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SOME REMARKS ON THE GENERALIZATION OF ORTHOGONALITY IN TERMS OF OPERATORS

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Abstract. This paper deals with a generalization of orthogonality in terms of bounded linear operators on a Banach space. The goal is to find a relation between orthogonality of images and orthogonality of elements. We prove that if the images of a bounded linear operator are orthogonal in the Pythagorean sense, then the elements are orthogonal in the sense of Birkhoff's definition. In the case of Robert's orthogonality in terms of bounded linear operators under the restriction that any element belongs to the intersection of the norm attainment set of $T_1 + \lambda T_2$ and $T_1 - \lambda T_2$, if the images are orthogonal, then it implies that the operators are also orthogonal. Furthermore, some results in relation to the Carlsson, isosceles, and approximate Birkhoff-James orthogonality have been obtained.

У цій роботі розглядяється узагальнення ортогональності в термінах обмежених лінійних операторів на банаховому просторі. Метою роботи є знайти співвідношення між ортогональністю зображень і ортогональністю елементів. Доведено, що якщо образи обмеженого лінійного оператора ортогональні в піфагоровому сенсі, то елементи є ортогональний у сенсі визначення Біркгофа. У випадку ортогональності Роберта в термінах обмежених лінійних операторів із умовою, що будь-який елемент належить до перетину множин де оператори $T_1 + \lambda T_2$ і $T_1 - \lambda T_2$ досягають норми, з ортогональності образів випливає, що оператори є також ортогональні. Також отримано деякі результати про ортогональність в сенсі Карлссона, рівнобічної ортогональності та наближеної ортогональності в сенсі Біркгофа-Джеймса.

1. Introduction

As a generalization of orthogonality from an inner product space to a normed linear space, a new concept of orthogonality in terms of bounded liner operators has been used by different researchers. Bhatia and Semrl in [1] introduced a concept of orthogonality in terms of matrices by taking the concept of Birkhoff-James orthogonality. For any matrices A and B they use the symbol $\|A\|$ to denote the operator norm of A, and A is orthogonal to B in the sense of Birkhoff-James if and only if for any complex number z, $\|A + zB\| \geq \|A\|$. A matrix operator A is orthogonal to a matrix operator B iff there exists a unit vector x in a the Hilbert space H such that $\|Ax\| = \|A\|$, and the inner-product of the image of the operator A at x and the image of operator B at x is zero. i.e. $\langle Ax, Bx\rangle = 0$ [1]. They also introduced a Birkhoff-James orthogonality in [1] by $A \perp B$ if and only if $\|A + zB\|_p \geq \|A\|_p$, where $\|A\|_p$ denotes the Schatten p-norm of A defined by $\|A\|_p = [\sum_{j=1}^{n} S_j(A)^p]^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $S_1(A) \geq \ldots \geq S_n(A)$ are singular values of A.

Taking the special case for $p = 2$, Bhatia and Semrl also proved that the given orthogonality is equivalent to the usual Hilbert space condition, $\langle A, B\rangle = 0$, which defines an inner-product on the space of matrices as $\langle A, B\rangle = tr(A^*B)$ [1]. Benitz et al. [8] proved that X is an an inner-product space if and only if for any linear operators A and C in a finite dimensional normed space X, $A \perp C \Leftrightarrow \exists u \in S_X : \|Au\| = \|u\|, Au \perp Cu$, where $S_X = \{x \in X : \|x\| = 1\}$ and "$\perp$" denotes the Birkhoff-James orthogonality. Sain

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et al. in 2018 studied the problem of characterizing the Birkhoff-James orthogonality for bounded linear operators and prove separate necessary and sufficient conditions for smoothness of bounded linear operators on infinite dimensional normed linear space [7].

Mall et al. studied parallelism and approximate parallelism in the space of bounded linear operators defined on a Hilbert space and Bifkhoff-James orthogonality in a space of compact linear operators and bounded linear operators by using the concept of approximate parallelism [10]. For any compact linear operator from a a reflexive Banach space X to Y with $M_T = D \cup (-D)$, where D is a compact connected subset of S_X and M_T is the norm attainment set of T, Paul et al. in [6] proved that the operator T is Birkhoff orthogonal to an operator A if and only if Tx is Birkhoff orthogonal to Ax for some $x \in M_T$. In the case of a Hilbert space, an operator T is Birkhoff orthogonal to an operator A if and only if the inner-product of the image of T at x with the image of A at x is zero. i. e., $< Tx, Ax >= 0$. Botazzi et al. studied Birkhoff-James orthogonality and isosceles orthogonality of bounded linear operators between Hilbert spaces and Banach spaces. They also explore a relation between Birkhoff-James orthogonality and isosceles orthogonality in general Banach spaces [9].

Motivated by results of Botazzi et al., we develop a relation between orthogonality of images of a non-zero bounded linear operator in the Pythagorean sense with the Birkhoff orthogonality of elements. We also apply such a concept in relation to the Carlsson's orthogonality and approximate Birkhoff-James orthogonalit in terms of bounded linear operators.

2. Definition, notation and preliminary results

Throughout the paper, an operator $T : X \to Y$ is taken as a bounded linear operator from a normed space X to Y, H denotes a Hilbert space, and $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$, i.e., it is the set of all unit vectors in S_X at which T attains its norm. Similarly, $\langle x, y\rangle$ denotes the inner-product of x and y, $M_{T_1+\lambda T_2}$ and $M_{T_1-\lambda T_2}$ are norm attainment set of $T_1 + \lambda T_2$ and $T_1 - \lambda T_2$, respectively. Furthermore, $M_{q_k T_1 + r_k T_2}$ is the norm attainment set of $q_k T_1 + r_k T_2$, and $T_1 \perp_C^O T_2$ indicates that T_1 is Carlsson orthogonal to T_2 in terms of operators. Let $\perp^\varepsilon$, $\perp_P$, $\perp_B$ and $\perp_I$ denote the approximate orthogonality, Pythagorean, Bifkhoff-James, and isosceles orthogonalities, respectively.

Definition 2.1. [2] Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then x is said to be orthogonal to$y$ in the sense of Birkhoff-James if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{K}$, where $\mathbb{K}$ is a scalar field.

It is obvious that an orthogonality in an inner-product space is symmetric (i. e. $x \perp y$ implies $y \perp x$). However in a normed linear space, all orthogonalities may not be symmetric. For instance, the Birkhoff orthogonality is not symmetric, but the Pythagorean and isosceles orthogonalities are symmetric in any normed linear space.

Definition 2.2 ([4], [5])**.** Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then x is said to be isosceles orthogonal to y if and only if

$$\|x - y\| = \|x + y\|.$$

Definition 2.3 ([4], [5])**.** Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then x is said to be Pythagorean orthogonal to y if and only if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition 2.4. [3] Let $(X, \|.\|)$ be a normed linear space and $a_k, b_k, c_k, k = 1, \dots m$, a fixed collection of real numbers satisfying

$$\sum_{k=1}^{m} a_k b_k c_k = 1, \qquad \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0.$$

An element $x \in X$ is said to be Carlsson orthogonal to $y \in X$ if

$$\sum_{k-1}^{m} a_k \|b_k x + c_k y\|^2 = 0.$$

Definition 2.5. [3] Carlsson orthogonality is said to have property (H) in a normed linear space x if for any positive integer n, $x \perp y$ implies that

$$\lim_{n \to +\infty} n^{-1} \sum_{n=1}^{m} n a_k \|b_k x + c_k y\|^2 = 0.$$

The Carlsson orthogonality may not be symmetric in all cases. i.e., $x \perp y$ may not implies $y \perp x$. If the Carlsson orthogonality has the property (H) in a normed space X, then it is symmetric.

Definition 2.6. [9] Let T_1 and T_2 be norm attaining bounded linear operators in a Banach space X. Then T_1 is said to be isosceles orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\| = \|(T_1 + T_2)(h)\|. \tag{2.1}$$

The definition of Pythagorean orthogonaliti in terms of operators is given in [9] as follows: T_1 is said to be Pythagorean orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2 \text{or} \|(T_1 + T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2. \tag{2.2}$$

Definition 2.7. [10] Let $\epsilon \in [0, 1)$ and $x, y \in X$. Then x is said to be approximate Bifkhoff-James orthogonal to y if $(1 - \epsilon)\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{K}$.

Proposition 2.8. [10] *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be normed linear spaces. Let T be a bounded linear operator from X to Y, and let $x \in M_T$. Then for any $\epsilon \in [0, 1)$, x is said to be approximate parellel to y (denoted by $x\|^\epsilon y$) if and only if there exists a linear functional $f \in_{S_{X^*}}$ such that $f(x) = d \leq \|x\|$ and $f(y) = 0$.*

Using the definition of approximate Birkhoff-James orthogonality, Zamani and Molehian in [11] have introduced the notion of approximate parallelism as follows:

Definition 2.9. ([10], [11])) Let $(X.\|.\|)$ be a normed linear space and $x, y \in X$, $\epsilon \in [0, 1)$. Then x is said to be approximate parallel to y, written as $x\|^\epsilon y$ if $\inf\{\|x + \lambda y\| : \lambda \in \mathbb{K}\} \leq \epsilon\|x\|$.

Theorem 2.10. [10] *Let T and A be compact linear operators from a reflexive Banach space X to any normed linear space Y. Then the operator T is norm parallel to the operator A, written as $T\|A$, if and only if there exists $x \in M_T \cap M_A$ such that $Tx\|Ax$.*

Theorem 2.11. [10] *Let H be an infinite dimensional Hilbert space and $B(H)$ be the set of bounded linear operators on H. Let $\epsilon \in [0, 1)$ and $T \in B(H)$. Then $(1) \Rightarrow (2)$, where*

1. *For any $A \in B(H)$, $T\|^\epsilon A$ if and only if there exists $x \in M_T \cap M_A$ such that $Tx \perp^\epsilon Ax$.*
2. *There exists an finite-dimensional subspace H_0 of H such that $M_T = S_{H_0}$ and $\|T\|_{H_0^\perp} < \|T\|$.*

The following proposition related to approximate Birkhoff-James orthogonality has a vital role to support our main result. A slightly different definition of approximate Birkhoff-James orthogonality is given by Dragomir, written as $x \perp_D^\epsilon y$, and id defined as follows:

Definition 2.12. [10] Let $\epsilon \in [0, 1)$ and let $x, y \in X$. Then x is said to be approximate Birkhoff-James orthogonal to y if

$$\|x + \lambda y\| \geq (1 - \epsilon)\|x\|, \qquad \forall \lambda \in \mathbb{K}.$$

Proposition 2.13. [10] *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be normed linear spaces. Let T be a bounded linear operator from X to Y, and $x \in M_T$. Then for any $\epsilon \in [0, 1)$ and $y \in X$, $Tx \perp_D^\epsilon Ty \Rightarrow x \perp_D^\epsilon y$.*

3. Main Result

Proposition 3.1. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be normed linear spaces. Let T be a non-zero bounded linear operator from X to Y, and let $x \in M_T$. Then for any $y \in X$ and $\lambda \in \mathbb{K}, Tx \perp_P \lambda Ty \Rightarrow x \perp_B y$.*

Proof. Suppose $Tx \perp_P \lambda Ty$. Then

$$\begin{aligned}\|Tx + \lambda Ty\|^2 &= \|Tx\|^2 + \|\lambda Ty\|^2 \\ \Rightarrow \|Tx\|^2 &\leq \|Tx + \lambda Ty\|^2 \\ \Rightarrow \|Tx\| &\leq \|Tx + \lambda Ty\| \\ &\leq \|T\|\|x + \lambda y\|.\end{aligned}$$

Since $x \in M_T$, we can write $\|Tx\| = \|T\|\|x\|$, and therefore, we can write the above inequality as

$$\begin{aligned}\|T\|\|x\| &\leq \|T\|\|x + \lambda y\| \\ \Rightarrow \|x\| &\leq \|x + \lambda y\|\end{aligned}$$

This shows that x is Birkhoff orthogonal to y. □

Proposition 3.2. *Let T_1 and T_2 be bounded linear operators from a Banach space X to Y. Then for any $x \in M_{T_1+\lambda T_2} \cap M_{T_1-\lambda T_2}$, $T_1(x) \perp_R T_2(x) \Rightarrow T_1 \perp_R T_2$.*

Proof. Let $x \in M_{T_1+\lambda T_2} \cap M_{T_1-\lambda T_2}$. Suppose $T_1(x) \perp_R T_2(x)$. Then, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\|T_1(x) - \lambda T_2(x)\| &= \|T_1(x) + \lambda T_2(x)\| \\ \Rightarrow \quad \|(T_1 - \lambda T_2)x\| &= \|(T_1 + \lambda T_2)x\| \\ \Rightarrow \quad \|T_1 - \lambda T_2\| &= \|T_1 + \lambda T_2\|.\end{aligned}$$

□

Proposition 3.3. *Let T_1 and T_2 be bounded linear operators from a Banach space X to Y, and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $x \in M_{T_1+\lambda T_2} \cap M_{T_1-\lambda T_2}$*

$$T_1(x_n) \perp_R T_2(x_n) \Rightarrow T_1 \perp_R T_2.$$

Proof. Let $x \in M_{T_1+\lambda T_2} \cap M_{T_1-\lambda T_2}$. Suppose $T_1(x_n) \perp_R T_2(x_n)$. Then for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\|T_1(x_n) - \lambda T_2(x_n)\| &= \|T_1(x_n) + \lambda T_2(x_n)\| \\ \Rightarrow \quad \lim_{n\to\infty} \|T_1(x) - \lambda T_2(x_n)\| &= \lim_{n\to\infty} \|T_1(x_n) + \lambda T_2(x_n)\| \\ \Rightarrow \quad \|T_1(x_n) - \lambda T_2(x)\| &= \|T_1(x) + \lambda T_2(x)\| \\ \Rightarrow \quad \|T_1 - \lambda T_2\| &= \|T_1 + \lambda T_2\|.\end{aligned}$$

□

Corollary 3.4. *Let T_1 and T_2 be bounded linear operators from a Banach space X to Y. Then for any $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$, $T_1(x) \perp_I T_2(x) \Rightarrow T_1 \perp_I T_2$.*

Proof. Let $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$. Suppose $T_1(x) \perp_I T_2(x)$. Then,

$$\begin{aligned}\|T_1(x) - T_2(x)\| &= \|T_1(x) + T_2(x)\| \\ \Rightarrow \quad \|(T_1 - T_2)x\| &= \|(T_1 + T_2)x\| \\ \Rightarrow \quad \|T_1 - T_2\| &= \|T_1 + T_2\|.\end{aligned}$$

□

Corollary 3.5. *Let T_1 and T_2 be bounded linear operators from a Banach space X to Y, and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$*

$$T_1(x_n) \perp_I T_2(x_n) \Rightarrow T_1 \perp_I T_2.$$

Proof. Let $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$. Suppose $T_1(x_n) \perp_I T_2(x_n)$. Then,

$$\begin{aligned} \|T_1(x_n) - T_2(x_n)\| &= \|T_1(x_n) + T_2(x_n)\| \\ \Rightarrow \quad \lim_{n\to\infty} \|T_1(x) - T_2(x_n)\| &= \lim_{n\to\infty} \|T_1(x_n) + T_2(x_n)\| \\ \Rightarrow \quad \|T_1(x_n) - T_2(x)\| &= \|T_1(x) + T_2(x)\| \\ \Rightarrow \quad \|T_1 - T_2\| &= \|T_1 + T_2\|. \end{aligned}$$

□

Theorem 3.6. *Let T_1 and T_2 be bounded linear operators from a Banach space X to Y. Then for any $x \in M_{q_kT_1+r_kT_2}$, if the images of T_1 and T_2 are orthogonal in the sense of Carlsson orthogonality in terms of bounded linear operators, then the operators are also orthogonal.*

Proof. Let $x \in M_{q_kT_1+r_kT_2}$ and $T_1(x) \perp_C^O T_2(x)$. Then

$$\sum_{k=1}^{n} p_k \|q_kT_1(x) + r_kT_2(x)\|^2 = 0 \tag{3.3}$$

satisfyes the conditions

$$\sum_{k=1}^{n} p_kq_kr_k = 1, \quad \sum_{k=1}^{n} p_kq_k^2 = \sum_{k=1}^{n} p_kr_k^2 = 0 \tag{3.4}$$

Since $x \in M_{q_kT_1+r_kT_2}$, we must have $\|q_kT_1(x) + r_kT_2(x)\| = \|(q_kT_1 + r_kT_2)x\| = \|q_kT_1 + r_kT_2\|$. Therefore, from relation (3.3) and (3.4), we may conclude that

$$\sum_{k=1}^{n} p_k \|q_kT_1 + r_kT_2\|^2 = 0.$$

This shows that T_1 is Carlsson orthogonal to T_2 in terms of operators. □

Theorem 3.7. *Let $T_1, T_2 \in B(X)$ with $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$. Then $T_1(x) + T_2(x) \perp_B T_2(x)$ and $T_1(x) - T_2(x) \perp_B T_2(x)$ implies $T_1 \perp_I T_2$.*

Proof. Since $T_1(x)+T_2(x) \perp_B T_2(x)$, we have $\|T_1(x)+T_2(x)\| \leq \|T_1(x)+T_2(x)+\lambda T_2(x)\|$ for all $\lambda \in \mathbb{K}$. Taking $\beta = 1 + \lambda$, we have

$$\begin{aligned} \|T_1(x) + T_2(x)\| &\leq \|T_1(x) + T_2(x) + (\beta - 1)T_2(x)\| \\ &= \|T_1(x) + T_2(x) + \beta T_2(x) - T_2(x)\| \\ &= \|T_1(x) + \beta T_2(x)\| \\ \Rightarrow \|(T_1 + T_2)x\| &\leq \|(T_1 + \beta T_2)x\|. \end{aligned}$$

In particular for $\beta = -1$, we have $\|(T_1 + T_2)x\| \leq \|(T_1 - T_2)x\|$. Since $x \in M_{T_1+T_2} \cap M_{T_1(x)+T_2(x)}$, we must have $\|(T_1+T_2)\| \leq \|(T_1-T_2)\|$. Similarly, if $T_1(x)-T_2(x) \perp_B T_2(x)$, we obtain $\|T_1 - T_2\| \leq \|T_1 + T_2\|$ and, therefore, by combining these inequalities we get the desired result. □

Corollary 3.8. *Let $T_1, T_2 \in B(X)$, and assume that $T_1 + T_2 \perp_B T_2$ and $T_1 - T_2 \perp_B T_2$. Then $T_1 \perp_I T_2$.*

Proof. Since $T_1 + T_2 \perp_B T_2$, we have $\|T_1 + T_2\| \leq \|T_1 + T_2 + \lambda T_2\|$ for all $\lambda \in \mathbb{K}$. Taking $\beta = 1 + \lambda$, we have

$$\begin{aligned} \|T_1 + T_2\| &\leq \|T_1 + T_2 + (\beta - 1)T_2\| \\ &= \|T_1 + T_2 + \beta T_2 - T_2\| \\ &= \|T_1 + \beta T_2\|. \end{aligned}$$

In particular, for $\beta = -1$, we have $\|T_1 + T_2\| \leq \|T_1 - T_2\|$. Similarly, if $T_1 - T_2 \perp_B T_2$, we obtain $\|T_1 - T_2\| \leq \|T_1 + T_2\|$ and therefore, by combining these inequalities, we get the desired result. □

Proposition 3.9. *Let X and Y be normed linear spaces, T be a non-zero bounded linear operator acting from X to Y, and $\{x_n\}$, $\{y_n\}$ be sequences in X such that $x_n \to x \in M_T$, $y_n \to y$, $Tx_n \to Tx$. Then for any $\epsilon \in [0,1)$ $Tx_n \perp_B^\epsilon Ty_n$ for each n implies that $x \perp_B^\epsilon y$.*

Proof. Let $x \in M_T$ and $\epsilon \in [0,1)$. Since $Tx_n \perp_B^\epsilon Ty_n$, we have for any $\lambda \in \mathbb{K}$,

$$\begin{aligned}
&(1-\epsilon)\|Tx_n\| \leq \|Tx_n + \lambda T y_n\| \\
&\Rightarrow (1-\epsilon)\|Tx_n\| \leq \|T\|\|x_n + \lambda y_n\| \\
&\Rightarrow (1-\epsilon)\lim_{n\to\infty}\|Tx_n\| \leq \|T\|\lim_{n\to\infty}\|x_n + \lambda y_n\| \\
&\Rightarrow (1-\epsilon)\|Tx\| \leq \|T\|\|x + \lambda y\|. \qquad (3.5)
\end{aligned}$$

Since $x \in M_T$, we must have $\|Tx\| = \|T\|\|x\|$, and therefore the inequality (3.5) can be written as $(1-\epsilon)\|T\|\|x\| \leq \|T\|\|x + \lambda y\|$. Hence $\|(1-\epsilon)\|x\| \leq \|x + \lambda y\|$ is the desired result. □

Conclusion. The concept of Birkhoff-James and isosceles orthogonalities in terms of bounded linear operators in Banach and Hilbert spaces has been studied and characterized during the last decades. We conclude that there may be various chances of characterizing others like: Pythagorean, Carlsson and Robert orthogonalities in terms of bounded linear operators under the restriction for any element to belong to the norm attainment set of operators.

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