Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 4, pp. 308–319
https://doi.org/10.31392/MFAT-npu26_4.2021.03

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

GENERALIZED MITTAG-LEFFLER KERNELS AND GENERALIZED SCALING OPERATORS IN MITTAG-LEFFLER ANALYSIS

WOLFGANG BOCK AND ANG ELYN GUMANOY

Abstract. Generalized scaling operators and generalized Gauss kernels are fundamental concepts in Gaussian analysis with application to path integrals and PDEs via the Feynman-Kac formula. In non-Gaussian analysis, particularly in Mittag-Leffler analysis, i.e., in the case when compared to a Gaussian characteristic function the exponential is replaced by a Mittag-Leffler function, these concepts are unknown. In view of this, we elaborate in this article the generalized scaling and generalized Mittag-Leffler kernels and prove a form of a Wick-type product formula. We give some first examples for generalized scaling.

Узагальнені оператори масштабування та узагальнені ядра Гаусса становлять фундаментальні поняття гаусового аналізу та мають застосування до інтегралів за шляхами та рівнянь у частинних похідних з використанням формули Фейнмана-Каца. Це є новим в негаусівського аналізу, зокрема в аналізі Міттага-Леффлера, тобто у випадку якщо в гаусовій характеристичній функції експонента замінюється функцією Міттага-Леффлера. З огляду на це, в статті детально розглянуто ядра узагальненого масштабування та узагальнені ядра Міттага-Леффлера, та доведено форма формулу добутку Віковського типу. Наведено кілька перших прикладів узагальненого масштабування.

1. Introduction

Generalized Gauss kernels and generalized scaling operators play a fundamental role in different aspects and applications of Gaussian analysis. Gaussian analysis and white noise analysis became a rapidly developing theory with various applications such as mathematical physics as statistical mechanics, quantum field theory, quantum mechanics and polymer physics as well as in applied mathematics and Stochastic Analysis, Dirichlet forms, Stochastic (Partial) Differential equations and finance, see e.g. the monographs [10, 22, 13]. Various characterization theorems [24, 14, 6] are proven to build up a strong analytical foundation. Almost at the same time, first attempts were made to introduce a non-Gaussian infinite dimensional analysis, by transferring properties of the Gaussian measure to the Poisson measure [11], with the help of a biorthogonal generalized Appell systems [2, 1, 16]. This approach is suitable for many measures, like the Gaussian measure and the Poisson measure [15]. In [4] a similar concept was used to establish the Mittag-Leffler Analysis. The grey noise measure [27, 21] is included as a special case in the class of Mittag-Leffler measures, which offers the possibility to apply the Mittag-Leffler analysis to fractional differential equations, in particular to fractional diffusion equations [26, 27], which carry numerous applications in science, like relaxation type differential equations or viscoelasticity. There is a well-known connection between PDEs and stochastic processes, provided by the Feynman-Kac formula. By investigating a heat equation, where the time derivative is a Caputo derivative of fractional order, Schneider introduced grey Brownian motion (gBm) in [27]. He showed that a solution to the time-fractional heat equation is given in terms of grey Brownian motion like in the Feynman-Kac case. The link between grey Brownian motion and fractional differential equations is also studied by Mura and

2020 *Mathematics Subject Classification.* 26E15, 46F25, 47B38.
Keywords. Mittag-Leffler Analysis, Scaling Operators, Wick Formula.

Mainardi in the framework of fractional diffusion equations in [21]. In [5] also a relation between the fractional heat kernel in this setting and the associated process grey Brownian motion was proved.

In this article, we define a set of generalized Mittag-Leffler kernels in Mittag-Leffler analysis which plays the role of generalized Gauss kernels in Gaussian analysis. We show its relation to generalized scaling operators and provide a version of a generalized Wick formula. At the end of the paper, we study the application of a generalized scaling operator to Donsker's Delta function in Mittag-Leffler Analysis.

2. The Mittag-Leffler Measure

Let $d \in \mathbb{N}$ and L^2_d be the Hilbert space of vector-valued square integrable measurable functions

$$L^2_d := L^2(\mathbb{R}) \otimes \mathbb{R}^d.$$

The space L^2_d is unitary isomorphic to a direct sum of d identical copies of $L^2 := L^2(\mathbb{R})$, (i.e., the space of real-valued square integrable measurable functions with Lebesgue measure). Any element $f \in L^2_d$ may be written in the form

$$f = (f_1 \otimes e_1, \dots, f_d \otimes e_d), \tag{2.1}$$

where $f_i \in L^2(\mathbb{R})$, $i = 1, \dots, d$ and $\{e_1, \dots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. The inner product in L^2_d is given by

$$(f, g)_0 = \sum_{k=1}^{d} (f_k, g_k)_{L^2} = \sum_{k=1}^{d} \int_{\mathbb{R}} f_k(x) g_k(x)\, dx,$$

where $g = (g_1 \otimes e_1, \dots, g_d \otimes e_d)$, $f_k \in L^2$, $k = 1, \dots, d$, f as given in (2.1). The corresponding norm in L^2_d is given by

$$|f|_0^2 := \sum_{k=1}^{d} |f_k|^2_{L^2} = \sum_{k=1}^{d} \int_{\mathbb{R}} f_k^2(x)\, dx.$$

As a densely embedded nuclear Fréchet space in L^2_d we choose $S_d := S(\mathbb{R}) \otimes \mathbb{R}^d$, where $S(\mathbb{R})$ is the Schwartz test function space. An element $\varphi \in S_d$ has the form

$$\varphi = (\varphi_1 \otimes e_1, \dots, \varphi_d \otimes e_d), \tag{2.2}$$

where $\varphi_i \in S(\mathbb{R})$, $i = 1, \dots, d$. Together with the dual space $S'_d := S'(\mathbb{R}) \otimes \mathbb{R}^d$ we obtain the basic Gel'fand triple

$$S_d \subset L^2_d \subset S'_d.$$

The dual pairing between S'_d and S_d is given as an extension of the scalar product in L^2_d by

$$\langle f, \varphi \rangle_0 = \sum_{k=1}^{d} (f_k, \varphi_k)_{L^2},$$

where f and φ as in (2.1) and (2.2), respectively. In S'_d we choose the Borel σ-algebra $\mathcal{B}$ generated by the cylinder sets.

Define the operator $M_-^{\frac{\alpha}{2}}$ on $S(\mathbb{R})$ by

$$M_-^{\frac{\alpha}{2}} \varphi := \begin{cases} K_{\frac{\alpha}{2}} D_-^{-(\alpha-1)/2} \varphi, & \alpha \in (0,1), \\ \varphi, & \alpha = 1, \\ K_{\frac{\alpha}{2}} I_-^{(\alpha-1)/2} \varphi, & \alpha \in (1,2), \end{cases}$$

where the normalization constant $K_{\frac{\alpha}{2}} := \sqrt{\alpha \sin(\frac{\alpha\pi}{2})\Gamma(\alpha)}$ and D_-^r, I_-^r denote the left-side fractional derivative and fractional integral of order r in the sense of Riemann-Liouville, respectively:

$$\begin{aligned}(D_-^r f)(x) &= \frac{-1}{\Gamma(1-r)}\frac{d}{dx}\int_x^\infty f(t)(t-x)^{-r}\,dt\\ (I_-^r f)(x) &= \frac{1}{\Gamma(r)}\int_x^\infty f(t)(t-x)^{r-1}\,dt, \quad x\in\mathbb{R}.\end{aligned}$$

We refer to [25] or [12] for the details on these operators. It is possible to obtain a larger domain of the operator $M_-^{\frac{\alpha}{2}}$ in order to include the indicator function $1\!\!1_{[0,t)}$ such that $M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)} \in L^2$, for the details we refer to Appendix A in [5]. We have the following

Proposition 2.1 (Corollary 3.5 in [5])**.** *For all $t, s \geq 0$ and all $0 < \alpha < 2$ it holds that*

$$\left(M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)}, M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,s)}\right)_{L^2} = \frac{1}{2}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right). \tag{2.3}$$

Note that this coincides with the covariance of the fractional Brownian motion with Hurst parameter $H = \frac{\alpha}{2}$.

In order to construct ggBm we will use the Mittag-Leffler function which is introduced by G. Mittag-Leffler in a series of papers [18, 19, 20].

Definition 2.1 (Mittag-Leffler function)**.** For $\beta > 0$ the Mittag-Leffler function E_β is defined as an entire function by the following series representation

$$E_\beta(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta n+1)}, \quad z\in\mathbb{C}. \tag{2.4}$$

Here Γ denotes the well-known Gamma function which is an extension of the factorial to complex numbers such that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

Note that for $\beta = 1$, the Mittag-Leffler function coincides with the classical exponential function, i.e, $E_1(z) = e^z$ for all $z \in \mathbb{C}$. We also consider the so-called the M-Wright function M_β for $0 < \beta \leq 1$ (in one variable) where its series expansion is given by

$$M_\beta(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}.$$

For the choice $\beta = \frac{1}{2}$ the corresponding M-Wright function reduces to the Gaussian density

$$M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right). \tag{2.5}$$

The Mittag-Leffler function E_β and the M-Wright are related through the Laplace transform

$$\int_0^\infty e^{-s\tau} M_\beta(\tau)\,d\tau = E_\beta(-s). \tag{2.6}$$

The Mittag-Leffler measures μ_β, $0 < \beta \leq 1$ is a family of probability measures on S_d' whose characteristic functions are given by Mittag-Leffler functions, see Definition 2.1. Using the Bochner-Minlos theorem, see [3] or [9], we obtain the following definition.

Definition 2.2 (cf. [4])**.** For any $\beta \in (0,1]$ the Mittag-Leffler measure is defined as the unique probability measure μ_β on S_d' by fixing its characteristic functional

$$\int_{S_d'} e^{i\langle w,\varphi\rangle_0}\,d\mu_\beta(w) = E_\beta\left(-\frac{1}{2}|\varphi|_0^2\right), \quad \varphi\in S_d. \tag{2.7}$$

Remark 2.1. (1) The measure μ_β is also called grey noise (reference) measure, cf. [4] and [5].

(2) The range $0 < \beta \leq 1$ ensures the complete monotonicity of $E_\beta(-x)$, see Pollard [23], i.e., $(-1)^n E_\beta^{(n)}(-x) \geq 0$ for all $x \geq 0$ and $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. In other words, it is sufficient to show that

$$S_d \ni \varphi \mapsto E_\beta\left(-\frac{1}{2}|\varphi|_0^2\right) \in \mathbb{R}$$

is a characteristic function in S_d.

By $L^2(\mu_\beta) := L^2(S'_d, \mathcal{B}, \mu_\beta)$ we denote the complex Hilbert space of square integrable measurable functions defined on S'_d with scalar product

$$(\!(F, G)\!)_{L^2(\mu_\beta)} := \int_{S'_d} F(w)\bar{G}(w)\, d\mu_\beta(w), \quad F, G \in L^2(\mu_\beta).$$

The corresponding norm is denoted by $\|\cdot\|_{L^2(\mu_\beta)}$. It follows from (2.7) that all moments of μ_β exists and we have

Lemma 2.1. *For any $\varphi \in S_d$ and $n \in \mathbb{N}_0$ we have*

$$\int_{S'_d} \langle w, \varphi\rangle_0^{2n+1}\, d\mu_\beta(w) = 0,$$

$$\int_{S'_d} \langle w, \varphi\rangle_0^{2n}\, d\mu_\beta(w) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)}|\varphi|_0^{2n}.$$

In particular, $\|\langle\cdot, \varphi\rangle\|^2_{L^2(\mu_\beta)} = \frac{1}{\Gamma(\beta+1)}|\varphi|_0^2$ and by polarization for any $\varphi, \psi \in S_d$ we obtain

$$\int_{S'_d} \langle w, \varphi\rangle_0 \langle w, \psi\rangle_0\, d\mu_\beta(w) = \frac{1}{\Gamma(\beta + 1)}\langle \varphi, \psi\rangle_0.$$

3. Generalized grey Brownian motion in dimension d

For any test function $\varphi \in S_d$ we define the random variable

$$X^\beta(\varphi) : S'_d \longrightarrow \mathbb{R}^d,\ w \mapsto X^\beta(\varphi, w) := \big(\langle w_1, \varphi_1\rangle, \ldots, \langle w_d, \varphi_d\rangle\big).$$

As a consequence of Lemma 2.1 and the characteristic function of μ_β which is given in (2.7), the random variable $X^\beta(\varphi)$ possess the following properties:

Proposition 3.1. *Let $\varphi, \psi \in S_d$, $k \in \mathbb{R}^d$ be given. Then*

(1) *The characteristic function of $X^\beta(\varphi)$ is given by*

$$\mathbb{E}\big(e^{i(k, X^\beta(\varphi))}\big) = E_\beta\left(-\frac{1}{2}\sum_{j=1}^d k_j^2 |\varphi_j|^2_{L^2}\right). \tag{3.8}$$

(2) *The characteristic function of the random variable $X^\beta(\varphi) - X^\beta(\psi)$ is*

$$\mathbb{E}\big(e^{i(k, X^\beta(\varphi) - X^\beta(\psi))}\big) = E_\beta\left(-\frac{1}{2}\sum_{i=1}^d k_j^2 |\varphi_j - \psi_j|^2_{L^2}\right). \tag{3.9}$$

(3) *The expectation of the $X^\beta(\varphi)$ is zero and*

$$\|X^\beta(\varphi)\|^2_{L^2(\mu_\beta)} = \frac{1}{\Gamma(\beta + 1)}|\varphi|_0^2. \tag{3.10}$$

(4) *The moments of* $X^\beta(\varphi)$ *are given by*

$$\int_{S'_d} \left|X^\beta(\varphi, w)\right|^{2n+1} d\mu_\beta(w) = 0,$$

$$\int_{S'_d} \left|X^\beta(\varphi, w)\right|^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n\Gamma(\beta n+1)}|\varphi|_0^{2n}.$$

Remark 3.1. (1) The property (3.10) of $X^\beta(\varphi)$ gives the possibility to extend the definition of X^β to any element in L^2_d, in fact, if $f \in L^2_d$, then there exists a sequence $(\varphi_k)_{k=1}^\infty \subset S_d$ such that $\varphi_k \longrightarrow f$, $k \to \infty$ in the norm of L^2_d. Hence, the sequence $\left(X^\beta(\varphi_k)\right)_{k=1}^\infty \subset L^2(\mu_\beta)$ forms a Cauchy sequence which converges to an element denoted by $X^\beta(f) \in L^2(\mu_\beta)$.

(2) For $\beta = 1$ property (3.10) yields the Itô isometry.

We define $1\!\!1_{[0,t)} \in L^2_d$, $t \geq 0$, by

$$1\!\!1_{[0,t)} := (1\!\!1_{[0,t)} \otimes e_1, \ldots, 1\!\!1_{[0,t)} \otimes e_d)$$

and consider the process $X^\beta(1\!\!1_{[0,t)}) \in L^2(\mu_\beta)$ such that the following definition makes sense.

Definition 3.1. For any $0 < \alpha < 2$ we define the process

$$\begin{aligned} S'_d \ni w \mapsto B^{\beta,\alpha}(t, w) &:= \left(\langle w, (M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)}) \otimes e_1\rangle, \ldots, \langle w, (M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)}) \otimes e_d\rangle\right) \\ &= \left(\langle w_1, M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)}\rangle, \ldots, \langle w_d, M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)}\rangle\right), \ t > 0 \end{aligned} \tag{3.11}$$

as an element in $L^2(\mu_\beta)$. This process is called a version of d-dimensional generalized grey Brownian motion (ggBm). Its characteristic function has the form

$$\mathbb{E}\left(e^{i(k, B^{\beta,\alpha}(t))}\right) = E_\beta\left(-\frac{|k|^2}{2}t^\alpha\right), \ k \in \mathbb{R}^d. \tag{3.12}$$

Remark 3.2. (1) By Remark 3.1 the d-dimensional ggBm exist as a $L^2(\mu_\beta)$-limit and hence the map $S'_d \ni w \mapsto \langle w, 1\!\!1_{[0,t)}\rangle$ yields a version of ggBm, $\mu_\beta - a.s.$, but not in the pathwise sense.

(2) For a fixed $0 < \alpha < 2$ one can show by the Kolmogorov-Centsov continuity theorem that the paths of the process are $\mu_\beta - a.s.$ continuous.

Proposition 3.2. *For any* $0 < \alpha < 2$, *the process* $B^{\beta,\alpha} := \{B^{\beta,\alpha}(t),\ t \geq 0\}$, *is* $\frac{\alpha}{2}$ *self-similar with stationary increments.*

Remark 3.3. The family $\{B^{\beta,\alpha}(t),\ t \geq 0,\ \beta \in (0,1],\ \alpha \in (0,2)\}$ forms a class of $\frac{\alpha}{2}$ self-similar process with stationary increments ($\frac{\alpha}{2}$-sssi) which includes:

(1) For $\beta = \alpha = 1$, the process $\{B^{1,1}(t),\ t \geq 0\}$ is a standard d-dimensional Brownian motion.

(2) For $\beta = 1$ and $0 < \alpha < 2$, $\{B^{1,\alpha}(t),\ t \geq 0\}$ is a d-dimensional fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$.

(3) For $\alpha = 1$, $\{B^{\beta,1}(t),\ t \geq 0\}$ is $\frac{1}{2}$ self-similar non Gaussian process with

$$\mathbb{E}\left(e^{i\left(k, B^{\beta,1}(t)\right)}\right) = E_\beta\left(-\frac{|k|^2}{2}t\right), \quad k \in \mathbb{R}^d. \tag{3.13}$$

(4) For $0 < \alpha = \beta < 1$, the process $\{B^\beta(t) := B^{\beta,\beta}(t),\ t \geq 0\}$ is $\frac{\beta}{2}$ self-similar and is called d-dimensional grey Brownian motion (gBm for short). Its characteristic function is given by

$$\mathbb{E}\left(e^{i\left(k, B^\beta(t)\right)}\right) = E_\beta\left(-\frac{|k|^2}{2}t^\beta\right), \quad k \in \mathbb{R}^d. \tag{3.14}$$

For $d=1$, gBm was introduced by W. Schneider in [26, 27].

4. Distributions and characterization theorems

With the help of the Appell systems, a test function and a distribution space in non-Gaussian analysis can be constructed, the details of this construction can be found in [16], [4], [5] and references therein. In between the many choices of triples which can be constructed, we choose the Kondratiev triple

$$(S_d)^1_{\mu_\beta} \subset (H_p)^1_{q,\mu_\beta} \subset L^2(\mu_\beta) \subset (H_{-p})^{-1}_{-q,\mu_\beta} \subset (S_d)^{-1}_{\mu_\beta}.$$

The space $(H_p)^1_{q,\mu_\beta}$ is defined as the completion of the $\mathcal{P}\big(S'_d\big)$ (the space of smooth polynomials on S'_d) w.r.t. the norm $\|\cdot\|_{p,q,\mu_\beta}$ given by

$$\|\varphi\|^2_{p,q,\mu_\beta} := \sum_{n=0}^{\infty}(n!)^2 2^{nq}|\varphi^{(n)}|^2_p, \quad p,q\in\mathbb{N}_0,\ \varphi\in\mathcal{P}\big(S'_d\big).$$

The dual space $(H_{-p})^{-1}_{-q,\mu_\beta}$ is a subset of $\mathcal{P}'\big(S'_d\big)$ such that if $\Phi\in(H_{-p})^{-1}_{-q,\mu_\beta}$, then

$$\|\Phi\|^2_{-p,-q,\mu_\beta} := \sum_{n=0}^{\infty}2^{-nq}|\Phi^{(n)}|^2_{-p} < \infty, \quad p,q\in\mathbb{N}_0.$$

The dual pairing between $\big(S'_d\big)^{-1}_{\mu_\beta}$ and $\big(S_d\big)^{1}_{\mu_\beta}$, denoted by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mu_\beta}$ is a bilinear extension of scalar product in $L^2(\mu_\beta)$. For any $\varphi\in\big(S_d\big)^{1}_{\mu_\beta}$ and $\Phi\in\big(S'_d\big)^{-1}_{\mu_\beta}$ we have

$$\langle\!\langle\Phi,\varphi\rangle\!\rangle_{\mu_\beta} = \sum_{n=0}^{\infty} n!\langle\Phi^{(n)},\varphi^{(n)}\rangle.$$

The set of μ_β-exponentials

$$\left\{ e_{\mu_\beta}(\cdot,\varphi) := \frac{e^{\langle\cdot,\varphi\rangle}}{\mathbb{E}\big(e^{\langle\cdot,\varphi\rangle}\big)},\ \varphi\in S_{d,\mathbb{C}},\ |\varphi|_p<2^{-q}\right\}$$

forms a total set in $(H_p)^1_{q,\mu_\beta}$ and for any $\varphi\in S_{d,\mathbb{C}}$ such that $|\varphi|_p<2^{-q}$ we have $\|e_{\mu_\beta}(\cdot,\varphi)\|_{p,q,\mu_\beta}<\infty$.

Let us introduce an integral transform, the S_{μ_β}-transform, which is used to characterize the spaces $(S_d)^1_{\mu_\beta}$ and $(S_d)^{-1}_{\mu_\beta}$. For any $\Phi\in(S_d)^{-1}_{\mu_\beta}$ and $\varphi\in U\subset S_{d,\mathbb{C}}$, where U is a suitable neighborhood of zero, we define

$$S_{\mu_\beta}\Phi(\varphi) := \frac{\langle\!\langle\Phi,e^{\langle\cdot,\varphi\rangle}\rangle\!\rangle_{\mu_\beta}}{\mathbb{E}\big(e^{\langle\cdot,\varphi\rangle}\big)} = \frac{1}{E_\beta\big(\frac{1}{2}\langle\varphi,\varphi\rangle\big)}\langle\!\langle\Phi,e^{\langle\cdot,\varphi\rangle}\rangle\!\rangle_{\mu_\beta}.$$

The characterization theorem for the space $(S_d)^{-1}_{\mu_\beta}$ via the S_{μ_β}-transform is done using the spaces of holomorphic functions on $S_{d,\mathbb{C}}$. We denote by $\mathrm{Hol}_0\big(S_{d,\mathbb{C}}\big)$ the space of holomorphic functions at zero where we identify two functions which coincides in a neighborhood of zero. The space $\mathrm{Hol}_0\big(S_{d,\mathbb{C}}\big)$ is given as the inductive limit of a family of normed spaces, see [16] for the details and the proof of the following characterization theorem.

Theorem 4.1 (cf. [16, Theorem 8.34])**.** *The S_{μ_β}-transform is a topological isomorphism from $(S_d)^{-1}_{\mu_\beta}$ to $\mathrm{Hol}_0\big(S_{d,\mathbb{C}}\big)$.*

As a corollary from the characterization theorem the following integration result can be deduced.

Theorem 4.2. *Let $(T,\mathcal{B},\nu)$ be a measure space and $\Phi_t\in(S_d)^{-1}_{\mu_\beta}$ for all $t\in T$. Let $\mathcal{U}\subset S_{d,\mathbb{C}}$ be an appropriate neighbourhood of zero and $0<C<\infty$, such that*

(1) *$S_{\mu_\beta}\Phi.(\xi): T \to \mathbb{C}$ is measurable for all $\xi \in \mathcal{U}$.*
(2) *$\int_T \left|S_{\mu_\beta}\Phi_t(\xi)\right| \, d\nu(t) \leq C$ for all $\xi \in \mathcal{U}$.*

Then, there exists $\Phi \in (S_d)^{-1}_{\mu_\beta}$ such that for all $\xi \in \mathcal{U}$

$$S_{\mu_\beta}\Psi(\xi) = \int_T S_{\mu_\beta}\Phi_t(\xi)\, d\nu(t).$$

We denote Ψ by $\int_T \Phi_t \, d\nu(t)$ and call it the weak integral of Φ.

In the following we will use the T_{μ_β}-transform which is defined as follows.

Lemma 4.1. *Let $\Phi \in (S_d)^{-1}_{\mu_\beta}$ and $p, q \in \mathbb{N}$ such that $\Phi \in (H_{-p})^{-1}_{-q,\mu_\beta}$. Then, the T_{μ_β}-transform given by*

$$T_{\mu_\beta}\Phi(\varphi) = \langle\!\langle \Phi, \exp\left(i\langle\cdot, \varphi\rangle\right)\rangle\!\rangle_{\mu_\beta}$$

is well-defined for $\varphi \in U_{p,q}$ and we have

$$T_{\mu_\beta}\Phi(\varphi) = E_\beta\left(-\frac{1}{2}\langle\varphi, \varphi\rangle\right) S_{\mu_\beta}\Phi(i\varphi).$$

In particular, $T_{\mu_\beta}\Phi \in \mathrm{Hol}_0\big(S_{d,\mathbb{C}}\big)$ if and only if $S_{\mu_\beta} \in \mathrm{Hol}_0\big(S_{d,\mathbb{C}}\big)$. Moreover, Theorem 4.2 also holds if the S_{μ_β}-transform is replaced by the T_{μ_β}-transform.

For details and proofs we refer to [4]. A direct application of the integral theorem is the existence of a generalization of the well-known Donsker's Delta.

Theorem 4.3. [5] *Let $0 \neq \eta \in L^2_d$ and $a \in \mathbb{R}$ arbitrary. Then*

$$\delta_a(\langle\cdot, \eta\rangle) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(is(\langle\cdot, \eta\rangle - a))\, ds$$

exists in $(S_d)^{-1}_{\mu_\beta}$ as a weak integral in the sense of Theorem 4.2 and it is called Donsker's Delta in $a \in \mathbb{R}$. Its T_{μ_β} transform is given by

$$\begin{aligned} &T_{\mu_\beta}(\delta_a(\langle\cdot, \rangle))(\xi) \\ &\quad = \frac{1}{\sqrt{2\pi\langle\eta, \eta\rangle}}\int_0^\infty M_\beta(r) r^{-\frac{1}{2}} \exp(-\frac{1}{2} r(\langle\xi, \xi\rangle - \frac{\langle\xi, \eta\rangle^2}{\langle\eta, \eta\rangle}) + ia\frac{\langle\xi, \eta\rangle}{\langle\eta, \eta} - \frac{a^2}{2r\langle\eta, \eta\rangle}) dr. \end{aligned}$$

5. Operators in Mittag-Leffler Analysis

5.1. **Generalized Mittag-Leffler Kernels.** Let $(e_n)_{n\in\mathbb{N}} \subset S_d$ be an orthonormal basis of L^2_d. Let P_{e_n} be the bounded linear operator on L^2_d defined by

$$P_{e_n} f = \langle f, e_n\rangle e_n, \quad \text{with } f \in L^2_d.$$

The projection to the one-dimensional subspace spanned by e_n is a continuous mapping on S_d and can be extended to a continuous mapping on S'_d, see e.g. [7].

Definition 5.1. Let K be a normal compact operator on L^2_d with eigenvalues λ_n and corresponding eigenvectors $(e_n)_{n\in\mathbb{N}} \subset L^2_d$, then we define the quadratic form

$$\langle w, Kw\rangle := \sum_{n=1}^\infty \lambda_n \langle e_n, w\rangle^2.$$

We now define a special class of Mittag-Leffler distributions which are defined by their T_{μ_β}-transform, similarly to generalized Gauss kernels in white noise analysis e.g. [7], [10]. Let $\mathcal{B}$ be the set of all continuous bilinear mappings $\mathfrak{B}: S_d \times S_d \to \mathbb{C}$. Then the functions

$$S_d \ni \xi \mapsto E_\beta\left(-\frac{1}{2}\mathfrak{B}(\xi, \xi)\right) \in \mathbb{C}$$

for all $\mathfrak{B} \in \mathcal{B}$ are in $\mathrm{Hol}_0\big(S_{d,\mathbb{C}}\big)$. Therefore, by using the characterization of Mittag-Leffler distributions in Theorem 4.1, the inverse T_{μ_β}-transform of these functions

$$\Phi_{\mathfrak{B}} := T_{\mu_\beta}^{-1} E_\beta\left(-\frac{1}{2}\mathfrak{B}(\cdot,\cdot)\right)$$

are elements of $(S_d)_{\mu_\beta}^{-1}$.

Definition 5.2. The set of *generalized Mittag-Leffler kernels* is defined by

$$GMLK := \{\Phi_{\mathfrak{B}},\ \mathfrak{B} \in \mathcal{B}\}.$$

In case the continuous bilinear form is given via the dual pairing and an operator $B : S_{d,\mathbb{C}} \to S'_{d,\mathbb{C}}$ we write $\Phi_B = T^{-1} E_\beta\left(-\frac{1}{2}\langle \cdot, B\cdot\rangle\right)$

Remark 5.1. For positive definite operator B, one can also consider Φ_B as a generalized version of a Radon-Nikodym derivative. The corresponding measure is then $\mu_\beta(\sqrt{B}^{-1}.)$, if B is self-adjoint w.r.t $\langle \cdot, \cdot \rangle$. In the case the root of the operator does not exists, as in the case of non-positive definite operators B, one can still consider Φ_B as a kind of change of measure, although, the "covariance matrix" might not be positive definite.

Example 5.1. Let B be the orthogonal projection on the complement of the subspace spanned by $\eta \in L^2(\mathbb{R})$, with $(\eta, \eta) = 1$. Then

$$\begin{aligned} T_{\mu_\beta}(\Phi_B)(\xi) &= E_\beta(-\frac{1}{2}\langle \xi, B\xi\rangle) \\ &= E_\beta(-\frac{1}{2}\langle \xi, \xi - \langle \xi, \eta\rangle \eta\rangle) \\ &= E_\beta(-\frac{1}{2}\langle \xi - \langle \xi, \eta\rangle \eta, \xi - \langle \xi, \eta\rangle \eta\rangle) \end{aligned}$$

As the T_{μ_0}-transform of Φ_B coincides with the one from the well-known Donsker's delta function $T_{\mu_0}(\delta_0(\langle ., \eta\rangle))$ for when $\beta = 0$, the distributions are the same. Compare also [10]. In the case $\beta \neq 0$ this is not the case see e.g. [5].

5.2. **Generalized Scaling Operators.** Pointwise multiplication with a generalized Mittag-Leffler kernel can be considered as a measure transformation. In a view to the previous sections, we want to generalize the notion of scaling to bounded operators. More precisely, we investigate for which kind of linear mappings $B \in L(S'_d, S'_d)$ there exists some operator $\sigma_B : (S_d)^1_{\mu_\beta} \to (S_d)^1_{\mu_\beta}$ such that

$$\Phi_{(BB^*)} \cdot \varphi := \sigma_B^\dagger \sigma_B \varphi.$$

where B^* is the dual operator of B with respect to $\langle \cdot, \cdot \rangle$. Furthermore we state a generalization of the Wick formula to Mittag-Leffler kernels. We start with the definition of σ_B.

Proposition 5.1. *Let $B \in L(S'_d, S'_d)$ and $\varphi \in (S_d)^1$ given by its continuous version. Then we define*

$$\sigma_B \varphi(w) = \varphi(Bw),$$

for $w \in S'_d$.

This can be proved directly by an explicit calculation on the the set of exponentials, a density argument and a verifying of pointwise convergence, compare [22, Proposition 4.6.7, p. 104], for the Gaussian case.

Next we show the continuity of the generalized scaling operator.

Proposition 5.2. *Let $B : S'_d \to S'_d$ be a bounded operator. For $\varphi, \psi \in (S_d)^1$ the following equation holds*

$$\sigma_B(\varphi\psi) = (\sigma_B\varphi)(\sigma_B\psi).$$

Since we consider a continuous mapping from $(\mathcal{S})^1$ into itself one can define the dual scaling operator with respect to $\langle\cdot,\cdot\rangle$, $\sigma_B^\dagger : (S_d)^{-1}_{\mu_\beta} \to (S_d)^{-1}_{\mu_\beta}$ by

$$\left\langle\!\left\langle \sigma_B^\dagger \Phi, \psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \Phi, \sigma_B\psi \right\rangle\!\right\rangle.$$

The Wick formula in White Noise Analysis was stated in [8] for Donsker's delta function. A similar result is true for Mittag-Leffler-kernels

Proposition 5.3 (Generalized Wick formula)**.** *Let $\Psi, \Xi \in (S_d)^{-1}$, $\varphi, \psi \in (S_d)^1$ and $B \in L(S'_d, S'_d)$. We define*

$$\Psi \diamond \Xi := S^{-1}_{\mu_\beta}(S_{\mu_\beta}(\Psi)S_{\mu_\beta}(\Xi).$$

Then we have

(i)

$$\sigma_B^\dagger \Psi = \Phi_{BB^*} \diamond \Gamma_{B^*}\Psi,$$

where ${\Gamma_B}^*$ is defined by

$$S_{\mu_\beta}(\Gamma_{B^*}\Psi)(\xi) = S_{\mu_\beta}(\Psi)(B^*\xi), \quad \xi \in S_d.$$

In particular we have

$$\sigma_B^\dagger \mathbb{1} = \Phi_{BB^*}.$$

(ii) $\Phi_{BB^*} \cdot \varphi = \sigma_B^\dagger(\sigma_B\varphi)$.
(iii) $\Phi_{BB^*} \cdot \varphi = \Phi_{BB^*} \diamond (\Gamma_{B^*} \circ \sigma_B(\varphi))$.

Proof. Proof of (i): Let $\Psi \in (S_d)^{-1}$ and $\xi \in S_d$ then we have

$$\begin{aligned} S(\sigma_B^\dagger\Psi)(\xi) &= \frac{\langle\!\langle \sigma_B^\dagger\Psi, \exp(\langle\cdot,\xi\rangle)\rangle\!\rangle}{E_\beta(\frac{1}{2}\langle\xi,\xi\rangle)} \\ &= \frac{\langle\!\langle \Psi, \sigma_B \exp(\langle\cdot,\xi\rangle)\rangle\!\rangle}{E_\beta(\frac{1}{2}\langle\xi,\xi\rangle)} \\ &= \frac{\langle\!\langle \Psi, \exp(\langle\cdot,B^*\xi\rangle)\rangle\!\rangle}{E_\beta(\frac{1}{2}\langle\xi,\xi\rangle)} \\ &= \frac{\langle\!\langle \Psi, \exp(\langle\cdot,B^*\xi\rangle)\rangle\!\rangle}{E_\beta(\frac{1}{2}\langle B^*\xi,B^*\xi\rangle)} \cdot \frac{E_\beta(\frac{1}{2}\langle B^*\xi,B^*\xi\rangle)}{E_\beta(\frac{1}{2}\langle\xi,\xi\rangle)} \\ &= S_{\mu_\beta}(\Gamma_{B^*}\Psi)(\xi) \cdot S_{\mu_\beta}(\Phi_{BB^*})(\xi) \end{aligned}$$

Proof of (ii): First we have $\sigma_B^\dagger\mathbb{1} = \Phi_{BB^*} \diamond \Gamma_{B^*}\mathbb{1} = \Phi_{BB^*}$.
Thus for all $\varphi, \psi \in (S_d)^1_{\mu_\beta}$

$$\langle\!\langle \Phi_{BB^*}\varphi, \psi\rangle\!\rangle = \langle\!\langle \sigma_B^\dagger\mathbb{1}, \varphi\cdot\psi\rangle\!\rangle = \langle\!\langle \mathbb{1}, (\sigma_B\varphi)(\sigma_B\psi)\rangle\!\rangle = \langle\!\langle (\sigma_B\varphi), (\sigma_B\psi)\rangle\!\rangle = \langle\!\langle \sigma_B^\dagger(\sigma_B\varphi), \psi\rangle\!\rangle.$$

Proof of (iii): Immediate from (i) and (ii). □

Remark 5.2. The scaling operator can be considered as a linear measure transform. Let $\varphi \in (S_d)^1_{\mu_\beta}$ and B a real bounded operator on S'_d. Then we have

$$\int_{S'_d} \sigma_B\varphi(w)\, d\mu(w) = \int_{S'_d} \varphi(Bw)\, d\mu(w) = \int_{S'_d} \varphi(w)\, d\mu(B^{-1}w).$$

Moreover we have

$$\int_{S'_d} \exp(i\langle w, \xi\rangle)\, d\mu(B^{-1}w) = E_\beta(-\frac{1}{2}\langle B^*\xi, B^*\xi\rangle),$$

which is a characteristic function of a probability measure by the Theorem of Bochner-Minlos-Sazonov. Furthermore

$$\int_{S'_d} \exp(i\langle \xi, w)\, d\mu(B^{-1}w) = T_{\mu_B}(\sigma_B^\dagger 1\!\!1)(\xi),$$

such that Φ_{BB^*} is represented by the positive measure $\mu \circ B^{-1}$.

6. Examples for Generalized Scaling

Example 6.1 (Fractional Scaling)**.** Let $M_-^{\frac{\alpha}{2}}$ denote the operator defined in Section 2. In this example we use generalized scaling operators to scale the underlying noise and hence the processes up to some certain maximal Hölder continuity. Let p be a polynomial on $\mathbb{R}$. Let $\varphi \in S(\mathbb{R})$. Then

$$\sigma_{M_-^{\frac{\alpha}{2}*}}(p(\langle \cdot, \varphi\rangle_0)) = p(\langle \cdot, M_-^{\frac{\alpha}{2}}\varphi\rangle_0) = p(\langle \cdot, M_-^{\frac{\alpha}{2}}\varphi\rangle_0),$$

which is again a polynomial in $(S)^1_{\mu_\beta}$, since $M_-^{\frac{\alpha}{2}}$ leaves $S(\mathbb{R})$ invariant, see e.g. [17]. Consider now the the random variable

$$p(\langle \cdot, 1\!\!1_{[0,t)}\rangle_0) \in (L^2).$$

Then formally as before

$$\sigma_{M_-^{\frac{\alpha}{2}*}}(p(\langle \cdot, 1\!\!1_{[0,t)}\rangle_0)) = p(\langle \cdot, M_-^{\frac{\alpha}{2}} 1\!\!1_{[0,t)}\rangle_0).$$

Indeed this can be made rigorously if we consider a sequence φ_n converging to $1\!\!1_{[0,t)}$ with $\varphi_n \in S(\mathbb{R})$ and the use of the convergence theorem for Mittag-Leffler distributions similarly to the findings in [4].

Let us see what would be the representation with the help Proposition 22:

$$\sigma^\dagger_{M_-^{\frac{\alpha}{2}*}} 1\!\!1 = \Phi_{M_-^{\frac{\alpha}{2}*} M_-^{\frac{\alpha}{2}}}.$$

Furthermore we have that

$$T_{\mu_\beta}(\Phi_{M_-^{\frac{\alpha}{2}*} M_-^{\frac{\alpha}{2}}})(\xi) = E_\beta\left(-\frac{1}{2}\langle M_-^{\frac{\alpha}{2}}\xi, M_-^{\frac{\alpha}{2}}\xi\rangle\right),$$

which is the characteristic function of the generalized grey Mittag-Leffler measure.

Example 6.2 (Scaling with an Orthogonal Operator)**.** Let $O \in L(S'_d, S'_d)$ be an orthogonal operator, i.e.

$$O^* = O^{-1}.$$

Such operators are for examples rotations and reflections. Then we have:

$$\sigma_O^\dagger 1\!\!1 = \Phi_{OO^*}.$$

And

$$T_{\mu_\beta}(\Phi_{OO^*})(\xi) = E_\beta\left(-\frac{1}{2}\langle \xi, \xi\rangle\right).$$

This shows that indeed the characteristic function and hence the measure is invariant under orthogonal transformations.

Example 6.3 (Pointwise Products of GMLK with Donsker's Delta).
Let $\varphi = \exp(i\lambda\langle w,\xi\rangle)$, $\xi \in S_d$. Let $B \in L(S'_d, S'_d)$. Then

$$\sigma_B^\dagger(\sigma_B\varphi) = \sigma_B^\dagger(\sigma_B \exp(i\lambda\langle w,\xi\rangle)) = \Phi_{BB^*} \cdot \exp(i\lambda\langle w,\xi\rangle)$$

As in Theorem 15, we can define Donsker's Delta as

$$\frac{1}{2\pi}\int_{\mathbb{R}} \exp(i\lambda\langle w,\xi\rangle)d\,\lambda.$$

Since Φ_{BB^*} is not dependent on λ we can formally write

$$\int_{\mathbb{R}} \sigma_B^\dagger(\sigma_B \exp(i\lambda\langle w,\xi\rangle))d\,\lambda = \int_{\mathbb{R}} \Phi_{BB^*} \cdot \exp(i\lambda\langle w,\xi\rangle)d\,\lambda = \Phi_{BB^*}\delta(\langle w,\xi\rangle).$$

On the other hand using Theorem 15 (iii) we have with the same arguments and using Fubini with $finS_d$

$$S_{\mu_\beta}(\Phi_{BB^*}\delta(\langle w,\xi\rangle))(f) = \int_{\mathbb{R}} S_{\mu_\beta}(\Phi_{BB^*})(f) \cdot S_{\mu_\beta}(\exp(i\lambda\langle Bw,\xi\rangle)(B^*f)d\,\lambda.$$

Now we obtain

$$S_{\mu_\beta}(\Phi_{BB^*}\delta(\langle w,\xi\rangle))(f) = S_{\mu_\beta}(\Phi_{BB^*})(f)\int_{\mathbb{R}} S_{\mu_\beta}(\exp(i\lambda\langle w,B^*\xi\rangle)(B^*f)d\,\lambda$$

which gives

$$E_\beta(\langle B^*f, B^*f\rangle) \cdot S_{\mu_\beta}(\delta(\langle w,B^*\xi\rangle))(B^*f)$$

References

[1] S. Albeverio, Y. Daletzky, Y. G. Kondratiev, and L. Streit. Non-Gaussian infinite dimensional analysis. *J. Funct. Anal.*, 138:311–350, 1996.

[2] Yu. L. Daletsky. A biorthogonal analogy of the Hermite polynomials and the inversion of the Fourier transform with respect to a non-Gaussian measure. *Funct. Anal. Appl.*, 25:68–70, 1991.

[3] I. M. Gel'fand and N. Ya. Vilenkin. *Generalized functions. Vol. 4: Applications of harmonic analysis.* Translated by Amiel Feinstein. Academic Press, New York, 1964.

[4] M. Grothaus, F. Jahnert, F. Riemann, and J. L. Silva. Mittag-Leffler Analysis I: Construction and characterization. *J. Funct. Anal.*, 268(7):1876–1903, April 2015.

[5] M. Grothaus and F. Jahnert. Mittag-Leffler Analysis II: Application to the fractional heat equation. *J. Funct. Anal.*, 270(7):2732–2768, April 2016.

[6] M. Grothaus, Y. G. Kondratiev, and L. Streit. Complex Gaussian analysis and the Bargmann-Segal space. *Methods Funct. Anal. Topology*, 3(2):46–64, 1997.

[7] M. Grothaus and L. Streit, *Quadratic actions, semi-classical approximation, and delta sequences in Gaussian analysis*, Rep. Math. Phys. **44** (1999), no. 3, 381–405.

[8] M. Grothaus, L. Streit, and A. Vogel, *The complex scaled Feynman–kac formula for singular initial distributions*, Stochastics **84** (2012), no. 2-3, 347–366.

[9] T. Hida. *Stationary Stochastic Processes.* Princeton University Press, 1970.

[10] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit. *White Noise. An Infinite Dimensional Calculus.* Kluwer Academic Publishers, Dordrecht, 1993.

[11] Y. Ito. Generalized Poisson functionals. *Prob. Th. Rel. Fields*, 77:1–28, 1988.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*, volume 204 of *North-Holland Mathematics Studies.* Elsevier Science B.V., Amsterdam, 2006.

[13] H.-H. Kuo. *White Noise Distribution Theory.* CRC Press, Boca Raton, New York, London, Tokyo, 1996.

[14] Yu.G. Kondratiev, P. Leukert, J. Potthoff, L. Streit, and W. Westerkamp. Generalized functionals in Gaussian spaces: The characterization theorem revisited. *J. Funct. Anal.*, 141(2):301–318, 1996.

[15] Y. G. Kondratiev, J. L. Silva, and L. Streit. Differential geometry on compound Poisson space. *Methods Funct. Anal. Topology*, 4(1):32–58, 1998.

[16] Y. G. Kondratiev, L. Streit, W. Westerkamp, and J.-A. Yan. Generalized functions in infinite dimensional analysis. *Hiroshima Math. J.*, 28(2):213–260, 1998.

[17] Y. Mishura. Stochastic calculus for fractional Brownian motion and related processes. *Vol. 1929 of Lecture Notes in Mathematics, Springer Science & Business Media.*, 410 p., 2008.

[18] G. M. Mittag-Leffler. Sur la nouvelle fonction $e_\alpha(x)$. *CR Acad. Sci. Paris*, 137(2):554–558, 1903.

[19] G. M. Mittag-Leffler. Sopra la funzione $e_\alpha(x)$. *Rend. Accad. Lincei*, 5(13):3–5, 1904.

[20] G. M. Mittag-Leffler. Sur la représentation analytique d'une branche uniforme d'une fonction monogène. *Acta Math.*, 29(1):101–181, 1905.

[21] A. Mura and F. Mainardi. A class of self-similar stochastic processes with stationary increments to model anomalous diffusion in physics. *Integral Transforms Spec. Funct.*, 20(3-4):185–198, 2009.

[22] N. Obata. *White Noise Calculus and Fock Spaces*, volume 1577 of LNM. Springer Verlag, Berlin, Heidelberg, New York, 1994.

[23] H. Pollard. The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. *Bull. Amer. Math. Soc.*, 54:1115–1116, 1948.

[24] J. Potthoff and L. Streit. A characterization of Hida distributions. *J. Funct. Anal.*, 101:212–229, 1991.

[25] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikol'skiĭ, Translated from the 1987 Russian original, Revised by the authors.

[26] W. R. Schneider. Grey noise. In S. Albeverio, G. Casati, U. Cattaneo, D. Merlini, and R. Moresi, editors, *Stochastic Processes, Physics and Geometry*, pages 676–681. World Scientific Publishing, Teaneck, NJ, 1990.

[27] W. R. Schneider. Grey noise. In S. Albeverio, J. E. Fenstad, H. Holden, and T. Lindstrøm, editors, *Ideas and Methods in Mathematical Analysis, Stochastics, and Applications (Oslo, 1988)*, pages 261–282. Cambridge Univ. Press, Cambridge, 1992.

[28] W. R. Schneider, W. Wyss. Fractional diffusion and wave equations. *J. Math. Phys.* 30(1), 134-144, 1989.

[29] W. Westerkamp. *Recent Results in Infinite Dimensional Analysis and Applications to Feynman Integrals*. PhD thesis, University of Bielefeld, 1995.

W. Bock: bock@mathematik.uni-kl.de
Department of Mathematics, TU Kaiserslautern (TUK), D-67663 Kaiserslautern, Germany

A. Gumanoy: angelyn.gumanoy@g.msuiit.edu.ph
Department of Mathematics, MSUIIT, Iligan, The Philippines

Received 24/06/2021; Revised 04/07/2021