$$

corresponding to the eigenvalue $\tilde{\nu}$. Here $\tilde{u}_0$ is the amplitude of the point which lies at the distance $\tilde{l} = \sum\limits_{k=0}^{n_1-1} a_k^{(1)} + \tilde{a}_{n_1}^{(1)}$ from the left end of the interval $(0, l)$.

Let us consider the subtrees T_j $(j = 2, 3, ..., d^-(v_1) + 1)$ rooted at v_1. Denote by N_j the number of masses, by $n_0^{(j)}$ the maximal length of the paths in T_j starting from v_1 measured in number of masses $(j = 2, 3...., d^-(v_1) + 1)$. Then due to (3.8) we can present $(f_1(z))^{-1}$ in the form

$$(f_1(z))^{-1} = \sum_{j=2}^{d^-(v_2)+1} F_j(z)$$

where

$$F_j(z) = B_j + \frac{A^{(j)}}{z-\tilde{\nu}} + \sum_{r=1}^{n_0^{(j)}} \frac{A_r^{(j)}}{z-\tau_r},$$

with $B_j > 0$. $\sum\limits_{j=2}^{d^-(v_1)+1} B_j = B$, $A^{(j)} > 0$, $\sum\limits_{j=2}^{d^-(v_1)+1} A^{(j)} = A$, $\sum\limits_{j=2}^{d^-(v_1)+1} A_r^{(j)} = A_r$ and $A_r^{(j)} = 0$ for $r > n_0^{(j)}$, $A_r^{(j)} > 0$ for $r \le n_0^{(j)}$.

Thus, $(F_j(z))^{-1}$ is an S_0-function with $n_0^{(j)}$ zeros and the same number of poles.

We apply this procedure to each of those of the subtrees T_j which are not just an edge. If a subtree T_j is a star then using once more the above procedure we finish with this subtree. If a subtree T_j is more complicated than a star graph we continue this procedure more times. □

4. Second largest multiplicity

Suppose the maximal multiplicity $\omega = p_{pen} - 1$ is possessed by an eigenvalue of problem (2.7)–(2.11) on a tree. In this section we show how to find the second largest possible multiplicity by a certain algorithm.

Definition 4.1. If an edge is incident with a pendant vertex which is not the root then we call it a *leaf*. If all but one edges incident with the vertex v are leaves then we call v a *distance-one* vertex.

Definition 4.2. A star subgraph S of a tree T centered at a distance-one vertex of T is said to be a *peripheral star graph* if all but one pendant vertices of S are pendant vertices of T.

Definition 4.3. We call an edge heavy (light) if it bears at least one mass (a massless edge).

Definition 4.4. A peripheral star subgraph is said to be light if it has at most one heavy leaf.

Let $n_j \ge 1$ for all j and let there be an eigenvalue of multiplicity ω. Then we can calculate the second largest possible multiplicity via the following algorithm.

Algorithm

Step 1. Consider the tree $T_1^{(1)}$ obtained from T by deleting one mass from each edge. If, after this, there has not appeared any light edge then the second largest multiplicity is again $\omega = p_{pen} - 1$.

If after deleting one mass from each edge there appeared light edges in $T_1^{(1)}$ then we proceed to

Step 2. Let v_i be a distance-one vertex of the tree $T_1^{(1)}$. If there are $p \ge 2$ heavy edges among the leaves of $T_1^{(1)}$ incident with v_i then we

(i) attribute a summand $p-1$ to the second greatest multiplicity which we denote by ω_2,

(ii) call the vertex v_i a separating vertex,

(iii) delete all leaves incident with v_i.

Having done it for all the distance-one vertices we obtain a new tree $T_2^{(1)}$ (see Fig. 1).

If $T_2^{(1)}$ has at least one distance-one vertex with at least two heavy leaves then we repeat step 2 and obtain a new tree $T_3^{(1)}$. We repeat step 2 until obtain a tree $T_{r_1}^{(1)}$ each distance-one vertex of which has less than two heavy leaves. If $T_{r_1}^{(1)}$ is a path then we add a summand 1 (or 0) to the second greatest multiplicity if there is a mass (no masses) on the path and finish. If $T_{r_1}^{(1)}$ is not a path then we proceed to

Step 3. We change one by one peripheral (light) star subgraph each for one edge bearing a mass and obtain a new tree $\tilde{T}_{r_1}^{(1)}$ (see Fig. 2).

Then we repeat the steps 2 and 3 for the tree $\tilde{T}_{r_1}^{(1)} := T_1^{(2)}$ and obtain $\tilde{T}_{r_2}^{(2)} := T_1^{(3)}$. We continue this procedure until come to 1) an isolated vertex or to 2) a graph P_s (paths with s edges with some $s \in \mathbb{N}$) , or 3) to a star graph. If it is P_s with at least one mass we add 1 to the second largest multiplicity. If it is P_s with no masses or just one vertex then we have finished. If it is a star graph then we add the maximum multiplicity of this star graph (which is less by one than the number of heavy edges of the star graph) to the second largest multiplicity.

Theorem 4.5. *Let a tree of Stieltjes strings be given and let*

(i) $n_j \geq 1$ for all j $(j = 1, 2, ..., g)$.

(ii) problem (2.7)–(2.11) have an eigenvalue of multiplicity $\omega = p_{pen} - 1$,

Then the second largest possible multiplicity ω_2 of an eigenvalue of (2.7)–(2.11) can be found using the above Algorithm.

Proof. First, let us show that this value can be reached.

If $n_j \geq 2$ for all j then we can choose $\nu_1 > 0$ and $\nu_2 > \nu_1$ and construct the edges of our tree such that ν_1 and ν_2 are eigenvalues of the Dirichlet-Dirichlet problem on each edge. Then the multiplicity of ν_1 and the multiplicity of ν_2 as eigenvalues of problem (2.7)–(2.11) is equal to ω (see [1] how to construct the corresponding eigenvectors).

Now let $n_j = 1$ for some values of j. We will prove existence of a tree of given form with given numbers of masses such that ν_1 is the eigenvalue of problem (2.7)–(2.11) of multiplicity ω and $\nu_2 \neq \nu_1$ is an eigenvalue of multiplicity ω_2 obtained by the Algorithm.

According to Theorem 3.3 we can construct a tree T such that 1) ν_1 is an eigenvalue of the Dirichlet-Dirichlet problem on each edge of T (and therefore ν_1 is an eigenvalue of multiplicity ω of problem (2.7)–(2.11) and an eigenvalue of multiplicity $p_{pen}^{(j)} - 1$ of problem (2.7)–(2.11) on each subtree T_j), 2) $\nu_2 \neq \nu_1$ is an eigenvalue of the Dirichlet-Dirichlet problem on each heavy leave and an eigenvalue of problem (2.7)–(2.11) on each of the subtrees (star subgraphs) which appear in Algorithm.

Now let us show that if an eigenvalue ν_1 is of the maximal multiplicity ω then the second largest multiplicity of an eigenvalue cannot be larger than the multiplicity ω_2 calculated by Algorithm.

The eigenvectors corresponding to the eigenvalue ν_1 of problem (2.7)–(2.11) on T of maximal multiplicity ω correspond to linearly independent paths in the tree. Each of eigenvectors have nodes (for the case of a tree we also call node a point of zero amplitude if there are points of nonzero amplitude in an arbitrary small neighborhood of this point) at all the interior vertices of the path (see [1]).

Each eigenvector corresponding to ν_2, the eigenvalue of the second largest multiplicity, has nodes at the separating vertices as it is described in Algorithm. To try to raise the second largest multiplicity we can suppose that an eigenvector has a zero at a vertex w which is not separating. If a star graph has masses only on one of its edges then its eigenvalues are all simple, moreover the central vertex is never a node. Thus, the multiplicity calculated by Algorithm is the second maximal. $\square$

Example 1. Let us consider the graph of Fig 1. The maximal possible multiplicity of the spectral problem correspondent to the tree T is $p_{pen} - 1 = 12$. Let this value be achieved by an eigenvalue. Now we use Algorithm to find the second largest possible multiplicity. Coming from $T_1^{(1)}$ to $T_2^{(1)}$ we assign a summand $+2$, coming from $T_2^{(1)}$ to $T_3^{(1)}$ we add $+1$, coming from $T_1^{(2)}$ to $T_2^{(2)}$ we add $+2$, coming from $T_1^{(3)}$ to $T_2^{(3)}$ we add $+1$. Thus all in all the second largest multiplicity is 6.

Example 2. Let us consider the graph of Fig 2. The maximal possible multiplicity is $p_{pen} - 1 = 7 - 1 = 6$. Let this value be achieved by an eigenvalue. Then applying

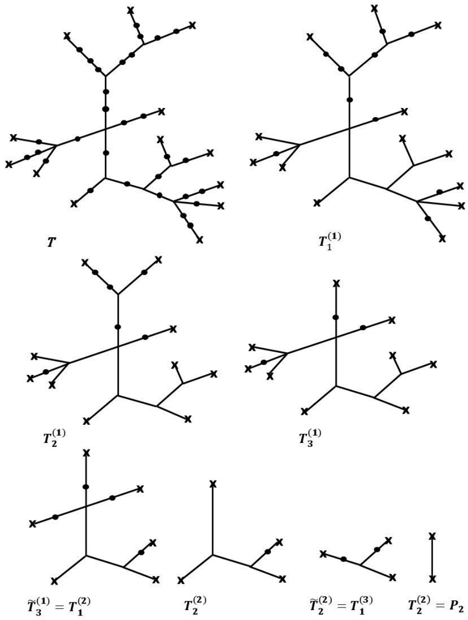

Fig.1

Algorithm we obtain $+4$ coming from $T_1^{(1)}$ to $T_2^{(1)}$ and $+1$ due to presence of a mass on $T_1^{(2)} = P_2$. Thus all in all the second largest possible multiplicity is 5.

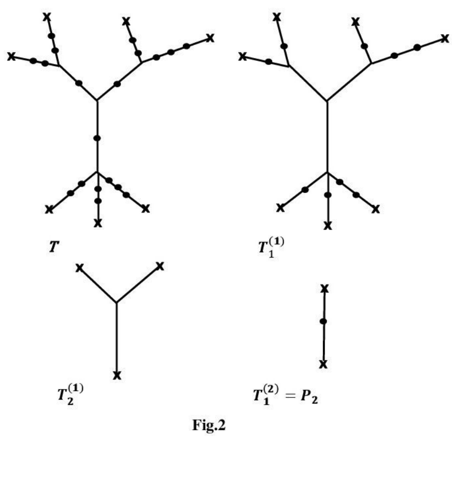

Fig.2

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