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V-SETS AND THE PROPERTY (VLD) IN BANACH SPACES

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Abstract. In this paper, we study the notion of V-sets in Banach spaces and Banach lattices, and we give some characterizations of it in terms of sequences. As an application, we establish new properties of unconditionally converging operators and 1-Schur property in Banach lattices. Next, by introducing the concept of the property (VLD) in Banach spaces, we investigate the Dunford-Pettis completely continuous property of unconditionally converging operator. Finally, we derive the relationships between the property (VLD) and the relatively compact Dunford-Pettis property (resp., the Pelczynski's property (V)), and we deduce some examples of Banach spaces with the property (VLD).

У цій роботі ми вивчаємо поняття V-множин у банахових просторах та банахових ґратках та даємо деякі їхні характеристики у термінах послідовностей. Як застосування, ми встановлюємо нові властивості безумовно збіжних операторів і властивість 1-Шура в банахових ґратках. Далі, вводячи поняття (VLD) властивості у банахових просторах, ми досліджуємо властивість цілковитої неперервності Данфорда-Петтіса безумовно збіжного оператора. Нарешті, ми виводимо зв'язки між (VLD) властивістю і властивістю відносної компактності Данфорда-Петтіса (відповідно, властивістю Пельчинського (V)), і виводимо деякі приклади банахових просторів із (VLD) властивістю.

1. Introduction

Recall that a series $\sum x_n$ in a Banach space X is weakly unconditionally converging if and only if $\sum |f(x_n)| < \infty$ for each $f \in X'$. Also, a series $\sum x_n$ is unconditionally converging if and only if $\sum x_{\sigma(n)}$ converges in the norm topology of X for every permutation σ of the natural numbers.

Note that a norm bounded subset A of a topological dual Banach space X' is called a V-subset if every weakly unconditionally converging series $\sum x_n$ in X converges uniformly to zero on A, that is,

$$\lim_{n\to\infty} \sup_{f\in A} f(x_n) = 0.$$

A norm bounded subset A of a Banach space X is said to be Dunford-Pettis set, if every weakly null sequence (f_n) in X' converges uniformly to zero on A, that is,

$$\lim_{n\to\infty} \sup_{x\in A} f_n(x) = 0.$$

Recall from [14] that a norm bounded subset A of a topological dual Banach space X' is an L-Dunford-Pettis if every weakly null sequence (x_n), which is a Dunford-Pettis subset of X converges uniformly to zero on A, that is,

$$\lim_{n\to\infty} \sup_{f\in A} f(x_n) = 0.$$

This paper is devoted to study V-sets in terms of sequences in Banach spaces and Banach lattices, several equivalent ways of presenting V-sets are given. Next, we establish a characterization of unconditionally converging operator from a Banach space into a

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Banach lattice in terms of V-sets (Theorem 4.4). Many authors are interested in the study of the unconditionally converging operators and their relations with other operators. In 1962, A. Pelczynski in his fundamental paper [13] studied Banach spaces on which every unconditionally converging operator is weakly compact. Later, Banach spaces on which every unconditionally converging operator is completely continuous have been studied by Joe Howard [10]. After that, Saab and Smith in their paper [17] looked at spaces on which every unconditionally converging operator is weakly completely continuous. Recently, Dehghani and Moshtaghioun ([5], Theorem 2.12) proved that the Banach space X has the 1-Schur property if and only if every operator $T : X \to Y$ is unconditionally converging for all Banach spaces Y. In the present paper, we continue on this path, and we are looking for the relationships between unconditionally converging operators and Dunford-Pettis completely continuous operators. Note that there is an unconditionally converging operator which is not Dunford-Pettis completely continuous. In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is unconditionally converging operator because $L^1[0,1]$ has the 1-Schur property (see Example 2.5 of [5] and Proposition 4.5), but it fail to be Dunford-Pettis completely continuous operator because $L^1[0,1]$ does not have the relatively compact Dunford-Pettis property (see [9], Proposition 3.3 and [15], Corollary 2.7). By introducing the property (VLD) in Banach space which is shared by those Banach space whose V-sets of his topological dual are L-Dunford-Pettis (Definition 5.1), we show that a Banach space X has the property (VLD) if and only if for any Banach space Y, every unconditionally converging operator $T : X \to Y$ is Dunford-Pettis completely continuous (Theorem 5.2). As an application, we get the relations between the property (VLD) and the relatively compact Dunford-Pettis property (resp., the property (V)) in Banach spaces. Finally, we find some examples about Banach spaces with the property (VLD). The notions of V-sets, L-Dunford-Pettis sets, unconditionally converging operators and Dunford-Pettis completely continuous operators play a consistent and important role in this study.

2. Definitions and notations

Recall from [6] that an operator T from a Banach space X into a another Banach space Y is called unconditionally converging if it maps weakly unconditionally converging series in X into unconditionally converging series in Y. An operator T from a Banach space X into a another Banach space Y is called Dunford-Pettis completely continuous (DPcc for short) if each weakly null sequence (x_n), which is a Dunford-Pettis set in X, we have $\|T(x_n)\| \to 0$, as $n \to \infty$ [18].

A Banach space X has

- the 1-Schur property if every weakly unconditionally converging (weakly 1-summable) series $\sum x_n$ in X, we have $\|x_n\| \to 0$, as $n \to \infty$ [5].
- the relatively compact Dunford-Pettis property (DPrcP for short) if every weakly null sequence, which is a Dunford-Pettis set in X, is norm null [18].
- the L-Dunford-Pettis property if every L-Dunford-Pettis set in X' is relatively weakly compact [14].
- the Pelczynski's property (V) (property (V) for short) if every V-set of X' is relatively weakly compact, equivalently, if for any Banach space Y, every unconditionally converging operator T from X into Y is weakly compact [13].

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. The sequence (x_n) of a Banach lattice E is disjoint if $|x_n| \wedge |x_m| = 0$ for $n \neq m$ (we denote by $x_n \perp x_m$). Recall that a nonzero element x of a vector lattice G is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice G is discrete, if it admits a complete disjoint system of discrete elements. We will use the term operator $T : X \longrightarrow Y$ between two Banach spaces to mean a bounded linear mapping, its dual operator T' is defined from Y' into X' by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and for each $x \in X$. The reader is referred to Aliprantis-Burkinshaw [1], Diestel [7] and Dunford-Schwartz [8] for undefined notation and terminology.

3. V-sets in a topological dual Banach space

We start this article by some characterizations of V-set in terms of sequences.

Proposition 3.1. *Let X be a Banach space and let A be a norm bounded subset of X'. The following statements are equivalent:*

1. *A is a V-set in X'.*
2. *For every sequence (f_n) in A and for every weakly unconditionally converging series $\sum x_n$ in X, we have $f_n(x_n) \to 0$ as $n \to \infty$.*

Proof. (2) $\Rightarrow$ (1) Assume by way of contradiction that A is not a V-set in X'. Then, there exists a weakly unconditionally converging series $\sum x_n$ in X such that $\sup_{f\in A} |f(x_n)| > \varepsilon > 0$ for some $\varepsilon > 0$ and each n. Hence, for every n there exists some f_n in A such that $|f_n(x_n)| > \varepsilon$, which is impossible from our hypothesis (2). This proves that A is a V-set in X'.

(1) $\Rightarrow$ (2) Let (f_n) be a sequence in A and $\sum x_n$ be a weakly unconditionally converging series in X. Since

$$|f_n(x_n)| \leq \sup_{f\in A} |f(x_n)|,$$

for every n, and A is a V-set in X', then $f_n(x_n) \to 0$ as $n \to \infty$. This finishes the proof. □

As a simple consequence, we obtain:

Corollary 3.2. *Let X be a Banach space and let (f_n) be a norm bounded sequence in X'. The following statements are equivalent:*

1. *The subset $\{f_n : n \in \mathbb{N}\}$ is a V-set in X';*
2. *For every sequence (g_m) of $\{f_n : n \in \mathbb{N}\}$ and for every weakly unconditionally converging series $\sum x_n$ in X, we have $g_m(x_m) \to 0$ as $m \to \infty$;*
3. *For every unconditionally converging series $\sum x_n$ in X, we have $f_n(x_n) \to 0$ as $n \to \infty$.*

In the next result, we establish the following characterization.

Proposition 3.3. *Let X be a Banach space. Then the following statements are equivalent:*

1. *every relatively weakly compact set in X' is a V-set;*
2. *every weakly null sequence (f_n) in X', the subset $\{f_n : n \in \mathbb{N}\}$ is a V-set;*
3. *for every unconditionally converging series $\sum x_n$ in X and every weakly null sequence (f_n) in X', we have $f_n(x_n) \to 0$ as $n \to \infty$.*

Proof. (1) $\Rightarrow$ (2) Let (f_n) be a weakly null sequence in X', so the subset $\{f_n : n \in \mathbb{N}\}$ is relatively weakly compact in X', and by our hypothesis we see that $\{f_n : n \in \mathbb{N}\}$ is V-set.

(2) $\Rightarrow$ (3) Let $\sum x_n$ be a weakly unconditionally converging series in X and (f_n) be a weakly null sequence in X', then the subset $\{f_n : n \in \mathbb{N}\}$ is V-set in X', and from Proposition 3.1 we deduce that $f_n(x_n) \to 0$ as $n \to \infty$.

(3) $\Rightarrow$ (1) Let K be a relatively weakly compact subset of X' and let $\sum x_n$ be a weakly unconditionally converging series in X, we show that (x_n) converges uniformly on K. Otherwise, there exist a sequence (f_n) of K and some $\varepsilon > 0$ such that $|f_n(x_n)| > \varepsilon$ for all n. As K is relatively weakly compact, there exists a subsequence (f_{n_m}) of (f_n) such that (f_{n_m}) converges weakly to f in X'. By our hypothesis, we get that

$$0 < \varepsilon < |f_{n_m}(x_{n_m})| \leq |(f_{n_m} - f)(x_{n_m})| + |f(x_{n_m})| \to 0$$

as $m \to \infty$, which is impossible, and this completes the proof. □

Now, we are in a position to give the following result about the order interval in Banach lattices.

Proposition 3.4. *Let T be an operator from a Banach space X into a Banach lattice F and let $f \in (F')^+$. The following statements are equivalent:*

(1) *$T'([-f, f])$ is a V-set in X'.*
(2) *For weakly unconditionally converging series $\sum x_n$ in X, $f(|T(x_n)|) \to 0$ as $n \to \infty$.*

Proof. Let (x_n) be a weakly unconditionally converging series $\sum x_n$ in X, then the result follows from the equality:

$$f(|T(x_n)|) = \sup \{|g(x_n)| : g \in T'([-f, f])\},$$

for every $f \in (F')^+$ and every $n \in \mathbb{N}$. □

As a consequences, we derive

Corollary 3.5. *For an operator T from a Banach space X into a Banach lattice F, the following statements are equivalent:*

(1) *$T'([-f, f])$ is a V-set in X', for each $f \in (F')^+$;*
(2) *for weakly unconditionally converging series $\sum x_n$ in X, $|T(x_n)| \to 0$ weakly in F.*

In particular, If we put $T = Id_E : E \to E$, we deduce

Corollary 3.6. *Let E be a Banach lattice, the following statements are equivalent:*

(1) *For each $f \in (E')^+$, $[-f, f]$ is a V-set in E'.*
(2) *For weakly unconditionally converging series $\sum x_n$ in E, $|x_n| \to 0$ weakly in E.*

Recall that the lattice operations in a Banach lattice E are weakly sequentially continuous whenever $x_n \to 0$ weakly in E implies $|x_n| \to 0$ weakly in E.

Remark 3.7. Let E be a Banach lattice.

(1) If the lattice operations of E are weakly sequentially continuous, then each order interval of E' is a V-set (see Corollary 3.6).
(2) It follows from Corollary of Chen-Wickstead [4] we see that if E is discrete with order continuous norm, then each order interval of E' is a V-set.

4. Some characterizations of unconditionally converging operators

In order to prove the next Theorem, we need the following Lemma.

Lemma 4.1. *Let E be a Banach lattice, and let (g_n) be a norm bounded sequence of E^+. Then the sequence defined for $n \geq 2$ by*

$$f_n = \left(g_n - 4^n \sum_{i=1}^{n-1} g_i - 2^{-n} \sum_{i=1}^{\infty} 2^{-i} g_i \right)^+,$$

is a disjoint sequence.

Proof. Let $n > m \geq 2$, then

$$0 \leq f_n \leq (g_n - 4^n g_m)^+,$$

and

$$\begin{aligned} 0 \leq 4^n f_m &\leq 4^n (g_m - 4^{-n} g_n)^+ \\ &= (4^n g_m - g_n)^+ \\ &= (g_n - 4^n g_m)^-. \end{aligned}$$

Since $(g_n - 4^n g_m)^+ \perp (g_n - 4^n g_m)^-$, we see that $f_n \perp f_m$, and we are done. □

Now, we establish some equivalent condition for $T'(A)$ to be V-set where A is a norm bounded solid subset of F' and $T : X \to F$ is an operator from a Banach space into a Banach lattice.

Theorem 4.2. *Let T be an operator from a Banach space X into a Banach lattice F, and let A be a norm bounded solid subset of F'. The following statements are equivalent:*

(1) *$T'(A)$ is a V-set in X';*
(2) *$T'([-f, f])$ and $\{T'(f_n), n \in \mathbb{N}\}$ are V-sets in X', for each $f \in A^+$ and for each disjoint sequence $(f_n) \subset A^+ = A \cap (F')^+$.*

Proof. (1) ⇒ (2) Let $f \in A^+$ and (f_n) be a disjoint sequence in A^+, then $T'([-f, f])$ and $\{T'(f_n), n \in N\}$ are two subsets of $T'(A)$, and by our hypothesis we see that $T'([-f, f])$ and $\{T'(f_n), n \in N\}$ are V-sets in X'.

(2) ⇒ (1) Let $\sum x_n$ be a weakly unconditionally converging series in X. To finish the proof, we have to show that $\sup_{g\in A} |T'(g)(x_n)| \to 0$ as $n \to \infty$. Assume by way of contradiction that $\sup_{g\in A} |T'(g)(x_n)|$ does not converge to 0 as $n \to \infty$. So there exists some $\varepsilon > 0$ such that $\sup_{g\in A} |T'(g)(x_n)| > \varepsilon$ for each n. Hence, there exists $g_n \in A^+$ such that $g_n(|T(x_n)|) > \varepsilon$ for all natural number n. Now, by our hypothesis we have that $T'([g, g])$ is a V-set in E' for every $g \in A^+$. By Proposition 3.4 we see that $g(|T(x_n)|) \to 0$ as $n \to \infty$ for every $g \in A^+$. Let $n_1 = 1$, since $g_{n_1}(|T(x_n)|) \to 0$ as $n \to \infty$, there exists some natural number n_2 such that $n_2 > n_1 = 1$ and $g_{n_1}(|T(x_{n_2})|) < \frac{\varepsilon}{2^{2\times 2+2}}$. Also, because $\sum_{k=1}^{2} g_{n_k}(|T(x_n)|) \to 0$ as $n \to \infty$, there exists some natural number n_3 such that $n_3 > n_2 > n_1 = 1$ and $\sum_{k=1}^{2} g_{n_k}(|T(x_{n_3})|) < \frac{\varepsilon}{2^{2\times 3+2}}$. By induction, we get a strictly increasing subsequence (n_k) of $\mathbb{N}$ such that

$$(\sum_{k=1}^{m-1} g_{n_k})(|T(x_{n_m})|) < \frac{\varepsilon}{2^{2m+2}}$$

for all $m \geq 2$. Now, let

$$h = \sum_{k=1}^{\infty} 2^{-k} g_{n_k}$$

and

$$f_m = (g_{n_m} - 4^m \sum_{k=1}^{m-1} g_{n_k} - 2^{-m} h)^+$$

for all $m \geq 2$. So by Lemma 4.1, we see that (f_m) is a disjoint sequence in $(F')^+$, as $0 \leq f_m \leq g_{n_m}$, $g_{n_m} \in A$ and A is a solid subset of F' then, $f_m \in A^+$. Hence, we have

$$\begin{aligned} f_m(|T(x_{n_m})|) &= (g_{n_m} - 4^m \sum_{k=1}^{m-1} g_{n_k} - 2^{-m} h)^+(|T(x_{n_m})|) \\ &\geq (g_{n_m} - 4^m \sum_{k=1}^{m-1} g_{n_k} - 2^{-m} h)(|T(x_{n_m})|) \end{aligned}$$

$$> \quad \varepsilon - \frac{\varepsilon}{4} - 2^{-m} h(|T(x_{n_m})|).$$

This proves that $f_m(|T(x_{n_m})|) > \frac{\varepsilon}{2}$ for m sufficiently large (because $2^{-m}h(|T(x_{n_m})|) \to 0$). Since $f_m(|T(x_{n_m})|) = \sup\{|T'(y)(x_{n_m})|, |y| \leq f_m\}$, for m sufficiently large there exist some $y_m \in F'$ such that $|y_m| \leq f_m$ and $|T'(y_m)(x_{n_m})| > \frac{\varepsilon}{2}$. It is clear that (y_m^+) and (y_m^-) are norm bounded disjoint sequences in A^+ and so, by our hypothesis we obtain

$$\begin{aligned} \frac{\varepsilon}{2} &< |T'(y_m)(x_{n_m})| \\ &\leq \left|T'(y_m^+)(x_{n_m})\right| + \left|T'(y_m^-)(x_{n_m})\right| \\ &\leq \sup_{k\in\mathbb{N}} \left|T'(y_k^+)(x_{n_m})\right| + \sup_{k\in\mathbb{N}} \left|T'(y_k^-)(x_{n_m})\right| \to 0, \end{aligned}$$

as $m \to \infty$. This leads to a contradiction, and we are done. □

As a consequence of Theorem 4.2, we get

Corollary 4.3. *Let E be a Banach lattice and let A be a norm bounded solid subset of E'. The following statements are equivalent:*

(1) *A is a V-set in E';*
(2) *$[-f, f]$ and $\{(f_n), n \in \mathbb{N}\}$ are V-sets in E', for each $f \in A^+$ and for each disjoint sequence $(f_n) \subset A^+ = A \cap (E')^+$.*

The next result characterizes unconditionally converging operator from a Banach space into a Banach lattice by V-set and sequences.

Theorem 4.4. *For an operator T from a Banach space X into a Banach space Y, the following statements are equivalent:*

(1) *T is unconditionally converging operator;*
(2) *$\|T(x_n)\| \to 0$ as $n \to \infty$ for every weakly unconditionally converging series $\sum x_n$ in X;*
(3) *$T'(B_{Y'})$ is a V-set in X'.*
If Y is a Banach lattice, we may add:
(4) *$T'([-f, f])$ and $\{T'(f_n), n \in \mathbb{N}\}$ are V-sets in X', for each $f \in B_{Y'}^+$ and for each disjoint sequence $(f_n) \subset B_{Y'}^+$;*
(5) *$|T(x_n)| \to 0$ weakly in Y and $f_n(T(x_n)) \to 0$ for every weakly unconditionally converging series $\sum x_n$ in X and for each disjoint sequence $(f_n) \subset B_{Y'}^+$.*

Proof. (1) ⇒ (2) Obvious.

(2) ⇒ (1) Suppose that T is not unconditionally converging. Then, there exist a weakly unconditionally converging series $\sum x_n$ in X, $\varepsilon > 0$, a permutation σ of $\mathbb{N}$, and intertwining sequences of positive integers (m_k) and (n_k) such that $\left\|\sum_{n=m_k}^{n_k} T(x_{\sigma(n)})\right\| > \varepsilon$ for each $k \in \mathbb{N}$. Now, if we take $z_k = \sum_{n=m_k}^{n_k} x_{\sigma(n)}$, we see that $\sum z_k$ is weakly unconditionally converging series in X and $\|T(z_k)\| = \left\|\sum_{n=m_k}^{n_k} T(x_{\sigma(n)})\right\| > \varepsilon$ for each $k \in \mathbb{N}$, which is impossible by our hypothesis.

(2) ⇔ (3) Follows from the equality $\sup_{f\in T'(B_{Y'})} |f(x_n)| = \|T(x_n)\|$ for every weakly unconditionally converging series $\sum x_n$ in X.

(3) ⇔ (4) Follows from Theorem 4.2.

(4) ⇔ (5) Follows from Proposition 3.4 and Corollary 3.5. □

Now, by Corollaries 2.9 and 2.12 of [5], Corollary 1.9 of [3] and Theorem 4.4, we derive the following characterization of 1-Schur property in Banach lattice.

Proposition 4.5. *Let X be a Banach space, the following statements are equivalent:*

(1) *X has the 1-Schur property;*
(2) *X contains no copy of c_0;*
(3) *for any Banach space Y, every operator from X into Y is unconditionally converging;*

(4) *every operator from X into ℓ^∞ is unconditionally converging;*
(5) $\|x_n\| \to 0$ *as* $n \to \infty$ *for every weakly unconditionally converging series* $\sum x_n$ *in* X*;*
(6) $B_{X'}$ *is a V-set in* X'.
If X *is a Banach lattice, we may add:*
(7) $[-f, f]$ *and* $\{(f_n), n \in \mathbb{N}\}$ *are V-sets in* X'*, for each* $f \in B^+_{X'}$ *and for each disjoint sequence* $(f_n) \subset B^+_{X'}$*;*
(8) $|x_n| \to 0$ *weakly in* X *and* $f_n(x_n) \to 0$ *for every weakly unconditionally converging series* $\sum x_n$ *in* X *and for each disjoint sequence* $(f_n) \subset B^+_{X'}$.

The following is a well-known result due to Pelczynski:

Lemma 4.6. *Let* X*,* Y *be Banach spaces and let* $T : X \to Y$ *be a bounded linear operator. Then* T *is unconditionally converging if and only if it does not fix a copy of* c_0 *(i.e., there is no subspace* $Z \subset X$ *isomorphic to* c_0 *such that* $T|_Z$ *is an isomorphism onto its image).*

See, e.g., ([7], p. 54, Exercise 8), a quantitative version was established in [[12], Theorem 3.6]. Now, in the following Theorem, we obtain a necessary condition such that an operator between two Banach spaces is not unconditionally converging.

Theorem 4.7. *Suppose that an operator* T *from a Banach space* X *into a Banach space* Y *is not unconditionally converging. Then, for each* $0 < \varepsilon < 1$*, there exist a sequence* $(f_n) \subset B_{Y'}$ *and a weakly unconditionally converging series* $\sum x_n$ *in* X *such that*

$$f_n(T(x_n)) = 1$$

and

$$\textstyle\sum_{k \neq n} |f_n(T(x_k))| < \frac{\varepsilon}{1-\varepsilon}$$

for each $n \in \mathbb{N}$.

Proof. Assume that T is not unconditionally converging. By Lemma 4.6 it fixes a copy of c_0. I.e., there is $Z \subset X$ isomorphic to c_0 such that $T|_Z$ is an isomorphism. Let (z_n) the c_0-basis of Z. Then $T(z_n)$ is a c_0-basis of $T(Z)$.

By James' distortion theorem for c_0 (see, e.g., [[11], Lemma 2.2]) for any positive number ε, there is a sequence (v_n) in the unit ball of $T(Z)$ such that

$$(1-\varepsilon)\sup_n |\alpha_n| < \Big\|\sum_n \alpha_n v_n\Big\| \le \sup_n |\alpha_n|$$

for each choice of scalars (α_n). Take $x_n \in Z \subset X$ with $T(x_n) = \frac{v_n}{\|v_n\|}$. Since (x_n) is a c_0-basis of Z, the series $\sum x_n$ is weakly unconditionally converging. Moreover, $\|T(x_n)\| = 1$, hence there is $(f_n) \subset B_{Y'}$ with $f_n(T(x_n)) = 1$ (by Hahn-Banach).

Now, Let $0 < \varepsilon < 1$, fix n and assume that the set $I \subset \{k \in \mathbb{N}/k \neq n\}$ is finite. Then (ϵ_k below are ± 1):

$$\begin{aligned}
\sum_{k\in I} |f_n(T(x_k))| &= -|f_n(x_n)| + \sum_{k \in I\cup\{n\}} |f_n(T(x_k))| \\
&= -1 + \sum_{k\in I\cup\{n\}} \epsilon_k f_n(T(x_k)) \\
&= -1 + f_n\Big(\sum_{k\in I\cup\{n\}} \epsilon_k T(x_k)\Big) \\
&\le -1 + \Big\|\Big(\sum_{k\in I\cup\{n\}} \epsilon_k T(x_k)\Big)\Big\| \\
&\le -1 + \sup_{k\in I\cup\{n\}} \frac{1}{\|v_k\|} \\
&< -1 + \frac{1}{1-\varepsilon} = \frac{\varepsilon}{1-\varepsilon}.
\end{aligned}$$

And this completes the proof. □

As a consequence, we get

Corollary 4.8. *Let X be a Banach space which contains a copy of c_0. Then, for each $0<\varepsilon<1$, there exist a sequence $(f_n)\subset B_{X'}$ and a weakly unconditionally converging series $\sum x_n$ in X such that*

$$f_n(x_n)=1$$

and

$$\textstyle\sum_{k\neq n}|f_n(x_k)|<\frac{\varepsilon}{1-\varepsilon}$$

for each $n\in\mathbb{N}$.

5. Banach spaces with the property (VLD)

Note that there exists a V-set in a topological dual Banach space which is not L-Dunford-Pettis. In fact, the closed unit ball $B_{L^\infty[0,1]}$ of $L^\infty[0,1]$ is V-set because $L^1[0,1]$ has the 1-Schur property (see Example 2.5 of [5] and Proposition 4.5), but $B_{L^\infty[0,1]}$ is not L-Dunford-Pettis because $L^1[0,1]$ does not have the relatively compact Dunford-Pettis property (see [9], Proposition 3.3 and [15], Corollary 2.7). Inspired by this fact, we introduce the notion of property (VLD) in Banach space.

Definition 5.1. Let X be a Banach space. Then we say that X has the property (VLD) if each V-set in X' is L-Dunford-Pettis.

Now, we are in a position to give the operators characterizations of the property (VLD) in a Banach space.

Theorem 5.2. *Let X be a Banach space. The following statements are equivalent;*

1. *X has the property (VLD).*
2. *Let Y be any Banach space, then an operator T from X to Y is Dunford-Pettis completely continuous whenever T is unconditionally converging.*
3. *Every unconditionally converging operator T from X to ℓ^∞ is Dunford-Pettis completely continuous.*

Proof. (1) $\Rightarrow$ (2) Assume that Y is a Banach space and operator T from X into Y is unconditionally converging. Now, let the series $\sum x_n$ in X be weakly unconditionally converging. Then, $\sum T(x_n)$ is an unconditionally converging in Y and $\|T(x_n)\|\to 0$ as $n\to\infty$.

Thus,

$$\|T(x_n)\|=\sup\{|T'f(x_n)| : f\in B_{Y'}\}\to 0$$

as $n\to\infty$, and hence $T'(B_{Y'})$ is a V-set in X'. Since X has the (VLD) property, then $T'(B_{Y'})$ is L-Dunford-Pettis set in X', and from Theorem 2.6 of [15] we see that the operator T is Dunford-Pettis completely continuous.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let A be a V-set in X' and let (f_n) be a sequence in A. Define the operator $T:\ell^1\to X'$ by

$$T(a)=\sum_{n=0}^{\infty}a_nf_n$$

for each $a=(a_n)\in\ell^\infty$, so the restriction in X of its adjoint $T'|_X : X\to\ell^\infty$ is defined by

$$T'|_X(x)=T'(x)=(f_m(x))$$

for each $x \in X$. Now, let $\sum x_n$ be a weakly unconditionally converging series in X, we have

$$\|T'(x_n)\| = \sup_{m \in \mathbb{N}} |f_m(x_n)| \leq \sup_{f \in A} |f(x_n)|$$

for each $n \in \mathbb{N}$. Since A is a V-set in X' we see that $\|T'(x_n)| \to 0$ as $n \to \infty$, this proves that the operator $T'|_X$ is unconditionally converging, and by our hypothesis we obtain that $T'|_X$ is Dunford-Pettis completely continuous operator.

Assume by way of contradiction that A is not an L-Dunford-Pettis set in X'. Then, there exists a weakly null sequence (y_n), which is a Dunford-Pettis subset of X such that $\sup_{g \in A} |g(y_n)| > \varepsilon > 0$ for some $\varepsilon > 0$ and each $n \in \mathbb{N}$. Hence, for every n there exists some g_n in A such that $|g_n(y_n)| > \varepsilon$. Now, we find

$$\|T'(y_n)\| = \sup_{m \in \mathbb{N}} |f_m(y_n)| \geq \sup_{f \in A} |f(y_n)| \geq |g_n(y_n)| > \varepsilon$$

for each $n \in \mathbb{N}$, which is a contradiction with the fact that $T'|_X$ is Dunford-Pettis completely continuous operator. Thus, A is L-Dunford-Pettis set in X', Therefore X has the property (VLD), and we are done. □

Example 5.3. (1) The Banach space ℓ^1 has the property (VLD). In fact, for any Banach space Y, we have every operator T from ℓ^1 into Y is Dunford-Pettis completely continuous (because ℓ^1 has the relatively compact Dunford-Pettis property), and we apply Theorem 5.2.
(2) By our Definition 5.1 the Banach space $L^1[0,1]$ does not have the property (VLD).

As a consequence we obtain:

Corollary 5.4. *Let X be a Banach space.*

(1) *If X has the property (VLD) and contains no copy of c_0 then it has the relatively compact Dunford-Pettis property.*
(2) *If X has the relatively compact Dunford-Pettis property then it has the property (VLD).*
(3) *Suppose that X contains no copy of c_0. Then, X has the property (VLD) if and only if it has the relatively compact Dunford-Pettis property.*

Proof. (1) Immediately From Theorem 5.2.

(2) Let X be a Banach space has the relatively compact Dunford-Pettis property, so by [14], Theorem 2.5, we see that every bounded subset of X' is L-Dunford-Pettis. In particular, we conclude that every V-set of X' is L-Dunford-Pettis, and X has the property (VLD).

(3) Immediately from (1) and (2). □

Remark 5.5. (1) Every Schur space has the relatively compact Dunford-Pettis property, and hence it has the property (VLD).
(2) Every discrete KB-space has the relatively compact Dunford-Pettis property (see [2], Corollary 3.10), and hence it has the property (VLD).

Corollary 5.6. *Let X be a Banach space.*

(1) *If X has the property (V) then it has the property (VLD).*
(2) *If X has the (VLD) and L-Dunford-Pettis properties then it has the property (V).*
(3) *Suppose that X has the L-Dunford-Pettis property. Then, X has the property (VLD) if and only if it has the property (V).*

Proof. (1) Let T be a unconditionally converging operator from a Banach space X to any Banach space Y, as X has the property (V) then T is weakly compact. By [4], Corollary 1.1 we conclude that T is Dunford-Pettis completely continuous, and by Theorem 5.2 we see that X has the property (VLD).

(2) Let T be a unconditionally converging operator from a Banach space X to any Banach space Y, since X has the property (VLD) we see that T is Dunford-Pettis completely continuous. Now, as X has the L-Dunford-Pettis property then by Theorem 2.7 of [14] we see that T is weakly compact, and X has the property (V).

(3) Immediately from (1) and (2). □

Example 5.7. (1) The Banach spaces c_0, ℓ^∞, $C(K)$ (K is a compact Hausdorff space) have the property (V) (see [10], page 7), and hence they have the property(VLD).

(2) Every reflexive Banach space has the property (V) [13], and hence it has the property (VLD).

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References

[1] C.D. Aliprantis, O. Burkinshaw, *Positive operators*, Reprint of the 1985 original. Springer, Dordrecht, 2006.

[2] B. Aqzzouz, K. Bouras, *Dunford-Pettis sets in Banach lattices*, Acta Math. Univ. Comenian. **81**(2) (2012), 185-196.

[3] M. E. Bahreini, E. M. Bator, I. Ghenciu, *Complemented subspaces of linear bounded operators*, Canad. Math. Bull. **55**(3) (2012), 449-461.

[4] Z. L. Chen, A. W. Wickstead, *Relative weak compactness of solid hulls in Banach lattices*, Indag. Math. (N.S.), **9**(2) (1998), 187-196.

[5] M. B. Dehghani, S. M. Moshtaghioun, *On the p-Schur property of Banach spaces*, Ann. Funct. Anal. **9** (2018), 123-136.

[6] J. Diestel, *A survey of results related to the Dunford-Pettis property in integration, topology, and geometry in linear spaces*, (Chapel Hill, NC, 1979), Contemp. Math. 2, Amer. Math. Soc., Providence, (1980), 15-60.

[7] J. Diestel, *Sequences and series in Banach spaces*, Grad Texts in Math, No. 92, Springer-Verlag, Berlin, 1984.

[8] N. Dunford, J.T. Schwartz, *Linear operators*, Part I: General Theory. Wiley-Interscience, New York, 1958.

[9] K. El Fahri, N. Machrafi and M. Moussa, *Banach lattices with the positive Dunford-Pettis relatively compact property*, extracta mathematicae. Vol. **30**, Num. 2 (2015), 161-179.

[10] J. Howard, *Banach spaces on which every unconditionally converging operator is completely continuous*, Commentationes Mathematicae Universitatis Carolinae. Vol. **16**, No. 4 (1975), 745-753.

[11] R.C. James, *Uniformly Non-Square Banach Spaces, Annals of Mathematics*, Nov., 1964, Second Series, Vol. **80**, No. 3 (Nov., 1964), pp. 542–550.

[12] H. Krulisova, *Quantification of Pelczynski's property (V)*, Mathematische Nachrichten (290) (2017), 2909–2924.

[13] A. Pelczynski, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Pol. Sci. **10** (1962), 641-648.

[14] A. Retbi, B. El Wahbi, *L-Dunford-Pettis property in Banach spaces*, Methods Func. Anal. Topol. **22** (2016), 387-392.

[15] A. Retbi, *A note on L-Dunford-Pettis sets in a topological dual Banach space*, Czechoslovak Mathematical Journal. (2020), 11 pages, DOI: 10.21136/CMJ.2020.0119-19.

[16] H.P. Rosenthal, *Characterization of Banach Spaces Containing* c_0, Journal of the American Mathematical Society 7 (1994), no. 3, 707–748.

[17] P. Saab, B. Smith, *Spaces on which every unconditionally converging operators are weakly completely continuous*, Rocky Mount J Math. **22**(3) (1992), 1001-1009.

[18] Y. Wen, J. Chen, *Characterizations of Banach spaces with relatively compact Dunford-Pettis Sets*, Advances in Mathematics (China), Adv. Math., Beijing, **45** (2014), 122-132.

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