

GREEN MEASURES FOR TIME CHANGED MARKOV PROCESSES

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ABSTRACT. In this paper we study Green measures for certain classes of random time change Markov processes where the random time change are inverse subordinators. We show the existence of the Green measure for these processes under the condition of the existence of the Green measure of the original Markov processes and they coincide. Applications to fractional dynamics is given.

У цій роботі досліджуються міри Гріна для деяких класів марківських процесів з випадковою заміною часу, де випадкова заміна часу є оберненим субординатором. Показано існування міри Гріна для цих процесів за умови існування міри Гріна вихідних марківських процесів і що вони збігаються. Також даються застосування отриманих результатів до динаміки процесів із дробовими похідними.

1. INTRODUCTION

One of the most important questions in the theory of random processes is related to the study of their asymptotic behavior. There are several possibilities to formulate such questions. For example, let $X(t)$, $t \geq 0$, be a random process in \mathbb{R}^d such that $X(0) = x \in \mathbb{R}^d$. Denote by μ_t^x the one-dimensional distribution of the process at time t . Then the natural question is the limiting behavior of μ_t^x for $t \rightarrow \infty$. Of course, we can expect a positive answer to this question only for certain particular classes of these processes. In the case of Markov processes, an essential technique to study the time asymptotic is related to the Fokker-Planck equation

$$\frac{\partial}{\partial t} \mu_t^x = L^* \mu_t^x,$$

where L is the generator of the Markov process. In that case, μ_t^x is nothing but the transition probability measure $P_t(x, dy)$ or the heat kernel for L which may be analyzed for certain particular cases, see e.g., [12, 10] and numerous references therein. On the other hand, even for very simple classes of processes the time-space behavior of $P_t(x, dy)$ may be very complicated, see [11] for an analysis of continuous time random walks (compound Poisson processes) in \mathbb{R}^d .

An alternative way is to consider averaged characteristics of Markov processes. In particular, we introduce the Green measure

$$\mathcal{G}(x, dy) := \int_0^\infty P_t(x, dy) dt.$$

The notion of the Green measure is closely related to the concept of potential in stochastic analysis, see [21] for details. In the latter paper we have shown the existence of Green measures for certain classes of Markov processes and analyzed their properties.

In this paper we are interested in transformations of Markov processes by means of independent random time changes. The resulting process is again a Markov process. In particular, as random time change we consider inverse of subordinators. In the literature most of the results in this direction are related to inverse stable subordinators, see e.g., [23, 3, 24, 2, 22]. In [15] the authors study the spectral heat contents for time changed

Brownian motions where the time change is given either by a subordinator or an inverse subordinator with the underlying Laplace exponent being regularly varying at infinity with index $\beta \in (0, 1)$. But it is also possible to consider more general inverse subordinators and study such random processes and related properties, see e.g., [6, 17, 16].

Let X be a Markov process which admits a Green measure $\mathcal{G}(x, dy)$ and Y its random time change by an inverse subordinator. Our aim is to study the asymptotic behavior of the process Y for different classes of inverse subordinators. The first main point concerns the existence of Green measure for these time changed processes, that is, applying the above definition of a Green measure leads to divergent integrals in all interesting cases. To overcome this difficulty we introduce the concept of renormalized Green measure

$$\mathcal{G}_r(x, dy) := \lim_{T \rightarrow \infty} \frac{1}{N(T)} \int_0^T \nu_t^x(dy) dt,$$

where ν_t^x is the marginal distribution of $Y(t)$. The renormalization $N(T)$ is uniquely defined by the inverse subordinator under consideration, see (2.16) and (4.21) below. Such kind of normalizations are well known in the theory of additive functionals for random processes. This enables us to state the main contribution of this paper as follows: If the initial Markov process has a Green measure, then the time changed process will have a renormalized Green measure which coincides with the Green measure for the Markov process, see Theorem 6 below. An interpretation of this result is very easy. In the time changed process the evolution is delayed by the random environment and hence as a result is slower. That means a slower decay in t leads to a divergent integral in the definition of the Green measure.

The paper is organized as follows. In Section 2 we describe the class of subordinators we are interested in as well as the corresponding inverse subordinators. The main assumption of these classes is given in terms of the corresponding Laplace exponent, see assumption (H) below. In addition, we recall a result on the asymptotic relating the density of the inverse subordinator and admissible kernels satisfying (H), see Theorem 4. We provide many examples which fulfill the assumptions in Example 2.1. Sections 3 and 4 we introduce the main object needed and show the main result of the paper, see Theorem 6. Finally, in Section 5 we make an application to fractional dynamics for the special class of Markov processes known as compound Poisson processes. More precisely, if $u(t, x)$ denotes the solution of the Kolmogorov equation and $v(t, x)$ is the solution of the associated fractional evolution equation, then the following average result holds, see Theorem 9

$$\frac{1}{N(t)} \int_0^t v(s, x) ds \sim \int_0^\infty u(s, x) ds = \int_{\mathbb{R}^d} f(y) \mathcal{G}(x, dy), \quad t \rightarrow \infty,$$

where f is a suitable initial data.

2. RANDOM TIMES AND FRACTIONAL ANALYSIS

In this section we introduce the classes of inverse subordinators we are interested in. Associated to these classes we define a kernel $k \in L_{\text{loc}}^1(\mathbb{R}_+)$ which is used to define a general fractional derivatives (GFD), see [19] for details and applications to fractional differential equations. These admissible kernels k are characterized in terms of their Laplace transforms $\mathcal{K}(\lambda)$ as $\lambda \rightarrow 0$, see assumption (H) below.

Let $S = \{S(t), t \geq 0\}$ be a subordinator without drift starting at zero, that is, an increasing Lévy process starting at zero, see [4] for more details. The Laplace transform of $S(t)$, $t \geq 0$ is expressed in terms of a Bernstein function $\Phi : [0, \infty) \rightarrow [0, \infty)$ (also known as Laplace exponent) by

$$\mathbb{E}(e^{-\lambda S(t)}) = e^{-t\Phi(\lambda)}, \quad \lambda \geq 0.$$

The function Φ admits the representation

$$\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda\tau}) d\sigma(\tau), \tag{2.1}$$

where the measure σ (called Lévy measure) has support in $[0, \infty)$ and fulfills

$$\int_{(0,\infty)} (1 \wedge \tau) d\sigma(\tau) < \infty. \tag{2.2}$$

In what follows we assume that the Lévy measure σ satisfy

$$\sigma((0, \infty)) = \infty. \tag{2.3}$$

Using the Lévy measure σ we define the kernel k as follows

$$k : (0, \infty) \longrightarrow (0, \infty), t \mapsto k(t) := \sigma((t, \infty)). \tag{2.4}$$

Its Laplace transform is denoted by \mathcal{K} , that is, for any $\lambda \geq 0$ one has

$$\mathcal{K}(\lambda) := \int_0^\infty e^{-\lambda t} k(t) dt. \tag{2.5}$$

The relation between the function \mathcal{K} and the Laplace exponent Φ is given by

$$\Phi(\lambda) = \lambda\mathcal{K}(\lambda), \quad \forall \lambda \geq 0. \tag{2.6}$$

In what follows we make the following assumption on the Laplace exponent $\Phi(\lambda)$ of the subordinator S .

(H): Φ is a complete Bernstein function (more precisely, the Lévy measure σ has a completely monotone density $\rho(t)$ with respect to the Lebesgue measure, that is, $(-1)^n \rho^{(n)}(t) \geq 0$ for all $t > 0$, $n = 0, 1, 2, \dots$) and the functions \mathcal{K} , Φ satisfy

$$\mathcal{K}(\lambda) \rightarrow \infty, \text{ as } \lambda \rightarrow 0; \quad \mathcal{K}(\lambda) \rightarrow 0, \text{ as } \lambda \rightarrow \infty; \tag{2.7}$$

$$\Phi(\lambda) \rightarrow 0, \text{ as } \lambda \rightarrow 0; \quad \Phi(\lambda) \rightarrow \infty, \text{ as } \lambda \rightarrow \infty. \tag{2.8}$$

Example 2.1. (1) A classical example of a subordinator S is the so-called α -stable process with index $\alpha \in (0, 1)$. Specifically, a subordinator is α -stable if its Laplace exponent is

$$\Phi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda\tau}) \tau^{-1-\alpha} d\tau.$$

In this case it follows that the Lévy measure is $d\sigma_\alpha(\tau) = \frac{\alpha}{\Gamma(1-\alpha)} \tau^{-(1+\alpha)} d\tau$, the corresponding kernel k_α has the form $k_\alpha(t) = g_{1-\alpha}(t) := \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $t \geq 0$ and its Laplace transform is $\mathcal{K}_\alpha(\lambda) = \lambda^{\alpha-1}$, $\lambda \geq 0$.

(2) The Gamma process $Y^{(a,b)}$ with parameters $a, b > 0$ is another example of a subordinator with Laplace exponent

$$\Phi_{(a,b)}(\lambda) = a \log \left(1 + \frac{\lambda}{b} \right) = \int_0^\infty (1 - e^{-\lambda\tau}) a\tau^{-1} e^{-b\tau} d\tau,$$

the second equality is known as the Frullani integral. The Lévy measure is given by $d\sigma_{(a,b)}(\tau) = a\tau^{-1} e^{-b\tau} d\tau$. The associated kernel $k_{(a,b)}(t) = a\Gamma(0, bt)$, $t > 0$ and its Laplace transform is $\mathcal{K}_{(a,b)}(\lambda) = a\lambda^{-1} \log(1 + \frac{\lambda}{b})$, $\lambda > 0$.

(3) Let $0 < \beta < 1$ and $0 < \alpha < 1$ be given and $S_{\alpha,\beta}(t)$, $t \geq 0$ the driftless subordinator with Laplace exponent given by

$$\Phi_{\alpha,\beta}(\lambda) = \lambda^\alpha + \lambda^\beta.$$

It is clear from item 1 above that the corresponding Lévy measure $\sigma_{\alpha,\beta}$ is the sum of two Lévy measures, that is,

$$d\sigma_{\alpha,\beta}(\tau) = d\sigma_\alpha(\tau) + d\sigma_\alpha(\tau) = \frac{\alpha}{\Gamma(1-\alpha)}\tau^{-(1+\alpha)} d\tau + \frac{\beta}{\Gamma(1-\beta)}\tau^{-(1+\beta)} d\tau.$$

Then the associated kernel $k_{\alpha,\beta}$ is

$$k_{\alpha,\beta}(t) := g_{1-\alpha}(t) + g_{1-\beta}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t > 0$$

and its Laplace transform is $\mathcal{K}_{\alpha,\beta}(\lambda) = \mathcal{K}_\alpha(\lambda) + \mathcal{K}_\beta(\lambda) = \lambda^{\alpha-1} + \lambda^{\beta-1}$, $\lambda > 0$.

Denote by D the inverse process of the subordinator S , that is,

$$D(t) := \inf\{s \geq 0 \mid S(s) \geq t\} = \sup\{s \geq 0 \mid S(t) \leq s\}. \tag{2.9}$$

For any $t \geq 0$ we denote by $G_t^k(\tau) := G_t(\tau)$, $\tau \geq 0$ the marginal density of $D(t)$ or, equivalently

$$G_t(\tau) d\tau = \partial_\tau P(D(t) \leq \tau) = \partial_\tau P(S(\tau) \geq t) = -\partial_\tau P(S(\tau) < t).$$

As the density $G_t(\tau)$ plays an important role in what follows, we collect the most important properties of it.

Remark 1. If S is the α -stable process, $\alpha \in (0, 1)$, then the inverse process $D(t)$ has a Mittag-Leffler distribution (cf. Prop. 1(a) in [5]), namely

$$\mathbb{E}(e^{-\lambda D(t)}) = \int_0^\infty e^{-t\tau} G_t(\tau) d\tau = \sum_{n=0}^\infty \frac{(-\lambda t^\alpha)^n}{\Gamma(n\alpha + 1)} = E_\alpha(-\lambda t^\alpha). \tag{2.10}$$

It follows from the asymptotic behavior of the Mittag-Leffler function E_α that $\mathbb{E}(e^{-\lambda D(t)}) \sim Ct^{-\alpha}$ as $t \rightarrow \infty$. Using the properties of the Mittag-Leffler function E_α , we can show that the density $G_t(\tau)$ is given in terms of the Wright function $W_{\mu,\nu}$, namely $G_t(\tau) = t^{-\alpha}W_{-\alpha,1-\alpha}(\tau t^{-\alpha})$, see [8] for more details.

For a general subordinator, the following lemma determines the t -Laplace transform of $G_t(\tau)$, with k and \mathcal{K} given in (2.4) and (2.5), respectively. For the proof see [19] or Lemma 3.1 in [27].

Lemma 2. *The t -Laplace transform of the density $G_t(\tau)$ is given by*

$$\int_0^\infty e^{-\lambda t} G_t(\tau) dt = \mathcal{K}(\lambda)e^{-\tau\lambda\mathcal{K}(\lambda)}. \tag{2.11}$$

The double (τ, t) -Laplace transform of $G_t(\tau)$ is

$$\int_0^\infty \int_0^\infty e^{-p\tau} e^{-\lambda t} G_t(\tau) dt d\tau = \frac{\mathcal{K}(\lambda)}{\lambda\mathcal{K}(\lambda) + p}. \tag{2.12}$$

For any $\alpha \in (0, 1)$ the Caputo-Dzhrbashyan fractional derivative of order α of a function u is defined by (see e.g., [14] and references therein)

$$(\mathbb{D}_t^\alpha u)(t) = \frac{d}{dt} \int_0^t k_\alpha(t-\tau)u(\tau) d\tau - k_\alpha(t)u(0), \quad t > 0, \tag{2.13}$$

where k_α is given in Example 2.1-(1), that is, $k_\alpha(t) = g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $t > 0$. In general, starting with a subordinator S and the kernel $k \in L^1_{\text{loc}}(\mathbb{R}_+)$ given as in (2.4), we may define a differential-convolution operator by

$$(\mathbb{D}_t^{(k)} u)(t) = \frac{d}{dt} \int_0^t k(t-\tau)u(\tau) d\tau - k(t)u(0), \quad t > 0. \tag{2.14}$$

The operator $\mathbb{D}_t^{(k)}$ is also known as generalized fractional derivative. The distributed order derivative $\mathbb{D}_t^{(\mu)}$ is an example of such operator, corresponding to

$$k(t) = \int_0^1 g_{1-\alpha}(t) d\alpha = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) d\alpha, \quad t > 0, \tag{2.15}$$

where $\mu(\tau)$, $0 \leq \tau \leq 1$ is a positive weight function on $[0, 1]$, see [1, 7, 13, 18, 9, 24] for applications.

Now we introduce a suitable class of admissible $k(t)$ an state and essential theorem which this class obeys, see Theorem 4 below.

Definition 3 (Admissible kernels - $\mathbb{K}(\mathbb{R}_+)$). The subset $\mathbb{K}(\mathbb{R}_+) \subset L^1_{loc}(\mathbb{R}_+)$ of admissible kernels k is defined by those elements in $L^1_{loc}(\mathbb{R}_+)$ satisfying (H) such that for some $s_0 > 0$

$$\liminf_{\lambda \rightarrow 0^+} \frac{1}{\mathcal{K}(\lambda)} \int_0^{s_0/\lambda} k(t) dt > 0 \tag{A1}$$

and

$$\lim_{\substack{t, r \rightarrow \infty \\ \frac{t}{r} \rightarrow 1}} \left(\int_0^t k(s) ds \right) \left(\int_0^r k(s) ds \right)^{-1} = 1. \tag{A2}$$

The following theorem establishes an asymptotic relation between the density $G_t(\tau)$ and the kernel $k \in \mathbb{K}(\mathbb{R}_+)$. For the proof, see [17].

Theorem 4. Let $\tau \in [0, \infty)$ be fixed and $k \in \mathbb{K}(\mathbb{R}_+)$ a given admissible kernel. Define the map $G_t(\tau) : [0, \infty) \rightarrow \mathbb{R}_+$, $t \mapsto G_t(\tau)$ such that $\int_0^\infty e^{-\lambda t} G_t(\tau) dt$ exists for all $\lambda > 0$. Then

$$\lim_{t \rightarrow \infty} \left(\int_0^t G_s(\tau) ds \right) \left(\int_0^t k(s) ds \right)^{-1} = 1 \tag{2.16}$$

or

$$M_t(G_t(\tau)) := \frac{1}{t} \int_0^t G_s(\tau) ds \sim \frac{1}{t} \int_0^t k(s) ds =: M_t(k(t)), \quad t \rightarrow \infty$$

and $M_t(G_t(\tau))$ is uniformly bounded in $\tau \in \mathbb{R}_+$.

3. MARKOV PROCESSES IN RANDOM TIME

Let $X = \{X(t), t \geq 0\}$ be a Markov process in \mathbb{R}^d such that $X(0) = x \in \mathbb{R}^d$ almost surely. We are interested in a new process $Y = \{Y(t), t \geq 0\}$ which is constructed by a random time change in X . Namely, if $D(t)$, $t \geq 0$ denotes (as in Section 2) the inverse of a subordinator S independent of X , then we define Y by

$$Y(t) := X(D(t)), \quad t \geq 0.$$

Note that inverse subordinators have found many applications in probability theory, see [22] for a detailed discussions and several related references. In particular, for their relationship with local times of some Markov processes, see [4]. Similarities between inverse subordinators and renewal processes also are well studied. There are important applications of inverse subordinators in finance and physics. We stress that random time processes may be considered as mathematical realizations of the general concept of biological time known in biology and ecology since the pioneering works of V. I. Vernadsky [28].

The first natural question which appear here concerns the possible relations between the characteristics of the processes $X(t)$ and $Y(t)$. To the best of our knowledge this question was for the first time discussed by A. Mura, M.S. Taqqu and F. Mainardi in [25]. The authors considered the diffusion processes with an implicitly defined class of random times $D(t)$. Later similar questions were discussed by several authors, see e.g., [27] and references therein.

The situation there may be described as follows. Define a function

$$u(t, x) := \mathbb{E}[f(X(t))], \quad t > 0, \quad x \in \mathbb{R}^d$$

for a proper $f : \mathbb{R}^d \rightarrow \mathbb{R}$. This is the solution of the Kolmogorov equation

$$\frac{\partial}{\partial t} u(t, x) = Lu(t, x), \quad (3.17)$$

$$u(0, x) = f(x),$$

where L is the generator of the process $X(t)$. Let us define a similar function for $Y(t)$:

$$v(t, x) = \mathbb{E}[f(Y(t))].$$

Then this function satisfies the following fractional evolution equation:

$$D_t^{(k)} v(t, x) = Lv(t, x). \quad (3.18)$$

Moreover, the subordination formula holds:

$$v(t, x) = \int_0^\infty u(\tau, x) G_t(\tau) \, d\tau, \quad (3.19)$$

where, as before, $G_t(\tau)$ is the density of the inverse subordinator $D(t)$.

If μ_t^x and ν_t^x denote the marginal distributions of $X(t)$ and $Y(t)$, respectively, then the subordination relations for these distributions is given by

$$\nu_t^x = \int_0^\infty \mu_\tau^x G_t(\tau) \, d\tau. \quad (3.20)$$

In the next section we use these relations to study the renormalized Green measure associated to the subordinated process Y .

4. RENORMALIZED GREEN MEASURES

Let X be a Markov process and Y be the time changed process as in Section 3 with all our notations from there. For every jump of the subordinator S there is a corresponding flat period of its inverse D . These flat periods represent trapping events in which the test particle gets immobilized in a trap. Trapping slows down the overall dynamics of the initial Markov process X . Our aim is to analyze how these traps will be reflected in the asymptotic behavior of the changed process Y .

To study the time asymptotic of random processes there is the useful notion of Green measures, see for example [21] for this notion. More precisely, if $Z(t)$, $t \geq 0$ is a random process in \mathbb{R}^d with $Z(0) = x \in \mathbb{R}^d$ and, for each $t \geq 0$, γ_t^x denotes its marginal distribution, then the Green measure of Z is defined by

$$\mathcal{G}(x, dy) := \int_0^\infty \gamma_t^x(dy) \, dt$$

if this integral converges. In [21] we have shown the existence of the Green measures for certain classes of Markov processes in \mathbb{R}^d with the necessary condition $d \geq 3$. For $d = 1, 2$ we have to modify this definition by means of a renormalized Green measure, namely

$$\mathcal{G}_r(x, dy) = \lim_{T \rightarrow \infty} \frac{1}{N(T)} \int_0^T \mu_t^x(dy) \, dt.$$

This approach (in a bit different framework) is well known in the theory of additive functionals for Markov processes, see [20] for an extended list of references.

The following lemma shows that the Green measure for $Y(t)$ does not exist for a general inverse subordinator and arbitrary Markov process $X(t)$.

Lemma 5. *Under the assumptions formulated above for any dimension d the Green measure for $Y(t)$ does not exist.*

Proof. Using the subordination formula (3.20) we obtain

$$\int_0^\infty \nu_t^x dt = \int_0^\infty \int_0^\infty \mu_\tau^x G_t(\tau) d\tau dt.$$

But we know that for each τ , it follows from (2.7), (2.8) and (2.11) that

$$\int_0^\infty G_t(\tau) dt = \mathcal{K}(0) = +\infty.$$

Therefore, the considered integral is divergent. □

As the Green measure does not exist for a general subordinated process Y , we have to consider instead a renormalized Green measure. More precisely, we would like to find the following limit

$$\mathcal{G}_r(x, dy) := \lim_{T \rightarrow \infty} \frac{1}{N(T)} \int_0^T \nu_t^x(dy) dt.$$

Theorem 6. *Assume that the Markov process $X(t)$ in \mathbb{R}^d , $d \geq 3$ has a Green measure $\mathcal{G}(x, dy)$ and define*

$$N(T) := \int_0^T k(s) ds, \quad T \geq 0. \tag{4.21}$$

Then the renormalized Green measure for $Y(t)$ exists and

$$\mathcal{G}_r(x, dy) = \mathcal{G}(x, dy).$$

Proof. Using the subordination relation (3.20) the renormalized Green measure $\mathcal{G}_r(x, dy)$ may be written as

$$\mathcal{G}_r(x, dy) = \lim_{T \rightarrow \infty} \frac{1}{N(T)} \int_0^T \int_0^\infty \mu_\tau^x(dy) G_t(\tau) d\tau dt.$$

Now using Fubini theorem and Theorem 4 it follows that

$$\mathcal{G}_r(x, dy) := \lim_{T \rightarrow \infty} \frac{1}{N(T)} \int_0^T \nu_t^x(dy) dt = \int_0^\infty \mu_t^x(dy) dt = \mathcal{G}(x, dy).$$

This shows the statement of the theorem and finish the proof. □

Remark 7. As we mentioned at the beginning of this section, random time produces trapping (or environments, or friction) effects in the Markov dynamics. That is one reason why in physics such processes are very useful. As the trapping slows down the Markov dynamics, then the usual definition of Green measures produces a divergent integral. To compensate this divergence we have to consider a renormalization with a time depended factor. The time asymptotic of the renormalized Green measure coincides with the Green measure of the initial Markov process.

5. APPLICATIONS TO FRACTIONAL DYNAMICS

Let $u(t, x)$ be the solution of equation (3.17) and $v(t, x)$ the corresponding solution of the fractional equation (3.18). Our goal is to compare the behaviors in t for these solutions. To this end, at first we restrict the class of Markov processes under considerations. Namely, let $a : \mathbb{R}^d \rightarrow \mathbb{R}$ be a fixed kernel with the following properties:

- (1) Symmetric, $a(-x) = a(x)$, for every $x \in \mathbb{R}^d$.
- (2) Positive, continuous and bounded, $a \geq 0$, $a \in C_b(\mathbb{R}^d)$.

(3) Integrable

$$\int_{\mathbb{R}^d} a(y) \, dy = 1.$$

Consider the generator L defined by

$$(Lf)(x) = \int_{\mathbb{R}^d} a(x - y)[f(y) - f(x)] \, dy = (a * f)(x) - f(x), \quad x \in \mathbb{R}^d.$$

In particular, $L^* = L$ in $L^2(\mathbb{R}^d)$ and L is a bounded linear operator in all $L^p(\mathbb{R}^d)$, $p \geq 1$. We call this operator the jump generator with jump kernel a . The corresponding Markov process is of a pure jump type and is known in stochastic as compound Poisson process, see [26].

We make the following assumptions on the kernel a .

(A): The jump kernel a is such that the Fourier transform $\hat{a} \in L^1(\mathbb{R}^d)$ and it has finite second moment, that is,

$$\int_{\mathbb{R}^d} |x|^2 a(x) \, dx < \infty.$$

Define the Banach space $CL(\mathbb{R}^d)$ as the set of all bounded continuous and integrable functions on \mathbb{R}^d , that is, $CL(\mathbb{R}^d) = C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The norm in this space is constructed as the sum of $C_b(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ norms. The following theorem was shown in [21].

Theorem 8. *Let a be a kernel which satisfies all the above assumptions and $d \geq 3$. Consider the solution $u(t, x)$ to (3.17) with an initial data $f \in CL(\mathbb{R}^d)$. Then*

$$\int_0^\infty u(t, x) \, dt = \int_{\mathbb{R}^d} f(y) \mathcal{G}(x, dy),$$

where $\mathcal{G}(x, dy)$ is the Green measure of the corresponding Markov jump process.

This result gives an averaged characteristic of the dynamics $u(t, x)$ corresponding to the Markov processes via the Kolmogorov equation. On the other hand, for the fractional dynamics $v(t, x)$ (the solution of equation (3.18)) we have only information about the Cesaro mean

$$M_t(f) = \frac{1}{t} \int_0^t v(s, x) \, ds.$$

The asymptotic of this mean was studied in [17]. In particular, when $\Phi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$ the kernel $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ and the GFD corresponds to the Caputo-Dzhrbashyan fractional derivative of order α . For this class of kernels we have

$$M_t(f) \sim Ct^{-\alpha}, \quad t \rightarrow \infty.$$

With the help of Theorem 8 we may also derive an average result for the fractional dynamics $v(t, x)$.

Theorem 9. *Under assumptions of Theorem 8 holds*

$$\frac{1}{N(t)} \int_0^t v(s, x) \, ds \sim \int_{\mathbb{R}^d} f(y) \mathcal{G}(x, dy), \quad t \rightarrow \infty.$$

Proof. Using (3.19) we have

$$\frac{1}{N(t)} \int_0^t v(s, x) \, ds = \frac{1}{N(t)} \int_0^t \int_0^\infty u(\tau, x) G_s(\tau) \, d\tau \, ds.$$

Again it follows from Fubini theorem, Theorem 4 and the definition of $N(t)$ that

$$\frac{1}{N(t)} \int_0^t v(s, x) \, ds = \int_0^\infty u(\tau, x) \left(\frac{1}{N(t)} \int_0^t G_s(\tau) \, ds \right) \, d\tau \xrightarrow{t \rightarrow \infty} \int_0^\infty u(\tau, x) \, d\tau.$$

Then the result of the theorem follows from Theorem 8. \square

Remark 10. (1) For concrete cases of jump kernels we have more information about space decay of the Green measures, see [21]. It gives the possibility to extend the statement of Theorem 9 to a wider class of the initial data f .

(2) The same result is true for the Brownian motion $B(t)$ in \mathbb{R}^d for $d \geq 3$, see [21].

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REFERENCES

- [1] T. M. Atanackovic, S. Pilipovic, and D. Zorica. Time distributed-order diffusion-wave equation. I., II. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 465, pages 1869–1891. The Royal Society, 2009.
- [2] B. Baeumer, M. Meerschaert, and E. Nane. Brownian Subordinators and Fractional Cauchy Problems. *Trans. Amer. Math. Soc.*, 361(7):3915–3930, 2009.
- [3] P. Becker-Kern, M. M. Meerschaert, and H.-P. Scheffler. Limit theorems for coupled continuous time random walks. *The Annals of Probability*, 32(1B):730–756, 2004.
- [4] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [5] N. H. Bingham. Limit theorems for occupation times of Markov processes. *Z. Wahrsch. verw. Gebiete*, 17:1–22, 1971.
- [6] J. L. da Silva, Y. G. Kondratiev, and A. N. Kochubei. Fractional statistical dynamics and fractional kinetics. *Methods Funct. Anal. Topology*, 22:197–209, 2016.
- [7] V. Daftardar-Gejji and S. Bhalekar. Boundary value problems for multi-term fractional differential equations. *J. Math. Anal. Appl.*, 345(2):754–765, 2008.
- [8] R. Gorenflo, Y. Luchko, and F. Mainardi. Analytical properties and applications of the Wright function. *Fract. Calc. Appl. Anal.*, 2(4):383–414, 1999.
- [9] R. Gorenflo and S. Umarov. Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations, Part one. *Z. Anal. Anwend.*, 24(3):449–466, 2005.
- [10] A. Grigor’yan, E. Hu, and J. Hu. Two-sided estimates of heat kernels of jump type dirichlet forms. *Adv. Math.*, 330:433–515, 2018.
- [11] A. Grigor’yan, Yu. G. Kondratiev, A. Piatnitski, and E. Zhizhina. Pointwise estimates for heat kernels of convolution-type operators. *Proc. Lond. Math. Soc. (3)*, 114(4):849–880, 2018.
- [12] A. Grigor’yan and A. Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.*, 40(3):1212–1284, 2012.
- [13] A. Hanyga. Anomalous diffusion without scale invariance. *J. Phys. A: Mat. Theor.*, 40(21):5551, 2007.
- [14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.
- [15] K. Kobayashi and H. Park. Spectral heat content for time-changed killed Brownian motions, Arxiv:2007.05776v1, 2020.
- [16] A. Kochubei, Yu. G. Kondratiev, and J. L. da Silva. From random times to fractional kinetics. *Interdisciplinary Studies of Complex Systems*, 16:5–32, 2020.
- [17] A. Kochubei, Yu. G. Kondratiev, and J. L. da Silva. Random time change and related evolution equations. Time asymptotic behavior. *Stochastics and Dynamics*, 4:2050034–1–24, 2020.
- [18] A. N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.*, 340(1):252–281, 2008.
- [19] A. N. Kochubei. General fractional calculus, evolution equations, and renewal processes. *Integral Equations Operator Theory*, 71(4):583–600, October 2011.
- [20] Y. Kondratiev, Y. Mishura, and G. Shevchenko. Limit theorems for additive functionals of continuous time random walks. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, pages 1–22, 2020.
- [21] Yu. G. Kondratiev and J. L. da Silva. Green Measures for Markov Processes. *Methods Funct. Anal. Topology*, 26(3):241–248, 2020.

- [22] M. Magdziarz and R. L. Schilling. Asymptotic properties of Brownian motion delayed by inverse subordinators. *Proceedings of the American Mathematical Society*, 143(10):4485–4501, 2015.
- [23] M. M. Meerschaert, D. A. Benson, H.-P. Scheffler, and B. Baeumer. Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E*, 65(4):041103, 2002.
- [24] M. M. Meerschaert and H.-P. Scheffler. Stochastic model for ultraslow diffusion. *Stochastic Process. Appl.*, 116(9):1215–1235, 2006.
- [25] A. Mura, M. S. Taqqu, and F. Mainardi. Non-Markovian diffusion equations and processes: analysis and simulations. *Phys. A*, 387(21):5033–5064, 2008.
- [26] A. V. Skorohod. *Random Processes with Independent Increments*, volume 47 of *Mathematics and its applications (Soviet series)*. Springer, 1991.
- [27] B. Toaldo. Convolution-type derivatives, hitting-times of subordinators and time-changed C_0 -semigroups. *Potential Anal.*, 42(1):115–140, 2015.
- [28] V. I. Vernadsky. *The Biosphere: Complete Annotated Edition*. Springer Science & Business Media, 1998.

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