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UNITARY REPRESENTATIONS OF POINCARÉ GROUP $\mathbf{P}(1,n)$ IN $\mathbf{SO}(1,n)$-BASIS

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Abstract. This paper concerns the problem of reduction of unitary irreducible representations of the Poincaré group $\mathrm{P}(1,n)$ with respect to representations of its subgroup $\mathrm{SO}(1,n)$. Based on a generalization of the Wigner–Eckart theorem, we obtain matrix elements of the shift operators in the $\mathrm{SO}(1,n)$-basis.

Робота присвячена проблемі редукції унітарних незвідних представлень групи Пуанкаре $P(1,n)$ відносно представлень її підгрупи $SO(1,n)$. На основі узагальнення теореми Вігнера-Еккарта отримано матричні елементи операторів зсуву в $SO(1,n)$-базисі.

Introduction

The problem of decomposition of an irreducible representation of a group G into irreducible components of its restriction on a closed subgroup H is of great importance in the representation theory. Though, nowadays it is still far from complete solution. There was almost no general result (excluding the case when G is a compact group [19]). However, in view of a great significance of this problem to the elementary particle theory, in recent decades a number of papers appeared, where certain particular cases of groups and representations are considered.

In [3] the problem of decomposition of unitary irreducible representations of the groups $\mathrm{P}(1,4)$, $\mathrm{P}(1,6)$, $\mathrm{P}(1,n)$, which are important for physical applications, with respect to the representations of Poincaré group was solved.

In the present paper we reduce (decompose) a representations of these groups with respect to representations of their maximal semisimple subgroups (i.e. perform the reduction $\mathrm{P}(1,n) \supset \mathrm{SO}_0(1,n) \supset \mathrm{SO}_0(n) \supset \cdots \supset \mathrm{SO}(2)$).

In the case $n=3$ this problem was considered in [1, 12]. In [1] the matrix elements of the shifts generators in the Lorentz basis were constructed. But this derivation is formal, and in [12] only the case of representations of zero mass was presented.

Section 1 provides some necessary information on the group $\mathrm{P}(1,n)$ and a classification of its unitary irreducible representations.

In Section 2 a classification of irreducible (in particular, unitary) representations of the group $\mathrm{SO}_0(1,n)$ is presented based on [7, 14, 16, 4, 18].

Section 3 concerns with the problem of decomposition of unitary irreducible representations of the group $\mathrm{P}(1,n)$ of the class $I^{\pm}$ with respect to the subgroup $\mathrm{SO}_0(1,n)$.

In the last section, we use a generalization of the Wigner–Eckart theorem for the case of non-compact groups [11] to obtain formulas for the actions of the generators of shifts P_μ in the $\mathrm{SO}(1,n)$-basis.

1. Poincaré group and its representations

Let X be a $(1+n)$-dimensional pseudo-Euclidean vector space with elements $x=(x_0,x_1,\dots,x_n)$ and the metric $g_{00}=-d_{11}=\cdots=-g_{nn}=1$. We refer to the set of linear

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transformations

$$x_\mu\colon x'_\mu = \hat{\lambda}^\nu_\mu x_\nu + a_\mu, \qquad a \in X, \qquad \mu, \nu = 0, 1, 2, \ldots, n,$$

that preserve the quadratic form $(x-y)^2 \equiv g^{\mu\nu}(x_\mu - y_\mu)(x_\nu - y_\nu)$ as generalized (complete) Poincaré group and denote it by $\tilde{\mathrm{P}}(1,n)$. Therefore, the matrix $\tilde{\Lambda}$ satisfies the condition $\tilde{\Lambda}^T_g \tilde{\Lambda} = g$, where Λ^T is the transposed matrix. The group $\tilde{\mathrm{P}}(1,n)$ also contains inversions I_μ of space X,

$$I_\mu x_\nu = \begin{cases} -x_\nu, & \mu = 0, \\ x_\nu, & \mu \neq \nu, \end{cases} \qquad I^2_\mu = 1, \qquad [I_\mu, I_\nu] = 0.$$

Due to this fact, the group $\tilde{\mathrm{P}}(1,n)$ has the subgroup of reflections,

$$R_a(1,n) = \{I_0, I_a, I_0, I_a, 1\}, \qquad a = 1, 2, \ldots, n.$$

This subgroup is obviously the maximal discrete divisor in $\tilde{\mathrm{P}}(1,n)$.

Factorizing $\tilde{\mathrm{P}}(1,n)$ by $R_a(1,n)$ one obtains the proper orthochronous group $\mathrm{P}(1,n)$, which is the semidirect product of the form

$$\mathrm{P}(1,n) = \mathrm{SO}_0(1,n) \otimes \mathrm{T}(1,n),$$

where $\mathrm{T}(1,n)$ is a $(1+n)$-parameter abelian group, and $\mathrm{SO}_0(1,n)$ is the identity of the group of real homogenous linear transformations of the variables $x_1, x_2, \ldots, x_n$, preserving the quadratic form $x_0^2 - x_1^2 - \cdots - x_n^2$. Moreover, this is the regular semidirect product [13], and, for this reason, the general theory of irreducible unitary representations of the group $\mathrm{P}(1,n)$ can be constructed using the method of induced representations.

Let $\mathcal{U}$ be a unitary irreducible representation of the group $\mathrm{P}(1,n)$, and P_μ, $J_{\mu\nu}$ its generators. Since the generators of shifts P_μ commute, there exists a basis in the space $\mathcal{H}$ such that all P_μ are simultaneously diagonal. Let $p = (p_0, p_1, \ldots, p_n)$ be the set of all possible eigenvalues of the operators $P_0, P_1, \ldots, P_n$, and we think of the real numbers $p_0, p_1, \ldots, p_n$ as of the components of a vector in a $(1+n)$-dimensional subspace. This subspace (p-space) is obviously pseudo-Euclidean with the metric $p_0^2 - p_1^2 - \cdots - p_n^2$.

Definition 1.1. The set σ of all possible p-vectors, the components of these vectors being the eigenvalues of the generators P_μ, is called the *orbit* of the representation $\mathcal{U}$, and $p = (p_0, p_1, \ldots, p_n) \in \sigma$ is a *spectral vector* of the representation $\mathcal{U}$.

From the general theory we know that the orbit of every irreducible representation of the group $\mathrm{P}(1,n)$ constitutes, in the p-space, a hyperplane that is homogenous and invariant with respect to all transformations from the group $\mathrm{SO}_0(1,n)$. Moreover, there exists an irreducible representation of the group $\mathcal{U}$ the orbit of which coincides with the group $\mathrm{SO}_0(1,n)$ that is an invariant and homogeneous hyperplane in a $(1+n)$-dimensional pseudo-Euclidean space.

This means that the orbit σ of any unitary irreducible representation can be uniquely determined by a single p-vector. We denote such vector by p^0, and call it a *determining vector* of the representation. Every vector $p \in \sigma$ can be expressed by the formula $p = L(p, p^0)p^0$, where $L(p, p^0) \in \mathrm{SO}_0(1,n)$. If we restrict the matrix $L(p, p^0)$ by the condition $\lim_{p \to p^0} L(p^0, p) = \lim_{p^0 \to p} L(p, p^0)$, then it is uniquely determined by the vectors p and p^0. In the case $n = 3$ this matrix is called the *Wigner rotation matrix.*

Definition 1.2. If every element of a subgroup $R_p \subset \mathrm{SO}(1,n)$ preserves the vector p, then this subgroup is called the *stationary (small) subgroup* of the vector p.

Stationary subgroups of vectors belonging to the same orbit are isomorphic.

Let $\mathcal{D}$ be an irreducible representation of a stationary subgroup R_{p_0}. Let $\mathcal{H}$ be the space of square-integrable functions $\varphi(p, \alpha)$, $p \in \sigma_{p_0}$, and $\alpha = (\alpha_1, \alpha_2, \ldots)$ be sets of numbers that enumerate the basis vectors of the space of the representation $\mathcal{D}$. Define the operators $U(a,\Lambda)$ by

$$(U(a,\Lambda)\varphi)(p,\alpha) = \exp i\langle a,p\rangle \sum_{\beta} \mathcal{D}_{\alpha\beta}(r_{p^0}(p,\Lambda))\varphi(\Lambda^{-1}p,\beta), \tag{1.1}$$

where $\mathcal{D}_{\alpha\beta}(r_{p^0})$ are the matrix elements of the operators $r_{p^0}(p,\Lambda) \in R_{p^0}$ of the representation $\mathcal{D}$ that satisfy the condition

$$r_{p^0}(p,\Lambda) = L^{-1}(p,p^0) \wedge L(\Lambda^{-1}p,p^0). \tag{1.2}$$

The inner product in $\mathcal{H}$ can be defined by

$$(\varphi_1,\varphi_2) = \int_\sigma \sum_{\alpha,\beta} \varphi_1^*(p,\alpha)\Delta_{\alpha\beta}\varphi_2(p,\beta)\mathrm{d}\mu(p),$$

where $\Delta_{\alpha\beta}$ are components of a R_{p^0}-invariant metric tensor $\Delta = \{\Delta_{\alpha\beta}\}$ in the subspace of the representation $\mathcal{D}$, $\mu(p)$ is a $\mathrm{SO}_0(1,n)$-invariant measure on σ.

Formula (1.1) defines a unitary irreducible representation of the group $\mathrm{P}(1,n)$ if and only if $\mathcal{D}$ is unitary and irreducible.

Results obtained by Mackey [13] involve that any unitary irreducible representation of the group $\mathrm{P}(1,n)$ is uniquely determined by its determining vector and the unitary irreducible representation of the corresponding stationary subgroup, and has the form (1.1).

Therefore, a classification of irreducible unitary representations of the group $\mathrm{P}(1,n)$ can be obtained by using a classification of $\mathrm{SO}_0(1,n)$-invariant orbits.

Table 1 presents all six classes of representations of $\mathrm{P}(1,n)$, the eigenvalues of the invariants $p^2 = g^{\mu\nu}P_\mu P_\nu$, $\operatorname{sign} P_0$, the determining vectors of the representations and stationary subgroups. The representations of the classes I^+ and II^+ are isomorphic to those of the classes I^- and II^-, respectively. The operator $\operatorname{sign} P_0$ is invariant only for the representations $O_I^\pm$ and $O_{II}^\pm$.

TABLE 1. Classes of representations of $\mathrm{P}(1,n)$

Class	Notation	p^2	sign P_0	Form of vector p_0	Orbit	St. subgroup
I^+	O_I^+	$\varkappa^2$	1	$(\varkappa,0,\dots,0)$, $\varkappa>0$	$\langle p,p\rangle = \varkappa^2$	$\mathrm{SO}(n)$
I^-	O_I^-	$\varkappa^2$	-1	$(\varkappa,0,\dots,0)$, $\varkappa<0$	$\langle p,p\rangle = \varkappa^2$	$\mathrm{SO}(n)$
II^+	O_{II}^+	0	1	$(1,0,\dots,1)$	$\langle p,p\rangle = 0$, $p_0>0$	$\mathrm{E}(n-1)$
II^-	O_{II}^-	0	-1	$(-1,0,\dots,1)$	$\langle p,p\rangle = 0$, $p_0<0$	$\mathrm{E}(n-1)$
III	O_{III}	$-\varkappa^2$	–	$(0,\dots,0,\varkappa)$, $\varkappa>0$	$\langle p,p\rangle = -\varkappa^2$	$\mathrm{SO}_0(1,n-1)$
IV	O_{IV}	0	–	$(0,0,\dots,0)$	$P_\mu = 0$	$\mathrm{SO}_0(1,n)$

The group $\mathrm{P}(1,n)$ is *tame* [9], i.e. every its unitary representation can be uniquely decomposed into a direct integral of irreducible ones.

As it was established before, the group $\mathrm{P}(1,n)$ has four different subgroups, which can be used to construct irreducible representations, namely $\mathrm{SO}(n)$, $\mathrm{E}(n-1)$, $\mathrm{SO}_0(1,n-1)$, and $\mathrm{SO}_0(1,n)$. In the first case this group is compact and tame in view of the Peter–Weyl theorem. In the second case it is a regular semidirect product of an abelian and a compact groups, hence it is tame too. In the other two cases we face a connected semisimple group, and, in accordance with the Harish-Chandra theorem, it is tame. Since the regular semidirect product is a tame group if and only if all stationary subgroups are tame, we conclude that $\mathrm{P}(1,n)$ is a tame group.

Now we find the generators P_μ and $J_{\mu\nu}$ of the representations that are induced from the group $\mathrm{SO}(n)$, i.e. the representations of the class $I^\pm$. In what follows we are interested only in these representations. The operators P_μ and $J_{\mu\nu}$ generate the Lie algebra of the group $\mathrm{P}(1,n)$ and satisfy the conditions

$$\begin{aligned} &[P_\mu, P_\nu] = 0,\\ &[P_\mu, J_{\alpha\beta}] = i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha),\\ &[J_{\mu\nu}, J_{\alpha\beta}] = i(g_{\mu\beta}I_{\nu\alpha} + g_{\nu\alpha}J_{\mu\beta} - g_{\mu\alpha}J_{\nu\beta} - g_{\nu\beta}J_{\mu\alpha}). \end{aligned} \tag{1.3}$$

Parametrizing the groups $\mathrm{SO}_0(1,n)$ and $\mathrm{T}(1,n)$ by the real parameters $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and a^μ, correspondingly, we can rewrite the elements $(a,\Lambda) \in \mathrm{P}(1,n)$ as

$$(a,\Lambda) = \exp(ia^\mu P_\mu + i\omega^{\mu\nu}J_{\mu\nu}), \tag{1.4}$$

where P_μ and $J_{\mu\nu}$ are the generators of the initial representation of the group $\mathrm{P}(1,n)$.

Observe, that the role of the determining vector of the representation of the class $I^\pm$ is played by a time-like vector

$$p^0 = (\varepsilon\varkappa, 0, \dots, 0), \qquad \varkappa > 0, \qquad \varepsilon = \pm 1, \tag{1.5}$$

and the stationary subgroup $\mathrm{SO}(n)$ is generated by the elements

$$r_{p^0} = \exp\left\{i\Omega^{ab}S_{ab}\right\}, \qquad a,b = 1,2,\dots,n, \tag{1.6}$$

where $S_{ab} = J_{ab}$, and Ω^{ab} are the parameters of the stationary subgroup. Now the operators of the induced representation can be rewritten as

$$(U(a,\Lambda)\varphi)(p) = \exp\left\{i\langle a,p\rangle + i\Omega^{ab}(p,\omega)S_{ab}\right\}\varphi(\Lambda^{-1}p),$$

where $\varphi(p)$ is a column vector with the dimension coinciding with the dimension of the representation $\mathcal{D}$, $\Omega^{ab}(p,\omega)$ are some functions of $p \in \sigma$, and σ is the orbit determined by vector (1.3) and parameters $\omega^{\mu\nu}$ that fix the element $\Lambda \in \mathrm{SO}_0(1,n)$. From (1.2) and (1.6) we get an equation for the functions $\Omega^{ab}(p,\omega)$, namely,

$$L^{-1}(p,p^0) \wedge L(\Lambda^{-1}p, p^0)q = \exp\left\{i\Omega^{ab}(p,\omega)S_{ab}\right\}q, \tag{1.7}$$

where q is an arbitrary vector of the p-space. Thinking of $\omega^{\mu\nu}$ and a^μ as infinitesimals, we can write

$$\begin{aligned} &\left(1 + ia^\mu P_\mu + i\omega^{\mu\nu}J_{\mu\nu}\right) = \varphi(p)\\ &\quad = \left[1 + ia^\mu P_\mu - i\omega^{\mu\nu}(S_{\mu\nu})^\sigma_\rho P_\sigma \frac{\partial}{\partial P_\rho} + i\Omega^{ab}(p,\omega)S_{ab}\right]\varphi(p). \end{aligned} \tag{1.8}$$

This leads to

$$\begin{aligned} (P_\mu\varphi)(p) &= P_\mu\varphi(p), \qquad p \in \sigma,\\ (J_{\mu\nu}\varphi)(p) &= \left\{-(S_{\mu\nu})^\sigma_\rho P_\sigma \frac{\partial}{\partial P_\rho} + \left(\frac{\delta\Omega^{ab}}{\delta\omega^{\mu\nu}}\right)_{\omega=0} S_{ab}\right\}\varphi(p). \end{aligned}$$

It is easy to show that

$$(S_{\mu\nu})^\sigma_\rho P_\sigma \frac{\partial}{\partial P_\rho} = i\left(P_\mu \frac{\partial}{\partial P_\nu} - P_\nu \frac{\partial}{\partial P_\mu}\right) \equiv M_{\mu\nu}.$$

If the vectors p, p^0 belong to σ, then $\langle p,p\rangle = \varkappa^2$, $\operatorname{sign} p^0 = \varepsilon$, $\varkappa^2 > 0$ and the "Wigner rotation" is determined by the transformation

$$\begin{gathered} q \to L(p,p^0)q = q + \frac{\langle p^0,q\rangle[\varkappa^2 + 2\langle p^0,p\rangle] - \varkappa^2\langle p,q\rangle}{\varkappa^2\varkappa^2 + \langle p^0,p\rangle}p - \frac{\langle p^0,q\rangle + \langle p,q\rangle}{\varkappa^2 + \langle p^0,p\rangle},\\ L^{-1}(p,p^0) = L(p^0,p). \end{gathered}$$

Hence, in view of the infinitesimality of the parameters, expression (1.7) can be rewritten as

$$\begin{aligned} q+\delta q-\frac{\langle p^0,q\rangle+\langle p,q\rangle}{\varkappa^2+\langle p^0,q\rangle}\delta p^0+\frac{\langle q,\delta p^0\rangle}{\varkappa^2+\langle p^0,p\rangle}(p^0+p) \\ =q+i\Omega^{ab}(p,\omega)S_{ab}q, \end{aligned} \tag{1.9}$$

where $\delta q \equiv \Lambda q - q$. Since q is an arbitrary vector, from (1.9) we have

$$\Omega^{ab}_{ab}(p,\omega)=\omega^{ab}-\frac{p^a\omega^{ab}}{\varepsilon\varkappa+P_\sigma}.$$

Therefore, the generators of the representations of the class $I^{\pm}$ have the form

$$\begin{aligned} &P_\mu=p_\mu, \qquad p\in\sigma, \\ &J_{ab}=M_{ab}+S_{ab}, \qquad J_{ab}=M_{aa}-\varepsilon\frac{P^bS_{ab}}{\varkappa+P_0}, \end{aligned} \tag{1.10}$$

where $\varkappa$, ε are determined by the vector p^0, and S_{ab} are generators of the unitary irreducible representation of the group $\mathrm{SO}(n)$. The operators (1.10) act on the functions that depend on the conditions

$$P^2\varphi(p)=\varkappa^2\varphi(p),$$

$$\operatorname{sign}P_0\varphi(p)=\varepsilon\varphi(p),$$

$$(\varphi,\varphi)=\int\varphi^+(p)\varphi(p)\delta(p^2-\varkappa^2)\theta(\varepsilon,p^0)\mathrm{d}p>0,$$

$$\mathrm{d}p=\prod_\mu \mathrm{d}p_\mu\colon \quad \sigma(z)=\begin{cases}1, & z>0,\\ 0, & x\leqslant 0.\end{cases}$$

The form (1.10) of the generators of $\mathrm{P}(1,n)$ is called the *canonical form*, and the corresponding basis is the *canonical basis*.

Definition 1.3. A basis of an irreducible unitary representation of the group $\mathrm{P}(1,n)$ that consists of Gelfand–Tsetlin bases of subspaces that are spaces of irreducible representations of the maximal compact subgroup $\mathrm{SO}(n)$, is called a $\mathrm{SO}_0(1,n)$-*basis*, if all Casimir operators of the group $\mathrm{SO}_0(1,n)$ are diagonal in it.

Our aim is to find an explicit form of the generators $J_{\mu\nu}$, P_μ in a $\mathrm{SO}(1,n)$-basis. We start with studying irreducible representations of the group $\mathrm{SO}_0(1,n)$

2. Representations of $\mathrm{SO}_0(1,n)$

Let KAN be the Iwasawa decomposition of the group $\mathrm{SO}_0(1,n)$, where K is $\mathrm{SO}(n)$, A is an abelian subgroup, N is the nilpotent subgroup, $M=Z_k(A)=\mathrm{SO}(n-1)$. A finite-dimensional irreducible representation μ of a subgroup M in the Hilbert space $\mathcal{H}_\mu$ and the complex linear form Λ on the homogenous linear space $a=\log\Lambda$ define, in a usual way, a representation of the parabolic subgroup $P=MAN$, which induces the representation $\Pi_{\mu\nu}$ of the basic (non-unitary) series of the group $\mathrm{SO}_0(1,n)$ in the space $\mathcal{H}_{\mu,\Lambda}$, i.e. the elementary representation of the group $\mathrm{SO}_0(1,n)$. The space of vector functions on K is usually used as a space $\mathcal{H}_{\mu,\Lambda}$.

Irreducible representations μ of the subgroup M are defined by the greatest weight Λ_μ with the components Λ_{μ_i}, $i=1,2,\dots,\left[\frac{n}{2}\right]^{-1}$,

$$\begin{aligned} &\Lambda_{\mu_1}\geqslant\Lambda_{\mu_2}\geqslant\dots\geqslant|\Lambda_{\mu p}|, \qquad n=2p+1, \\ &\Lambda_{\mu_1}\geqslant\Lambda_{\mu_2}\geqslant\dots\geqslant\Lambda_{\mu p}, \qquad n=2p. \end{aligned} \tag{2.1}$$

Irreducible representations ω of the subgroup K are defined by the greatest weight Λ_ω with the components Λ_{ω_i}, $i = 1, 2, \dots, \left[\frac{n}{2}\right]$,

$$\begin{aligned} &\Lambda_{\omega_1} \geqslant \Lambda_{\omega_2} \geqslant \cdots \geqslant \Lambda_{\omega p}, \qquad n = 2p+1, \\ &\Lambda_{\omega_1} \geqslant \Lambda_{\omega_2} \geqslant \cdots \geqslant |\Lambda_{\omega p}|, \qquad n = 2p. \end{aligned} \tag{2.2}$$

All the numbers Λ_{ω_i} and Λ_{μ_i} are simultaneously integers or half-integers.

Now we present the so-called p-coordinates instead of Λ_{μ_i}, $i = 1, 2, \dots, p$, i.e., the numbers

$$\begin{aligned} &\ell_i = \Lambda_{\mu_i} + p - i, \qquad n = 2p+1, \\ &\ell_i = \Lambda_{\mu i} + p - i, \qquad n = 2p. \end{aligned} \tag{2.3}$$

Since every linear form Λ on $a = \log \Lambda$ is defined in a unique way via its value on a normalized basis element of $\mathcal{H}$, we introduce the number $c = \Lambda - p$ instead of Λ. Thus, the representation $\Pi_{\mu\Lambda}$ is defined by the set of numbers $\ell \equiv (\ell_1, \ell_2, \dots, \ell_{[\frac{n-1}{2}]})$ and the number c. Therefore, we will write $\Pi_{\ell,c}$ instead of $\Pi_{\mu,\Lambda}$. Note that there exists a one-to-one correspondence between the numbers (ℓ, c) and the eigenvalues of the Casimir operators of the group $\mathrm{SO}_0(1, n)$.

The representation $\Pi_{\ell,c}$ of the group $\mathrm{SO}_0(1, 2p+1)$ under the restriction to K can be decomposed into those (and only those) irreducible representations of K whose greatest weights satisfy the inequality

$$\begin{aligned} &\ell_{1,2p+1} > \ell_1 \geqslant \ell_{2,2p+1} > \ell_2 \geqslant \cdots \geqslant \ell_{p,2p+1} > |\ell_p|, \\ &\ell_{i,2p+1} = \Lambda_{\omega_i} + p + 1 - i. \end{aligned} \tag{2.4}$$

In the case of the group $\mathrm{SO}_0(1, 2p)$, the condition (2.4) should be replaced with the condition

$$\begin{aligned} &\ell_{1,2p} \geqslant \ell_1 \geqslant \ell_{2,2p} \geqslant \ell_2 > \cdots \geqslant \ell_{p-1} > |\ell_{p,2p}|, \\ &\ell_{i,2p} = \Lambda_{\omega_i} + p - i. \end{aligned} \tag{2.5}$$

At the same time, every irreducible representation K is included in the decomposition not more than once.

A classification of irreducible[1] representations of the Li algebra of the group $\mathrm{SO}_0(1, n)$ is given in [7, 14, 16, 4, 18]. For convenience we present them here.

Case $\mathbf{SO_0(1, 2p+1)}$.

I. Representations $\mathcal{D}(\ell, c)$, where c is not an integer or half-integer simultaneously with ℓ, or $|c|$ coincides with one of the numbers ℓ_i, or $|c| < |\ell_p|$. The representation $\mathcal{D}(\ell, c)$ contains that (and only that) irreducible representations of the Li algebra $\mathfrak{so}(2p+1)$, the greatest weights of which satisfy the condition

$$\infty > \ell_{1,2p+1} > \ell_1 \geqslant \ell_{2,2p+1} > \ell_2 \geqslant \cdots > \ell_{p-1} \geqslant \ell_{p,2p+1} > |\ell_p|. \tag{2.6}$$

The representations $\mathcal{D}(\ell, c)$ and $\mathcal{D}(\ell_1, \ell_2, \dots, \ell_{p-1}, -\ell_p, c)$ are equivalent.

II. Representations $\mathcal{D}^j(\ell, c)$, $j = 1, 2, \dots, p-1$ (j-series), where c is an integer or half-integer simultaneously with ℓ, and $\ell_j > c > \ell_{j+1}$. The representation $\mathcal{D}^j(\ell, c)$, contains all those (and only those) irreducible representations of the Li algebra $\mathfrak{so}(2p+1)$, the greatest weights of which satisfy the conditions (2.6) and $c \geqslant \ell_{j+1,2p+1}$. All the representations of this class are mutually non-equivalent

III. Finite-dimensional representations F_μ, $\mu = (c, \ell)$, where c are integers or half-integers simultaneously with ℓ, and $c > \ell_1$. The ℓ-coordinates of the greatest weight of F_μ are $\mu = (c, \ell_1, \ell_2, \dots, > \ell_p)$.

Case $\mathbf{SO_0(1, 2p)}$.

[1]Here and below an irreducible representation means a quasi-simple irreducible representation, for which all the Casimir operators are multiples of the identity operator.

I. Representations $\mathcal{D}(\ell, c)$, where c is not an integer or half-integer simultaneously with ℓ, or one of the numbers c or $1-\ell$ coincides with one of ℓ_i, $i = 1, 2, \ldots, p-1$. The representation $\mathcal{D}(\ell, c)$ contains those and only thosee irreducible representations of the Li algebra $\mathfrak{so}(2p)$, the greatest weights of which satisfy the condition

$$\infty > \ell_{1,2p} \geqslant \ell_1 > \ell_{2,2p} \geqslant \cdots > \ell_{p-1,2p} \geqslant \ell_{p-1} > |\ell_{p,2p}|. \tag{2.7}$$

The representations $\mathcal{D}(\ell, c)$ and $\mathcal{D}(\ell, 1-c)$ are equivalent.

II. Representations $\mathcal{D}^j(\ell, c)$, where c and ℓ are integers or half-integers simultaneously with ℓ, and $\ell_j > c > \ell_{j+1}$, $j = 1, 2, \ldots, p-2$, or $\ell_{p-1} > c > 0$, $j = p-1$. The representation $\mathcal{D}^j(\ell, c)$, contains those and only those irreducible representations of the Li algebra of the group $\mathrm{SO}(2p)$, the greatest weights of which satisfy the conditions (2.7) and $c > |\ell_{j+1,2p}|$. All representations of this class are non-equivalent.

III. Representations $\mathcal{D}^{\pm}(\ell, c)$, where $\ell_{p-1} \geqslant 2$, c is an integer or half-integer simultaneously with ℓ, and $\ell_{p-1} > c > 0$. The representation $\mathcal{D}^{\pm}(\ell, c)$ contains those and only those representations of the Li algebra of the group $\mathrm{SO}(2p)$, the greatest weights of which satisfy the conditions (2.7) and $\pm\ell_{p,2p} \geqslant c$.

IV. Finite-dimensional representations F_μ, $\mu = (c, \ell)$, where $c > \ell$, and the ℓ-coordinates of the greatest weights F_μ are $\mu = (c, \ell_1, \ldots, \ell_{p-1})$. We will also denote infinitesimally irreducible representations of the group $\mathrm{SO}_0(1, n)$ that correspond to the aforementioned irreducible representations of the corresponding algebra by the symbols $\mathcal{D}(\ell, c)$, F_μ, $\mathcal{D}^j(\ell, c)$, $\mathcal{D}^{\pm}(\ell, c)$.

Unitary irreducible representations.

Case $\mathbf{SO_0(1, 2p)}$.

I. Major continuous series $\mathcal{D}(\ell, i\rho)$, where ρ is a real number. Under ℓ_ρ, $\rho \geqslant 0$, the representations $\mathcal{D}(\ell, i\rho)$ and $\mathcal{D}(\ell, -i\rho)$ are unitarily equivalent. The spectrum of the restriction of $\mathcal{D}(\ell, i\rho)$ to $\mathrm{SO}(2p+1)$ is defined by (2.6).

II. Additional continuous series $\mathcal{D}(\ell, \sigma)$, $0 < \sigma < t$, where t is an integer, $1 \leqslant t < p-1$, and $\ell_{p-v+1} = v-1$ for $v = 1, 2, \ldots, t$, and the numbers ℓ_i are integers such that $\ell_1 > \ell_2 > \cdots > \ell_{p-t} \geqslant t$. The spectrum of the restriction of $\mathcal{D}(\ell, \sigma)$ to $\mathrm{SO}(2p+1)$ is defined by condition (2.6).

III. j-series of representations, $\mathcal{D}^j(\ell, p-j)$, $j = 1, \ldots, p-1$, where all $\ell_1 > \ell_2 > \cdots > \ell_j \geqslant p-j+1$ are integers, and $\ell_v = p-v$ for $v = j+1, j+2, \ldots, p$. The spectrum of the restriction of $\mathcal{D}^j(\ell, p-j)$ to $\mathrm{SO}(2p+1)$ is defined by the conditions (2.6) and $\ell_{j+1,2p+1} = p-j$.

IV. Degenerate continuous series $\mathcal{D}(\ell, \tau)$ with $\ell_v = p-v$, $v = 1, 2, \ldots, p$, $0 < \tau < p$. The spectrum of the restriction of $\mathcal{D}(\ell, \tau)$ to $\mathrm{SO}(2p+1)$ is defined by (2.6).

Case $\mathbf{SO_0(1, 2p)}$.

I. Major continuous series, $\mathcal{D}(\ell, \frac{1}{2} + i\rho)$, where $\rho > 0$. All the numbers ℓ_v, $v = 1, 2, \ldots, p-1$ are integers or half-integers. The spectrum of the restriction of $\mathcal{D}(\ell, \frac{1}{2} + i\rho)$ to $\mathrm{SO}(2p)$ is defined by (2.7).

II. Additional continuous series $\mathcal{D}(\ell, \sigma)$ with $\frac{1}{2} \leqslant \sigma < t-1$, where t is an integer such that $\sigma \leqslant t \leqslant p-1$. All the numbers $\ell_{2p+1,v}$ are integers, $\ell_{p-v} = v-1$ for $v = 1, 2, \ldots, t$, and $\ell_1 > \ell_2 > \cdots > \ell_{p-t+1} \geqslant t$. The spectrum of the restriction of $\mathcal{D}(\ell, \sigma)$ to $\mathrm{SO}(2p)$ is defined by condition (2.7).

III. j-series of representations, $\mathcal{D}^j(\ell, p-j-1)$, $j = 1, \ldots, p-1$, where all the numbers $\ell_1 > \ell_2 > \cdots > \ell_j \geqslant p-j$ are integers, and $\ell_v = p-v-1$ for $v = j+1, j+2, \ldots, p-1$.

IV. Discrete series $\mathcal{D}^{\pm}(\ell, q)$, where q is an integer or half-integer simultaneously with ℓ_v and belongs to the interval $\ell_{p-1} > q \geqslant \frac{1}{2}$, $\ell_1 > \ell_2 > \cdots > \ell_{p-1}$. The spectrum of the restriction of $\mathcal{D}^{\pm}(\ell, q)$ to $\mathrm{SO}(2p)$ is defined by the conditions (2.7) and $\pm\ell_{p,2p} \geqslant q$. All the mentioned unitary irreducible representations of the group $\mathrm{SO}_0(1, 2p)$ are unitarily non-equivalent.

Now we describe the generators $J_{\mu\nu}$. In the case of the group $\mathrm{SO}_0(1,2p)$ it is sufficient to write the formulas of the action of $J_{2p+1,2p}$, and in the case of the group $\mathrm{SO}_0(1,2p+1)$ those of the action of $J_{2p+1,2p+1}$.

We have

$$J_{2p+1,2p}|\alpha\rangle = \sum_{j=1}^{p} A_{2p,j}(\alpha)|\alpha_{2p}^{+j}\rangle - \sum_{j=1}^{p} A_{2p,j}(\ell_{2p,j-1})|\alpha_{2p}^{-j}\rangle, \tag{2.8}$$

where

$$\begin{aligned} A_{2p,j}(\alpha) = \frac{1}{2}\Bigg[&\prod_{\nu=1}^{p-1}\big((\ell_{v,2p-1}-\tfrac{1}{2})^2-(\ell_{\nu,2p}+\tfrac{1}{2})^2\big) \\ &\times\prod_{\nu=1}^{p}\big((\ell_{v,2p+1}-\tfrac{1}{2})^2-(\ell_{\nu,2p}+\tfrac{1}{2})^2\big) \\ &\times\Bigg(\prod_{\nu\neq j}\big(\ell_{\nu,2p}^2-\ell_{j,2p}^2\big)\big(\ell_{\nu,2p}^2-(\ell_{j,2p}+1)^2\big)\Bigg)^{-1}\Bigg]^{\frac{1}{2}}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} J_{2p+2,2p+1}|\alpha\rangle &= \sum_{j=1}^{p} B_{2p+1,j}(\alpha)|\alpha_{2p+1}^{+j}\rangle \\ &- \sum_{j=1}^{p} B_{2p+1,j}(\ell_{j,2p+1}-1)|\alpha_{2p+1}^{-j}\rangle + C_{2p+2}|\alpha\rangle, \end{aligned}$$

where

$$\begin{aligned} B_{2p+1,j}(\alpha) = \Bigg[&\prod_{\nu=1}^{p}\big(\ell_{\nu,2p}^2-\ell_{j,2p+1}^2\big)\prod_{\nu=1}^{p+1}\big(\ell_{\nu,2p+2}^2-\ell_{j,2p+1}^2\big) \\ &\times\Bigg(\ell_{j,2p+1}^2(4\ell_{j,2p+1}^2-1) \\ &\times\prod_{\nu\neq j}\big(\ell_{\nu,2p+1}^2-\ell_{j,2p+1}^2\big)\big((\ell_{\nu,2p+1}-1)^2-\ell_{j,2p+1}^2\big)\Bigg)^{-1}\Bigg]^{\frac{1}{2}}, \end{aligned}$$

$$C_{2p+2} = \frac{\prod_{\nu=1}^{p}\ell_{\nu,2p}\prod_{\nu}^{p+1}\ell_{2p+2,\nu}}{\prod_{\nu=1}^{p}\ell_{2p+1,\nu}(\ell_{2p+1,\nu-1})}.$$

Here $|\alpha\rangle$ is the Gelfand–Tsetlin scheme [6], that corresponds to the basis element α.

3. Decomposition with respect to $\mathrm{SO}_0(1,n)$

To find out what representations (among the ones presented before) are generated by a unitary irreducible representation of the class $I^{\pm}$ of the group $\mathrm{P}(1,n)$, it is necessary to decompose the corresponding space into irreducible ones with respect to the subgroup $\mathrm{SO}(1,n)$ of this space. This problem was solved for the case $n=3$ in the papers [17, 2, 15].

A unitary irreducible representation of the class $I^{\pm}$ has the form

$$(U^{\varkappa}(a,\Lambda))\varphi(p,\alpha) = \exp\langle ia,p\rangle\sum_{\beta}\mathcal{D}_{\alpha\beta}(r_{p^0})(p,\Lambda)\varphi(\Lambda^{-1}p,\beta), \tag{3.1}$$

where $\mathcal{D}_{\alpha\beta}(r_{p^0})$ are matrix elements of the representation $\mathcal{D}(m_1, m_2, \dots, m_{[\frac{n}{2}]})$ of the group SO(n),

$$r_{p^0}(p,\Lambda) = L(p,p^0) \wedge L(\Lambda^{-1}p,p^0), \qquad L(p,p^0) \in \mathrm{SO}_0(1,n),$$
$$p \in \sigma = \{p\colon \langle p,p\rangle = \varkappa^2\}, \qquad p^0 = (\varkappa, 0, \dots, 0), \qquad \varkappa > 0.$$

The restriction of such a representation to $\mathrm{SO}_0(1,n)$ has the form

$$(U(\Lambda)\varphi)(p,\alpha) = \sum_{\beta} \mathcal{D}_{\alpha\beta}(r_{p^0})(p,\Lambda)\varphi(\Lambda^{-1}p,\beta). \tag{3.1$'$}$$

It is unitary and, therefore, can be decomposed in a unique way into the direct integral of unitary irreducible representations of the group $\mathrm{SO}_0(1,n)$.

Consider the case where $\mathcal{D}$ is the trivial representation. The decomposition of such representation can be performed using the general scheme described by I.M. Gelfand and M.I. Graev.

For each function we introduce its counterpart, the function

$$h(\xi,\alpha) = \int \varphi(p,\alpha)\delta(\langle p,\xi\rangle - \varkappa)\mathrm{d}p,$$

which is defined on the upper half of the cone $\langle \xi,\xi\rangle = 0$.

Herewith, the counterpart of $\varphi(\Lambda^{-1}p,\alpha)$ is the function

$$h_\Lambda(\xi,\alpha) = \int \varphi^{-1}(\Lambda^{-1}p,\alpha)\delta(\langle p,\xi\rangle - \varkappa)\mathrm{d}p = \int \varphi(p,\alpha)\xi(\langle \Lambda p,\xi\rangle - \varkappa)\mathrm{d}p$$
$$= \int \varphi(p,\alpha)\delta(\langle p,\Lambda^{-1}\xi\rangle - \varkappa)\mathrm{d}p = h(\Lambda^{-1}\xi,\alpha).$$

This means that the representation (3.1′) transforms to a quasiregular one,

$$\mathcal{U}_\Lambda h(\xi,\alpha) = h(\Lambda^{-1}\xi,\alpha),$$

in the space of functions on the cone.

In order to obtain a decomposition of such a representation, it is necessary to perform the Mellin transform of the function $h(\xi)$,

$$F_\sigma(\xi,\alpha) = \int_0^\infty h(t\xi,\alpha)t^{-\sigma-1}\mathrm{d}t.$$

These are homogenous functions with respect to ξ in the power σ. Therefore, irreducible representations can be performed on these functions.

Fourier components of the function $\varphi(p)$ are

$$F_\sigma(\xi,\alpha) = \int_0^\infty t^{-\sigma-1}\mathrm{d}t \int \varphi(p,\alpha)\delta(t\langle p,\xi\rangle - \varkappa)\mathrm{d}p.$$

Hence

$$F_\sigma(\xi,\alpha) = \int \varphi(p,\alpha)\langle p,\xi\rangle^\sigma \mathrm{d}p,$$

and we obtain a decomposition of the function $\varphi(p)$ using the Gelfand–Graev transform [5].

In the case $n = 2k+1$,

$$\varphi(p,\alpha) = \frac{(-1)^k}{2(2\pi)^{2k+1}} \int_{a-i\infty}^{a+i\infty} \mathrm{d}\sigma \int F_\sigma(\xi,\alpha)\delta^{(2k)}(\langle p,\xi\rangle - \varkappa)\mathrm{d}\xi.$$

In the case $n = 2k$

$$\varphi(p,\alpha) = \frac{(-1)^k\Gamma(2k)}{(2\pi)^{2k}} \int_{a-i\infty}^{a+i\infty} \mathrm{d}\sigma \int F_\sigma(\xi,\alpha)(\langle p,\xi\rangle - \varkappa)^{2k}\mathrm{d}\xi, \qquad a > 0.$$

Therefore, the decomposition of representation (3.1′) contains all the unitary representations of the major continuous series with $\ell = 0$, each of them appearing once and only once, i.e. the representations of the series $\mathcal{D}(0, i\rho)$ or $\mathcal{D}(0, \frac{1}{2} + i\rho)$.

Now let $\mathcal{D}$ be an arbitrary unitary irreducible representation.

For $\varphi(p, \alpha)$ we introduce the corresponding function

$$\chi(\Lambda\alpha) = \sum_{\beta} \mathcal{D}_{\alpha\beta}(r_{p^0})\varphi(p, \beta),$$

where $p = \Lambda^{-1}p_0$, $p_0 = (\varkappa, 0, \dots, 0)$. Moreover, the action of representation (3.1′) on the presented functions is governed by the formula

$$(U_{\Lambda_0}\chi)(\Lambda, \alpha) = \chi(\Lambda_0^{-1}\Lambda, \alpha). \tag{3.2}$$

The functions $\chi(\Lambda, \alpha)$ are defined on the group $\mathrm{SO}_0(1, n)$, and they are obviously square-integrable. Therefore, (3.2) defines a regular representation. Moreover, the functions χ satisfy the additional condition

$$\chi(k\Lambda, \alpha) = \sum_{\beta} \mathcal{D}_{\alpha\beta}(k)\chi(\Lambda, \beta), \qquad k \in \mathrm{SO}(n). \tag{3.3}$$

This is a consequence of the identity

$$r_{p^0}(k\Lambda, (k\Lambda)^{-1}p^0) = r_{p^0}(k, p^0)r_{p^0}(\Lambda, \Lambda^{-1}p^0).$$

Now, using the results of [8] and condition (3.3) we get the following results.

In the case $\mathrm{SO}_0(1, 2p)$

$$\begin{aligned}\chi(\Lambda, \alpha) = &\sum_{\bar{m}_1 \geqslant \ell_1 > \bar{m}_2} \sum_{\bar{m}_2 \geqslant \ell_2 > \bar{m}_3} \cdots \\ &\times \sum_{\bar{m}_{p-1} \geqslant \ell_{p-1} > |\bar{m}_p|} \int_{-\infty}^{\infty} iP(\ell_1, \dots, \ell_{p-1}, i\rho) \tanh(\pi p) S_p(\mathcal{U}_\Lambda^* \mathcal{U}_\chi) \mathrm{d}p \\ &+ \sum_{\ell_{p-1} > \ell_0 > \frac{1}{2}} \sum_{\bar{m}_1 > \ell_1 > \bar{m}_2} \sum_{\bar{m}_2 > \ell_2 > \bar{m}_3} \cdots \\ &\times \sum_{\bar{m}_{p-1} > \ell_{p-1} > |\bar{m}_p|} P(\ell_0, \ell_1, \dots, \ell_p) S_p'(\mathcal{U}_\Lambda^* \mathcal{U}_\chi),\end{aligned}$$

where

$$\ell_0 = q, \qquad \bar{m}_i = m_i + p - i,$$
$$P(x_1, \dots, x_p) = x_1 x_2 \dots x_p \prod_{1 \leqslant s < r \leqslant p} (x_r^2 - x_s^2),$$

and $S_p'(\dots)$ is the sum of the traces of two discrete series $\mathcal{D}^{\pm}$, see [8].

In the case $\mathrm{SO}_0(1, 2p+1)$,

$$\begin{aligned}\chi(\Lambda, \alpha) = &\sum_{\bar{m}_1 > \ell_1 \geqslant \bar{m}_2} \sum_{\bar{m}_2 > \ell_2 \geqslant \bar{m}_3} \cdots \\ &\times \sum_{\bar{m}_p > \ell_p \geqslant -\bar{m}_p} \int_{-\infty}^{\infty} P(\ell_1, \dots, \ell_p) S_p(\mathcal{U}_\Lambda^* \mathcal{U}_\chi) \mathrm{d}p,\end{aligned}$$

where

$$P(x_1, \dots, x_p) = \prod_{1 < s < r \leqslant p} (x_r^2 - x_s^2), \qquad \bar{m}_i = m_i + p - i + 1.$$

As a result, we state the following assertion.

Theorem 3.1. *The unitary irreducible representation* (3.1) *of the group* $\mathrm{P}(1,n)$ *has a unique decomposition into a direct integral of unitary irreducible representations of the group* $\mathrm{SO}_0(1,n)$*. The decomposition includes once each of the representations* $\mathcal{D}(\ell, \frac{1}{2}+i\rho)$, $\mathcal{D}^{\pm}(\ell,p)$, $\bar{m} \geqslant \ell_1 > \bar{m}_1 \geqslant \ell_2 > \cdots \geqslant \bar{m}_{p-1} \geqslant \ell_{p-1} > |\bar{m}_p|$ *in the case* $n=2p$ *and* $\mathcal{D}(\ell, i\rho)$, $\bar{m}_1 > \ell_1 \geqslant \bar{m}_2 > \ell_2 \geqslant \cdots \geqslant \bar{m}_p > \ell_p \geqslant -\bar{m}_p$ *in the case* $n=2p+1$.

4. Action in the $\mathrm{SO}(1,n)$ basis

In this section, using the Wigner–Eckart theorem [11], we present a construction of the matrix elements of the Lie algebra of the group $\mathrm{P}(1,n)$ in the $\mathrm{SO}_0(1,n)$-basis.

Definition 4.1. An operator P_t that depends on the index and acts in a Hilbert space $\mathcal{H}$ is called a *tensor operator*, if a (unitary or not-unitary) representation T_g of a group G acts in $\mathcal{H}$, such that

$$T_g P_t T_g^{-1} = \sum_{t'} \mathcal{D}^j_{tt'}(g) P_t, \tag{4.1}$$

where $\mathcal{D}^j_{tt'}(g)$ are matrix elements of a representation $\mathcal{D}^j$ of the group G.

Suppose that $\mathcal{D}^j$ is finite-dimensional, and T_g is unitary. Then $\mathcal{H}$ can be decomposed into the direct integral

$$\int \sum_{i_\chi} \mathcal{H}_{\chi, i_\chi} \mathrm{d}\mu(\chi) \tag{4.2}$$

of the Hilbert spaces $\mathcal{H}_{\chi,i_\chi}$ that admit irreducible representations T^χ_g, $\chi = (\ell, \rho)$. Here ℓ is a set of integer parameters, ρ is a set of real parameters. The same values of the index i_χ indicate multiple representations.

In each $\mathcal{H}_{\chi,i_\chi}$ we consider an orthonormal basis $|\chi, i_\chi, m\rangle$ consisting of bases of the subspaces that are the spaces of irreducible representations of the maximal compact subgroup K of the group G. Therefore, the vector-function $F(\chi, i_\chi)$ from $\mathcal{H}$ can be presented in the form

$$F(\chi, i_\chi) = \sum_m f(\chi, i_\chi, m) |\chi, i_\chi, m\rangle. \tag{4.3}$$

To continue, consider a space Φ that is everywhere dense and continuously embedded in $\mathcal{H}$ and consists of the vector-functions $F(\chi, i_\chi)$ such that the functions $f(\chi, i_\chi)$ can be analytically continued by the parameters of the index ρ into the whole complex space. Moreover, suppose that Φ is included in the domains of definition of all operators P_t, and these operators continuously map Φ into itself.

Each operator P_t determines a generalized function of the variables χ_2, i_{χ_2}, m_2, namely

$$P_t F(\chi, i_\chi) = \sum_{m'} \big(f(\chi_2, i_{\chi_2}, m_2) \langle \chi, i_\chi, m' | P_t | \chi_2, i_{\chi_2}, m_2 \rangle \big) |\chi, i_\chi, m'\rangle, \tag{4.4}$$

where $F \in \Phi$ and has the form (4.3).

The expression (4.4) obviously defines a vector-function.

According to the Wigner–Eckart theorem, see [11], the kernels $\langle \chi, i_\chi, m | P_t | \chi_2, i_{\chi_2}, m_2 \rangle$ of the tensor operator P_t are such that

$$\begin{aligned} P_t F(\chi, i_\chi) = \sum_{m'} \sum_{\chi_2, i_{\chi_2}, m_2} & f(\chi_2, i_{\chi_2}, m_2) \\ \times \sum_{v_2} & \langle \chi, m'; j^*, t | \chi_2, v_2, m_2 \rangle \langle \chi, i_\chi | \mathcal{D}^j | \chi_2, i_{\chi_2}, v_2 \rangle |\chi, i_\chi, m'\rangle, \end{aligned} \tag{4.5}$$

where for each fixed χ the summation is performed by those χ_2 for which the decompositions of the tensor product of the representations T_g^χ and $\mathcal{D}_g^j$ (Clebsch–Gordan series) contain the irreducible representations $T_g^{\chi_2}$, and the quantities $\langle\chi, i_\chi|\mathcal{D}^j|\chi_2, i_{\chi_2}, v_2\rangle$, which are the reduced matrix elements, not depending on m, m', m_2. Here $\langle\chi, m'; j^*, t|\chi_2, v_2, m_2\rangle$ are the Clebsch–Gordan coefficients that correspond to the Clebsch–Gordan decomposition (series).

If in (4.1), instead of g, we write a one-parameter differentiable subgroup $g(t)$, differentiate both the left hand-side and the right-hand side with respect to t and consider its value at the point $t = 0$, then we obtain

$$[T_a, P_t] = \sum_{t'} \mathcal{D}_{tt'}^j(a) P_{t'}, \tag{4.6}$$

where a is a tangent vector for $g(t)$ in the point 1, i.e. the element of the Lie algebra of the group G. This expression defines the tensor operator P_t in the infinitesimal form.

Observe that in this case the generators of shifts P_μ, $\mu = 0, 1, 2, \ldots, n$, are transformed by the vector representation $[1, 0, \ldots, 0] \equiv [1]_n$ of the group $\mathrm{SO}_0(1, n)$, and the representation T_g has the form (3.1′).

In view of the previous section, from (4.5) we have

$$\begin{aligned} P_\mu F(\chi) = \sum_{m'} \sum_{\chi_2, m_2} & f(\chi_2, m_2) \\ \times \sum_{v_2} & \langle\chi, m'; j^*, \mu|\chi_2, m_2\rangle\langle\chi|\mathcal{D}^j|\chi_2, v_2\rangle|\chi_2, m'\rangle, \end{aligned} \tag{4.7}$$

where $\chi = [\ell_{n+1}, c]$, $\mathcal{D}^j = \mathcal{D}^{[1]_{n+1}}$, and ℓ_{n+1} are the ℓ-coordinates of the signature of the representation of $\mathrm{SO}_0(1, n)$.

The Clebsch–Gordan series of the tensor product

$$T^{[\ell_{n+1}, C]} \otimes \mathcal{D}^{[1]_{n+1}}$$

has the form

$$\begin{aligned} [\ell_{n+1}, C] \otimes [1]_{n+1} &= \varepsilon_n^\ell [\ell_{n+1}, C] \\ &\oplus \sum_i \left\{[\ell_{n+1}^i, C] \oplus [\ell_{n+1}^{-i}, C]\right\} \oplus [\ell_{n+1}, C+1] \oplus [\ell_{n+1}, C], \end{aligned} \tag{4.8}$$

where

$$\varepsilon_n^\ell = \begin{cases} 0, & n = 2p+1, \\ 1, & n = 2p, \quad \ell_{p,2p+1} > 1, \\ 0, & n = 2p, \quad \ell_{p,2p+1} = 1, \end{cases}$$

see [10].

The coefficients of the Clebsch–Gordan decomposition (4.8) involved in (4.7) are as follows[2].

Case $\mathrm{SO}_0(1, 2p)$:

$$\left\langle \begin{matrix} [\ell_{2p+1}, C] \\ [\ell_{2p}] \end{matrix} \; \begin{matrix} [1] \\ 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p+1}^j, C] \\ [\ell_{2p}] \end{matrix} \right\rangle = \left| \prod_{\nu=1}^p \left(\ell_{\nu,2p}^2 - \ell_{j,2p+1}^2\right) \right|^{\frac{1}{2}}, \quad j = 1, 2, \ldots, p,$$

$$\left\langle \begin{matrix} [\ell_{2p+1}, C] \\ [\ell_{2p}] \end{matrix} \; \begin{matrix} [1] \\ 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p+1}, C] \\ [\ell_{2p}] \end{matrix} \right\rangle = -\left| \prod_{\nu=1}^p \left(\ell_{\nu,2p}^2 - (\ell_{j,2p+1} - 1)^2\right) \right|^{\frac{1}{2}},$$

[2]The Clebsch–Gordan coefficients are not normalized here

$$\left\langle \begin{matrix} [\ell_{2p+1}, C] & [1] \\ [\ell_{2p}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p+1}, C] \\ [\ell_{2p}] \end{matrix} \right\rangle = \prod_{\nu=1}^{p} \ell_{\nu,2p},$$

$$\left\langle \begin{matrix} [\ell_{2p+1}, C] & [1] \\ [\ell_{2p}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p+1}, C+1] \\ [\ell_{2p}] \end{matrix} \right\rangle = \left| \prod_{\nu=1}^{p} \left(\ell_{\nu,2p}^2 - c^2\right) \right|^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} [\ell_{2p+1}, C] & [1] \\ [\ell_{2p}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p+1}, C+1] \\ [\ell_{2p}] \end{matrix} \right\rangle = -\left| \prod_{\nu=1}^{p} \left(\ell_{\nu,2p}^2 - (c-1)^2\right) \right|^{\frac{1}{2}}.$$

Case $\mathrm{SO}_0(1, 2p-1)$:

$$\left\langle \begin{matrix} [\ell_{2p}, C] & [1] \\ [\ell_{2p-1}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p}^{i}, C] \\ [\ell_{2p-1}] \end{matrix} \right\rangle = \left| \prod_{\nu=1}^{p-1} \left((\ell_{\nu,2p-1} - \tfrac{1}{2})^2 - (\ell_{i,2p} + \tfrac{1}{2})^2\right) \right|^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} [\ell_{2p}, C] & [1] \\ [\ell_{2p-1}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p}^{-i}, C] \\ [\ell_{2p}] \end{matrix} \right\rangle = -\left| \prod_{\nu=1}^{p-1} \left((\ell_{\nu,2p-1} - \tfrac{1}{2})^2 - (\ell_{j,2p} - \tfrac{1}{2})^2\right) \right|^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} [\ell_{2p}, C] & [1] \\ [\ell_{2p-1}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p}, C+1] \\ [\ell_{2p-1}] \end{matrix} \right\rangle = \left| \prod_{\nu=1}^{p-1} \left((\ell_{\nu,2p-1} - \tfrac{1}{2})^2 - (c + \tfrac{1}{2})^2\right) \right|^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} [\ell_{2p}, C] & [1] \\ [\ell_{2p-1}] & 0 \end{matrix} \middle| \begin{matrix} [\ell_{2p}, C-1] \\ [\ell_{2p-1}] \end{matrix} \right\rangle = -\left| \prod_{\nu=1}^{p-1} \left((\ell_{\nu,2p-1} - \tfrac{1}{2})^2 - (c - \tfrac{1}{2})^2\right) \right|^{\frac{1}{2}}.$$

In view of these expressions, (4.7) leads to a formula of the action of the operator P_0 of the shift along the time axe.

Case $\mathrm{P}(1, 2p)$:

$$\begin{aligned} P_0 F([\ell_{n+1}, C]) &= P_0 \sum_{2p} f_{2p}(\ell_{2p+1}, C) |\ell_{2p}\rangle_{[\ell_{2p+1},C]} \\ &= \sum_{\ell'_{2p}} |\ell'_{2p}\rangle_{[\ell_{2p+1},C]} \left\{ \sum_{j=1}^{p-1} \left| \prod_{\nu=1}^{p} \left(\ell'^2_{\nu,2p} - \ell^2_{j,2p+1}\right) \right|^{\frac{1}{2}} \right. \\ &\qquad \times \rho_j(\ell_{2p+1}, c) f_{\ell'_{2p}}(\ell^{j}_{2p+1}, c) \\ &\quad - \sum_{j=1}^{p-1} \left| \prod_{\nu=1}^{p} \left(\ell'^2_{\nu,2p} - (\ell_{j,2p+1} - 1)^2\right) \right|^{\frac{1}{2}} \tau_j(\ell_{2p+1}, c) f_{2p}(\ell^{-j}_{2p+1}, c) \\ &\quad + \prod_{\nu=1}^{p} \ell'_{\nu,2p} \sigma(2p+1, c) f_{\ell'_{2p}}(\ell_{2p+1}, c) \\ &\quad + \left| \prod_{\nu=1}^{p} \left(\ell'_{\nu,2p} - c^2\right) \right|^{\frac{1}{2}} \rho_p(\ell_{2p+1}, c) f_{\ell'_{2p}}(\ell_{2p+1}, c+1) \\ &\quad \left. - \left| \prod_{\nu=1}^{p} \left(\ell'_{\nu,2p} - (c-1)^2\right) \right|^{\frac{1}{2}} \tau_p(\ell_{2p+1}, c) f_{\ell'_{2p}}(\ell_{2p+1}, c-1) \right\}. \end{aligned} \tag{4.9}$$

The summation is performed by all admissible signatures of the chain of the subgroups $\mathrm{SO}(2p) \supset \mathrm{SO}(2p-1) \supset \cdots \supset \mathrm{SO}(2)$, and the signature satisfies (2.5).

Case $\mathbf{P(1,2p+1)}$:

$$
\begin{aligned}
P_0F(\ell_{n+1},c) &= P_0\sum_{\ell_{2p+1}} f_{\ell_{2p+1}}(\ell_{2p+2},c)|\ell_{2p+1}\rangle_{[\ell_{2p+2},C]}\\
&= \sum_{\ell'_{2p+1}}|\ell'_{2p+1}\rangle_{[\ell_{2p+2},C]}\Bigg\{\sum_{j=1}^{p}\left|\prod_{\nu=1}^{p}\left((\ell'_{\nu,2p+1}-\tfrac{1}{2})^2-(\ell_{j,2p+2}+\tfrac{1}{2})^2\right)\right|^{\frac{1}{2}}\\
&\quad\times\rho_j(\ell_{2p+2},c)f_{\ell'_{2p+2}}(\ell^{j}_{2p+2},c)\\
&\quad-\sum_{j=1}^{p}\left|\prod_{\nu=1}^{p}\left((\ell'_{\nu,2p+1}-\tfrac{1}{2})^2-(\ell_{j,2p+2}-\tfrac{1}{2})^2\right)\right|^{\frac{1}{2}}\tau_j(\ell_{2p+2},c)\\
&\quad\times f_{\ell'_{2p+1}}(\ell^{-j}_{2p+2},c)+\left|\prod_{\nu=1}^{p}\left((\ell'_{\nu,2p+1}-\tfrac{1}{2})^2-(c+\tfrac{1}{2})^2\right)\right|^{\frac{1}{2}}\\
&\quad\times\rho_{p+1}(\ell_{2p+2},c)f_{\ell'_{2p+1}}(\ell_{2p+2},c+1)\\
&\quad-\left|\prod_{\nu=1}^{p}\left((\ell'_{\nu,2p+1}-\tfrac{1}{2})^2-(c-\tfrac{1}{2})^2\right)\right|^{\frac{1}{2}}\tau_{p+1}(\ell_{2p+2},c)f_{\ell'_{2p+1}}(\ell_{2p+2},c)\Bigg\}.
\end{aligned}
\tag{4.10}
$$

The summation is performed with respect to all admissible signatures of the chain of the subgroups $\mathrm{SO}(2p+1)\supset\mathrm{SO}(2p)\supset\cdots\supset\mathrm{SO}(2)$, and the signature ℓ_{2p+1} satisfies (2.4).

Using the commutation relation

$$[P_0,P_n]=[P_c,[J_{n+1,n},P_0]]=0$$

we obtain the following equations on the coefficients σ, ρ and τ involved in (4.14):

$$(\ell_{2p+1,j}-1)\sigma(\ell_{2p+1})=(\ell_{2p+1,j+1})\sigma(\ell'_{2p+1}),\tag{4.11}$$

$$
\begin{aligned}
&(\ell_{i,2p+1}-\ell_{j,2p+1}+1)\rho_j(\ell^{i}_{2p+1})\rho_i(\ell_{2p+1})\\
&\qquad=(\ell_{i,2p+1}-\ell_{j,2p+1}-1)\rho_i(\ell^{j}_{2p+1})\rho_j(\ell_{2p+1}),\\
&(\ell_{i,2p+1}+\ell_{j,2p+1})\tau_i(\ell^{j}_{2p+1})\rho_j(\ell_{2p+1})\\
&\qquad=(\ell_{i,2p+1}+\ell_{j,2p+1}-2)\rho_j(\ell^{-i}_{2p+1})\tau_j(\ell_{2p+1}),\\
&(\ell_{i,2p+1}-\ell_{j,2p+1}+1)\tau_i(\ell^{-j}_{2p+1})\tau_j(\ell_{2p+1})\\
&\qquad=(\ell_{i,2p+1}-\ell_{j,2p+1}-1)\tau_j(\ell^{-i}_{2p+1})\tau_i(\ell_{2p+1}),
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
&(\ell_{j,2p+1}+1)\sigma(\ell^{j}_{2p+1})\rho_j(\ell_{2p+1})=(\ell_{j,2p+1}-1)\rho_j(\ell_{2p+1})\sigma(\ell_{2p+1}),\\
&\ell_{j,2p+1}\tau_j(\ell_{2p+1})\sigma(\ell_{2p+1})=(\ell_{i,2p+1}-2)\sigma(\ell^{-j}_{2p+1}\tau_j)(\ell_{2p+1}),
\end{aligned}
\tag{4.13}
$$

$$\begin{aligned}\sum_i \frac{2\ell_{i,2p+1}+1}{\ell_{k,2p}^2-\ell_{k,2p+1}^2}\rho_i(\ell_{2p+1})\tau_i(\ell_{2p+1}^i)\prod_{\nu=1}^{p}(\ell_{\nu,2p}^2-\ell_{i,2p+1}^2)\\ +\frac{2\ell_{i,2p+1}-3}{(\ell_{i,2p+1}-1)^2-\ell_{k,2p}^2}\rho_j(\ell_{2p+1}^{-i})\tau_i(\ell_{2p+1})\\ \times\prod_{\nu=1}^{p}\left(\ell_{\nu,2p}^2-(\ell_{i,2p+1}-1)^2\right)\\ =\frac{1}{\ell_{k,2p}^2}\prod_{\nu}^{p}\ell_{\nu,2p}^2\sigma^2(\ell_{2p+1}).\end{aligned}\tag{4.14}$$

From (4.11) it follows that the expression $\ell_{j,2p+1}(\ell_{j,2p+1}-1)\sigma(\ell_{2p+1})$ does not depend of $\ell_{j,2p+1}$ for $j=1,2,\dots,p-1$, hence we can write

$$\sigma(\ell_{2p+1})\prod_{\nu=1}^{p-1}\ell_{\nu,2p+1}(\ell_{\nu,2p+1}-1)c(c-1)=\sigma_0(c),$$

where $\sigma_0(c)$ is a periodic function depending only on c with a period I. Therefore, if $\sigma_0(c)\neq 0$ then

$$\sigma(\ell_{2p+1})=\frac{\sigma_0(c)}{\prod\limits_{\nu=1}^{p-1}\ell_{\nu,2p+1}(\ell_{\nu,2p+1}-1)c(c-1)}.$$

If $\sigma_0(c)=0$ then $\sigma(\ell_{2p+1})=0$ or $c=0$. Hence the "diagonal" summands in (4.11) vanish, and the solutions of equations (4.11)–(4.14) are

$$\begin{aligned}\rho_j(\ell_{2p+1})=\varkappa\left|\frac{\prod\limits_{\nu=1}^{p}\left(\ell_{\nu,2p+2}^2-\ell_{2p+1,j}^2\right)}{\prod\limits_{\nu\neq j}^{p}(\ell_{2p+1,\nu}^2-\ell_{2p+1,j}^2)((\ell_{2p+1,\nu}-1)^2-\ell_{2p+1,j}^2)}\right|^{\frac{1}{2}}\\ \times\left|\ell_{2p+1,j}^2(4\ell_{2p+1,j}^2-1)\right|^{-\frac{1}{2}},\qquad j=1,2,\dots,p-1,\end{aligned}\tag{4.15}$$

$$\rho_p(\ell_{2p+1},c)=\varkappa\left|\frac{\prod\limits_{\nu=1}^{p}\left(\ell_{\nu,2p+2}^2-c^2\right)}{c^2(4c^2-1)\prod\limits_{\nu=1}^{p-1}(\ell_{\nu,2p+1}^2-c^2)((\ell_{\nu,2p+1}-1)^2-c^2)}\right|^{\frac{1}{2}},$$

$$\sigma(\ell_{2p+1})=\frac{\prod\limits_{\nu=1}^{p+1}\ell_{\nu,2p+2}}{\prod\limits_{\nu=1}^{p-1}\ell_{\nu,2p+2}(\ell_{\nu,2p+1}-1)c(c-1)},$$

$$\tau_j(\ell_{2p+1},c)=\rho_j(\ell_{2p+1}^{-j},c),$$

$$\tau_p(\ell_{2p+1},c)=\rho_p(\ell_{2p+1},c-1).$$

One can easily obtain expressions analogous to (4.11)–(4.14) for the group $\mathrm{P}(1,2p+1)$. We present here only an explicit solution of these equations,

$$\rho_j(\ell_{2p+2},c)=\frac{M}{2}\left|\frac{\prod\limits_{\nu=1}^{p}\left((\ell_{\nu,2p+3}-\frac{1}{2})^2-(\ell_{j,2p+2}+\frac{1}{2})^2\right)}{\prod\limits_{\nu\neq i}(\ell_{\nu,2p+2}^2-\ell_{j,2p+2}^2)(\ell_{\nu,2p+2}^2-(\ell_{j,2p+1}-1)^2)}\right|^{\frac{1}{2}},$$

$$\begin{gathered} j = 1, 2, \dots, p, \\ \tau_j(\ell_{2p+2}, c) = \rho_j(\ell_{2p+2}^{-j}, c), \\ \tau_{p+1}(\ell_{2p+2}, c) = \rho_{p+1}(\ell_{2p+2}, c-1), \end{gathered} \tag{4.16}$$

$$\rho_{p+1}(\ell_{2p+2}, c) = \frac{M}{2}\left|\frac{\prod\limits_{\nu=1}^{p}\left((\ell_{\nu,2p+3}-\frac{1}{2})^2-(c+\frac{1}{2})^2\right)}{\prod\limits_{\nu=1}^{p}(\ell_{\nu,2p+2}^2-c^2)(\ell_{\nu,2p+2}^2-(c+1)^2)}\right|^{\frac{1}{2}}.$$

Thus, the action of the shift operator P_0 in the case of the groups $\mathrm{P}(1,2p)$ and $\mathrm{P}(1,2p+1)$ is governed by the formulas (4.9) and (4.10) correspondingly, where the functions σ, ρ and τ have form (4.15) and (4.16).

The parameters $\{\ell_{1,2p+2}, \ell_{2,2p+2}, \dots, \ell_{p,2p+2}, \varkappa\}$ and $\{\ell_{1,2p+3}, \ell_{2,2p+3}, \dots, \ell_{p,2p+3}, \mu\}$ characterize the representations of the generalized Poincaré groups $\mathrm{P}(1,2p)$ and $\mathrm{P}(1,2p{+}1)$, correspondingly.

Formulas for actions of all the remaining shift operators are consequences of the commutation relations of P_0 and those operators of the subalgebra of $\mathfrak{so}_0(1,n)$ that act according to (2.8) and (2.9) in the $\mathrm{SO}_0(1,n)$-basis.

Consider the Poincaré group $\mathrm{P}(1,3)$ and the inhomogeneous de Sitter group $\mathrm{P}(1,4)$, which are important for physics.

Case $\mathbf{P(1,3)}$:

$$\begin{aligned} P_0 \sum_{j_0,m} f_{j,m}(j_0,c)|j,m\rangle_{j_0,c} &= \sum_{j',m'} |j',m'\rangle_{j_0,c} \\ \times\Big\{ \left|(j'+\tfrac{1}{2})^2-(j_0+\tfrac{1}{2})^2\right|^{\frac{1}{2}} &\rho_1(j_0,c) f_{j',m'}(j_0+1,c) \\ -\left|(j'+\tfrac{1}{2})^2-(j_0-\tfrac{1}{2})^2\right|^{\frac{1}{2}} &\rho_1(j_0-1,c) f_{j',m'}(j_0-1,c) \\ +\left|(j'+\tfrac{1}{2})^2-(c+\tfrac{1}{2})^2\right|^{\frac{1}{2}} &\rho_c(j_0-1,c) f_{j',m'}(j_0,c+1) \\ +\left|(c-\tfrac{1}{2})^2-(j'+\tfrac{1}{2})^2\right| &\rho_c(j_0,c-1) f_{j',m'}(j_0,c-1)\Big\}, \end{aligned} \tag{4.17}$$

where

$$\rho_1(j_0,c) = \frac{1}{2}\mu\left|\frac{(s-\frac{1}{2})^2-(j_0+\frac{1}{2})^2}{(c^2-j_0^2)(c^2-(j_0+1)^2)}\right|^{\frac{1}{2}},$$

$$\rho_c(j_0,c) = \frac{1}{2}\mu\left|\frac{(s-\frac{1}{2})^2-(c+\frac{1}{2})^2}{(j_0^2-c^2)(j_0^2-(c+1)^2)}\right|^{\frac{1}{2}}.$$

The action of the other operators P_a can be presented now utilizing the relationship

$$P_a = i(P_0 J_{0a} - J_{0a} P_0),$$

where

$$\begin{aligned} J_{01}\sum_{j,m} f_{j,m}(j_0,c)|j,m\rangle_{j_0,c} &= -\frac{i}{2}\sum_{j,m} f_{j,m}(j_a,c) \\ \times\Big\{ -\sqrt{(j-m)(j+m+1)}&\frac{ij_0c}{(j+1)j}|j,m+1\rangle_{j_0,c} \\ +\frac{i}{j}\left(\frac{(j^2-j_0^2)(j^2-c^2)(j-m)(j-m-1)}{4j^2-1}\right)^{\frac{1}{2}} &|j-1,m+1\rangle_{j_0,c} \end{aligned}$$

$$+\frac{i}{j+1}\left(\frac{\big((j+1)^2-j_0^2\big)\big((j+1)^2-c^2\big)(j+m+1)(j+m+2)}{4(j+1)^2-1}\right)^{\frac{1}{2}}$$

$$\times|j+1,m+1\rangle_{j_0,c}$$

$$-\frac{i}{j}\left(\frac{(j^2-j_0^2)(j^2-c^2)(j+m)(m+j-1)}{4j^2-1}\right)^{\frac{1}{2}}|j-1,m-1\rangle_{j_0,c}$$

$$-\frac{ij_0c}{j(j+1)}\big((m+j)(j-m+1)\big)^{\frac{1}{2}}|j,m-1\rangle_{j_0,c}$$

$$-\frac{i}{j+1}\left(\frac{(j-m+1)(j-m+2)\big((j+1)^2-j_0^2\big)\big((j+1)^2-c^2\big)}{4(j+1)^2-1}\right)^{\frac{1}{2}}$$

$$\times|j+1,m-1\rangle_{j_0,c}\Big\},$$

$$J_{02}\sum_{j,m}f_{j,m}(j_0,c)|j,m\rangle_{j_0,c}=\frac{1}{2}\sum_{j,m}f_{j,m}(j_0,c)$$

$$\times\Big\{\frac{i}{j}\left(\frac{(j^2-j_0^2)(j^2-c^2)(m+j)(m-j-1)}{4j^2-1}\right)^{\frac{1}{2}}|j-1,m-1\rangle_{j_0,c}$$

$$-\frac{ij_0c}{j(j+1)}\big((m+j)(j-m+1)\big)^{\frac{1}{2}}|j,m-1\rangle_{j_0,c}$$

$$-\frac{i}{j+1}\left(\frac{\big((j+1)^2-j_0^2\big)\big((j+1)^2-c^2\big)(j-m+1)(j-m+2)}{4(j+1)^2-1}\right)^{\frac{1}{2}}$$

$$\times|j+1,m-1\rangle_{j_0,c}$$

$$-\frac{ij_0c}{j(j+1)}\big((j-m)(j+m+1)\big)^{\frac{1}{2}}|j,m+1\rangle_{j_0,c}$$

$$=-\frac{i}{j}\left(\frac{(j^2-j_0^2)(j^2-c^2)(j-m)(j-m-1)}{4j^2-1}\right)^{\frac{1}{2}}|j-1,m+1\rangle_{j_0,c}$$

$$-\frac{i}{j+1}\left(\frac{(j+m-1)(j+m+2)\big((j+1)^2-j_0^2\big)\big((j+1)^2-c^2\big)}{4(j+1)^2-1}\right)^{\frac{1}{2}}$$

$$\times|j+1,m+1\rangle_{j_0,c}$$

$$-\frac{i}{j}\left(\frac{(j^2-j_0^2)(j^2-c^2)(m+j)(m+j-1)}{4j^2-1}\right)^{\frac{1}{2}}|j-1,m-1\rangle_{j_0,c}\Big\},$$

$$J_{03}\sum_{j,m}f_{j,m}(j_0,c)|j,m\rangle_{j_0,c}=\sum_{j,m}f_{j,m}(j_0,c)$$

$$\times\Big\{\frac{1}{j}\left(\frac{(j^2-j_0^2)(j^2-c^2)(j^2-m^2)}{4j^2-1}\right)^{\frac{1}{2}}|j-2,m\rangle_{j_0,c}$$

$$-\frac{j_0c}{j(j+1)}im|j,m\rangle_{j_0,c}$$

$$+\frac{i}{j+1}\left(\frac{\big((j+1)^2-j_0^2\big)\big((j+1)^2-c^2\big)\big((j+1)^2-m^2\big)}{(2j+3)(2j+1)}\right)^{\frac{1}{2}}$$

$$\times|j+1,m\rangle_{j_0,c}\Big\}.$$

The summation in (4.17) is performed by all j' and m' that the signatures $\{j_0 \pm 1, c\}$, $\{j_0, c \pm 1\}$ admit.

Observe, that the coefficients at the functions $f_{j',m}$ coincide with the matrix elements of the operator P_0 formally obtained in [1], if we take into account the relationship

$$(j_0,\lambda)_{\text{chakr}} \to (j_0, ic); \qquad s_{\text{chakr}} \to s$$

between the parameters of the representations.

Case $\mathbf{P(1,4)}$:

$$\begin{aligned} P_0 \sum_{M,j,m} f_{M,j,m}(\ell,c)|M,j,m\rangle_{\ell,c} &= \sum_{M',j',m'} |M',j',m'\rangle_{j_0,c} \\ \times\Big\{ \big|\big((m_1+1)^2-\ell^2\big)(m^2-\ell^2)\big|^{\frac12} \rho_1(\ell,c)|f_{M',j',m'}(\ell+1,c) \\ - \big|(m_1+1)^2-(\ell-1)^2\big|^{\frac12} \rho_1(\ell-1,c) f_{M',j',m'}(\ell-1,c) \\ + (m_1+1)m_2\sigma(\ell,c) f_{M',j',m'}(\ell,c) \\ + \big|\big((m_1+1)^2-c^2\big)\big(m_2^2-c^2\big)\big|^{\frac12} \rho_c(\ell,c) f_{M',j',m'}(\ell,c+1) \\ - \big|\big((m_1+1)^2-(c-1)^2\big)\big(m_2^2-(c-1)^2\big)\big|^{\frac12} \\ \times \rho_c(\ell,c-1) f_{M',j',m'}(\ell,c-1)\Big\}, \end{aligned} \tag{4.18}$$

where

$$\rho_1 = \varkappa\left|\frac{(k_1^2-\ell^2)(k_2^2-\ell^2)}{\ell^2(4\ell^2-1)(c^2-\ell^2)((c-1)^2-\ell^2)}\right|^{\frac12},$$

$$\rho_c = \varkappa\left|\frac{(k_1^2-c^2)(k_2^2-c^2)}{c^2(4c^2-1)(\ell^2-c^2)((\ell_1-1)^2-c^2)}\right|^{\frac12},$$

$$\sigma(\ell,c) = \frac{\varkappa k_1, k_2}{\ell(\ell-1)c(c-1)}.$$

The summation in (4.18) is performed in all M', j' and m' admitted with respect to the signatures $\{\ell,c\}$, $\{\ell\pm1,c\}$, $\{\ell,c\pm1\}$.

The parameters

$$\{k_1,k_2,\varkappa\} \to \{\ell,c\} \to \{m_1,m_2\} \to j \to m$$

define the representations of the groups in the chain $\mathrm{P}(1,4) \supset \mathrm{SO}_0(1,4) \supset \mathrm{SO}(4) \supset \mathrm{SO}(3) \supset \mathrm{SO}(2)$, respectively.

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