Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 4, pp. 320–327
https://doi.org/10.31392/MFAT-npu26_4.2021.04

RESULTS ON MATRIX TRANSFORMATION OF COMPLEX UNCERTAIN SEQUENCES VIA CONVERGENCE IN ALMOST SURELY

BIROJIT DAS, BINOD CHANDRA TRIPATHY, AND PIYALI DEBNATH

Abstract. In this paper, the concept of convergence of complex uncertain series is applied to study matrix transformation of complex uncertain sequences in terms of almost surely. We establish a necessary and sufficient condition under which an infinite matrix operator transforms a null complex uncertain sequence in almost surely into another null sequence and almost surely convergent complex uncertain sequence into a convergent sequence of same type. We further characterize this transformation by introducing boundedness of complex uncertain sequences. Some other results of matrix transformation in real sequence space are also established in an uncertainty space of sequences of complex uncertain variable.

У даній роботі концепція збіжності комплексних невизначених рядів застосовується для дослідження матричних перетворень комплексних невизначених послідовностей в термінах майже напевно. Встановлено необхідна і достатня умова за якої оператор нескінченої матриці перетворює нульову комплексну невизначену послідовність у іншу нульову послідовність в сенсі майже напевно, а також збіжну комплексну невизначену послідовність у збіжну послідовність такої ж типу в сенсі майже напевно. Надано характерізацію цього перетворення, вводячи поняття обмеженості комплексних невизначених послідовностей. Деякі інші результати для матричних перетворень в просторі дійсних послідовностей отримано також в просторі невизначеності комплексних послідовностей невизначених змінних.

1. Introduction

The theory of uncertainty is introduced by B. Liu [7] in the year 2007 and it has evolved in the last decade in a large scale. The basics of different field of mathematics viz. measure theory, programming, risk analysis, reliability analysis, propositional logic, entailment, set theory, inference, renewal process, calculus, differential equation, finance, statistics, chance theory have been studied in uncertain environment. As part of the study of uncertainty theory, Liu [7] introduced the concept of uncertain sequences and several types of convergences, namely convergence in mean, in measure, in distribution and in almost surely. You [8] extended this study by introducing a new convergence concept with respect to uniformly almost surely. To describe the complex uncertain quantities, the notions of complex uncertain variable and complex uncertainty distribution are presented by Peng [15]. Chen et al. [14] explored convergence of sequence of complex uncertain variables due to Peng and reported five convergence concepts, namely convergence in almost surely, convergence in measure, convergence in mean, convergence in distribution and convergence with respect to uniformly almost surely by establishing interrelationships among them. These convergence concept of complex uncertain sequence has also been generalised by Nath and Tripathy [12], Das et al. [1, 2, 3, 5, 6]. Authors [4] introduced the notion of convergence of complex uncertain series very recently. As extension of the work, in this article we characterize matrix transformation of complex uncertain sequences.

2020 *Mathematics Subject Classification.* 60B10, 60E05, 40A05, 40A30, 40F05.
Keywords. uncertainty space, complex uncertain series, complex uncertain sequence, matrix transformation.

A norm, denoted by $||\cdot||$ defined as follows, which can be found in the functional analysis books.

Definition 1.1. Let X be a linear space defined over the filed $\mathbb{K}$ of real or complex numbers. The function $||\cdot|| : X \to \mathbb{R}_+ \cup \{0\}$, where $\mathbb{R}_+$ denotes the set of positive real numbers is called norm if it satisfies the followings for all $x, y \in X$ and $\alpha \in \mathbb{K}$:

(i) $||x|| = 0$ if and only if $x = \theta$, the zero element of X.
(ii) $||x + y|| \leq ||x|| + ||y||$.
(iii) $||\alpha x|| = |\alpha| ||x||$.

Before going to the main section we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2. Preliminaries

In this section, we recall some related definitions as ready references for the present work.

Definition 2.1 ([7])**.** Let $\mathcal{L}$ be σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom). $\mathcal{M}\{\Gamma\}$=1;
Axiom 2: (Duality Axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\}$=1, for any $\Lambda \in \mathcal{L}$;
Axiom 3: (Subadditivity Axiom). For every countable sequence of elements Λ_j in $\mathcal{L}$, we have

$$\mathcal{M}\Big\{\bigcup_{j=1}^{\infty} \Lambda_j\Big\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event.

Definition 2.2 ([15])**.** A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\}$=$\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

Definition 2.3 ([14])**.** The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\}$=1 such that

$$\lim_{n\to\infty} |\zeta_n(\gamma) - \zeta(\gamma)| = 0$$

for every $\gamma \in \Lambda$. Throughout the article, the family of all convergent complex uncertain sequence in almost surely is denoted by $c(\Gamma_{a.s})$. Similarly, the collection of all null sequences in almost surely is denoted by $c_0(\Gamma_{a.s})$.

Definition 2.4 ([4])**.** Let $\zeta = \{\zeta_k\}$ be a complex uncertain sequence and $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space. Then the infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent in almost surely if $\{S_n(\gamma)\}$, where $\gamma \in \Gamma$ is any event, is convergent to some limit S in almost surely. Here, $\{S_n(\gamma)\}$ is the complex uncertain sequence of partial sums defined by $S_n(\gamma) = \sum_{k=1}^{n} \zeta_k(\gamma)$.

In this case, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \|S_n(\gamma) - S(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.

3. Matrix Transformation of Complex Uncertain Sequences

The study of sequence space through matrices are very much relevant in the current research flow. Interest in matrix transformation theory was stimulated in Summability theory by Cesaro, Borel and others. It was however, Toeplitz explored matrix transformations while working on methods of linear space theory in sequence spaces. He characterized such infinite real sequences $A = (a_{nk}) = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$, which maps the space c into itself. The study of matrix transformation is very very important because in many cases the most general linear operator on one sequence space into another is actually given by a matrix and so researchers made progress enormously in this particular direction [9, 10, 11]. The concept of uncertainty theory developed in large scale in the last decade. Mathematicians and researchers from different field are showing great interest in investigating results of real space in uncertainty space. As a part of reconnoitring of uncertain sequence, in this section we introduce the notion of matrix transformation of different types of complex uncertain sequences via convergence of complex uncertain series. In [4], authors studied matrix transformation of complex uncertain sequence to some extent by considering the concept of convergence of complex uncertain series with respect to uniformly almost surely. In this article, we confine our study only to the concept of almost surely convergence of complex uncertain sequences and series simultaneously.

To make this precise, at first let us consider the space $c_0(\Gamma_{a.s})$ of all null sequences in almost surely. Consider an infinite matrix $A = \{a_{nk}\} = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$ and a complex uncertain sequence $\zeta = \{\zeta_k\} \in c_0(\Gamma_{a.s})$. Then there exists event Λ with unit uncertain measure in which ζ is a null sequence.

Then the matrix operator is applied on the complex uncertain sequences by normal matrix multiplication as follows:

$$A\zeta(\gamma) = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \zeta_1(\gamma) \\ \zeta_2(\gamma \\ \dots \end{pmatrix} = \begin{pmatrix} a_{11}\zeta_1(\gamma) + a_{12}\zeta_2(\gamma) + \dots \\ a_{21}\zeta_1(\gamma) + a_{22}\zeta_2(\gamma) + \dots \\ \dots \quad \dots \quad \dots \end{pmatrix} \quad \forall\, \gamma \in \Lambda,$$

Thus it can be written as $(A\zeta)_n \equiv A_n(\zeta)$, where $A_n(\zeta(\gamma)) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$, provided that the infinite series converges with respect to almost surely, for each n.

An infinite matrix A is said to belong to (X, Y) if A transforms the uncertain sequences from the space X to Y.

Now, the question arises that in what condition a matrix operator transforms a null sequences (with respect to almost surely) into another null sequence (with respect to almost surely). We find a necessary and sufficient condition regarding this below.

Theorem 3.1. *Let $A = \{a_{nk}\}$ be a matrix such that $\lim_{n\to\infty} a_{nk} \to 0$ (uniformly for all $k \in \mathbb{N}$) and $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$ to be finite. Then A is said to be a bounded linear operator on $c_0(\Gamma_{a.s})$ into itself and $||A|| = M$.*

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\{\zeta_n(\gamma)\} \in c_0(\Gamma_{a.s})$. Then there exists uncertain event Λ with $\mathcal{M}(\Lambda) = 1$, such that $\{\zeta_n(\gamma)\}$ uniformly converges to $\zeta(\gamma) = 0$ in Λ. i.e., for any $\varepsilon > 0$, there exists $k > 0$ such that $||\zeta_n(\gamma)|| < \varepsilon$, for all $\gamma \in \Lambda$ and $n \geq k$. We now show that $A\zeta(\gamma) \in c_0(\Gamma_{a.s})$, which implies that the complex uncertain series $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ is absolutely convergent in almost surely for each n.

Now, for any $m \geq 1$ and $\gamma \in \Lambda$,

$$||A_n(\zeta(\gamma))|| = \sum_{k=1}^{\infty} ||a_{nk}\zeta_k(\gamma)|| = \sum_{k=1}^{m} ||a_{nk}\zeta_k(\gamma)|| + \sum_{k=m+1}^{\infty} ||a_{nk}\zeta_k(\gamma)||$$
$$\leq ||\zeta_k(\gamma)|| \sum_{k=1}^{m} |a_{nk}| + \max_{k \geq m+1} ||\zeta_k(\gamma)|| M.$$

Take m and n so large that for any arbitrary small $\varepsilon > 0$, $\max\{||\zeta_k(\gamma)|| : k \geq m+1, \gamma \in \Lambda\} < \varepsilon$ and $\sum_{k=1}^{m} |a_{nk}| < \varepsilon$, since $a_{nk} \to 0$ as $n \to \infty$ (k fixed). Therefore, $A(\zeta(\gamma)) \in c_0(\Gamma_{a.s})$ and hence A defines an operator from $c_0(\Gamma_{a.s})$ into $c_0(\Gamma_{a.s})$.

Also, for any uncertain event $\gamma \in \Lambda$,

$$||A(\zeta(\gamma))|| = \sup_n || \sum_k a_{nk}\zeta_k(\gamma)|| \leq ||\zeta(\gamma)|| \sup_n \sum_k |a_{nk}| = M||\zeta(\gamma)||,$$

for every $\zeta \in c_0(\Gamma_{a.s})$. Hence, $||A|| \leq M, \quad \forall\, \zeta \in c_0(\Gamma_{a.s})$ and so A is bounded.

For the reverse inequality, there exists $n = m(\varepsilon)$ such that $\sum_k |a_{mk}| > M - \frac{\varepsilon}{2}$ and since $\sum_k |a_{mk}|$ is finite, there exists $p = p(\varepsilon)$ such that $\sum_{k>p} |a_{mk}| < \frac{\varepsilon}{2}$.

For all $\gamma \in \Lambda$, define the uncertain null sequence $\zeta = \{\zeta_k\}$ with respect to almost surely by

$$\zeta_k(\gamma) = \begin{cases} sgn\ a_{nk} & 1 \leq k \leq p; \\ 0 & k > p. \end{cases}$$

Then $||\zeta(\gamma)|| = 1$ and

$$\frac{||A(\zeta(\gamma))||}{||\zeta_(\gamma)||} = \sup_n ||A_n(\zeta(\gamma))|| \geq ||A_n(\zeta(\gamma))|| > M - \varepsilon$$

It implies that $M = \sup\left\{\frac{||A(\zeta(\gamma))||}{||\zeta(\gamma)||} : \zeta(\gamma) \neq 0\right\} = ||A||$. Hence the theorem is proved. □

We now prove the converse case of the above theorem.

Theorem 3.2. *Let $A : c_0(\Gamma_{a.s}) \to c_0(\Gamma_{a.s})$ be any bounded linear operator. Then A determines a matrix (a_{mn}) such that $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$, for every $\gamma \in \Lambda$ and $||A|| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$. Also, $\lim_{n\to\infty} a_{nk} = 0$, uniformly for all k.*

Proof. Let $\zeta \in c_0(\Gamma_{a.s})$. Then $\zeta(\gamma) = \sum_k(\zeta_k(\gamma)e_k)$, where $\{e_n\}$ is a basis in $c_0(\Gamma_{a.s})$ which is given by $e_n = \{e_n^j\}$, and

$$e_n^j(\gamma) = \begin{cases} 1, & n = j; \\ 0, & otherwise; \end{cases}$$

Now

$$A\zeta(\gamma) = \sum_{k=1}^{\infty} \zeta_k(\gamma)Ae_k = \sum_{k=1}^{\infty} \zeta_k(\gamma)a_k^{(n)}, \qquad n \in \mathbb{N},$$

where Ae_k is a sequence $\left\{a_k^{(1)}, a_k^{(2)}, ...\right\} \in c_0(\Gamma_{a.s}); \quad k = 1, 2, 3, ...$ Then,

$$(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_k^{(n)}\zeta_k(\gamma), \qquad n = 1, 2, ...$$

Since each $e_k \in c_0(\Gamma_{a.s})$, therefore $Ae_k \in c_0(\Gamma_{a.s})$ also, for $k = 1, 2, 3, ...$ That implies, $\lim_{n\to\infty} a_{nk} = 0$, keeping k fixed. Thus, $\lim_{n\to\infty} A_n\zeta(\gamma) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$.

We now prove that $||A|| = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$.

For each n, we have, $||A_n\zeta(\gamma)|| \leq ||A\zeta(\gamma)|| \leq ||A||||\zeta||$. Since A is a bounded linear operator and $\zeta \in c_0(\Gamma_{a.s})$, then A_n is a bounded linear functional on $c_0(\Gamma_{a.s})$. Thus we have the sequence $\{A_n\} \in c_0^*(\Gamma_{a.s})$ such that $\lim_{n\to\infty} A_n(\zeta(\gamma)) = 0$. Then, by Banach-Steinhaus theorem, for all n, $||A_n|| \leq H$, for some constant H. By the table of dual spaces in page 110 of [11], $||A_n|| = \sum_k |a_{nk}|$. Then $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and by the above theorem $||A|| = M$. □

Definition 3.3. A complex uncertain sequence $\{\zeta_n\}$ in an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be bounded with respect almost surely if for any $\varepsilon > 0$, there exists event Λ with unit uncertain measure such that

$$\sup_n |\zeta_n(\gamma)| < \infty, \qquad \text{for all } \gamma \in \Lambda.$$

The set of all such types of sequences is denoted by $\ell_\infty(\Gamma_{a.s})$.

Theorem 3.4. *An infinite bounded matrix operator $A = (a_{nk})$ acts from $\ell_\infty(\Gamma_{a.s})$ into $\ell_\infty(\Gamma_{a.s})$ if and only if*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

Proof. Let $\zeta = \{\zeta_k\} \in \ell_\infty(\Gamma_{a.s})$. Then, there exists some event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that $\sup_n ||\zeta_n(\gamma)|| < \infty$, for all $\gamma \in \Lambda$. Suppose $A = (a_{nk})$ be an infinite bounded matrix operator such that $\sup_n \sum_{k=1}^{\infty} |a_{nk}|$ is finite. Since, $A = (a_{nk})$ is bounded uniformly for each n, $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists, for all $\gamma \in \Lambda$. Then,

$$\sup_n ||(A\zeta)_n|| = \sup_n ||\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)|| \leq ||\zeta(\gamma)|| \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty,$$

since $\zeta \in \ell_\infty(\Gamma_{a.s})$. Therefore, $A\zeta \in \ell_\infty(\Gamma_{a.s})$. Hence, A defines a bounded linear operator from $\ell_\infty(\Gamma_{a.s})$ into $\ell_\infty(\Gamma_{a.s})$.

Conversely, let $A \in (\ell_\infty(\Gamma_{a.s}), \ell_\infty(\Gamma_{a.s}))$. That is A transforms a complex uncertain sequence $\zeta \in \ell_\infty(\Gamma_{a.s})$ to another sequence $A\zeta \in \ell_\infty(\Gamma_{a.s})$. This implies

$$\sup_n ||(A\zeta)_n|| = \sup_n || \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)|| < \infty.$$

Then, by Banach-Steinhaus theorem, we have $\sup_n ||A_n|| = \sup_n |a_{nk}| < \infty$. □

Corollary 3.5. *For each bounded linear matrix operator $A = \{a_{nk}\}$ between the spaces $c_0(\Gamma_{a.s})$ to $\ell_\infty(\Gamma_{a.s})$; $c(\Gamma_{a.s})$ to $\ell_\infty(\Gamma_{a.s})$; $c_0(\Gamma_{a.s})$ to $c(\Gamma_{a.s})$; and $c(\Gamma_{a.s})$ to $c(\Gamma_{a.s})$,*

$$(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma), \text{ and } ||A|| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

Proof. Using the techniques same as above, this can established easily. □

Definition 3.6. Let $p = \{p_k\}$ be a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ is finite. The space $[\ell_\infty(p)]_{\Gamma_{a.s}}$ is defined as follows:

$$[\ell_\infty(p)]_{\Gamma_{a.s}} = \{\zeta = \{\zeta_k\} : \sup_{k\in\mathbb{N}} |\zeta_k(\gamma)|^{p_k} < \infty\},$$

where $\zeta = \{\zeta_k\}$ is a complex uncertain sequence and $\gamma \in \Lambda$, Λ being an event with unit uncertain measure.

Theorem 3.7. *An infinite real matrix operator $A \in \left([\ell_\infty(p)]_{\Gamma_{a.s}}, \ell_\infty(\Gamma_{a.s})\right)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}} < \infty, \ \forall\ N \in \mathbb{N}\ (N > 1)$$

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and Λ be an event with $\mathcal{M}\{\Lambda\} = 1$. To prove the necessary part, let us suppose the infinite matrix operator

$$A \in \left([\ell_\infty(p)]_{\Gamma_{a.s}}, \ell_\infty(\Gamma_{a.s})\right).$$

Suppose that there exists $N \in \mathbb{N}$ $(N > 1)$ be such that

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}} = \infty.$$

We consider the infinite matrix operator $B = (b_{nk})$ which is defined by

$$b_{nk} = a_{nk} N^{\frac{1}{p_k}}, \qquad \forall\ k, n \in \mathbb{N}.$$

It is obvious that $B \notin (\ell_\infty(\Gamma_{a.s}), \ell_\infty(\Gamma_{a.s}))$. Thus, we can find a complex uncertain sequence $\zeta = \{\zeta_k\} \in \ell_\infty(\Gamma_{a.s})$ with $||\zeta|| = 1$ such that

$$\sum_{k=1}^{\infty} b_{nk}\zeta_k(\gamma) \notin \ell_\infty(\Gamma_{a.s}), \text{ where } \gamma \in \Lambda.$$

Then,

$$\sum_{k=1}^{\infty} a_{nk} N^{\frac{1}{p_k}} \zeta_k(\gamma) \notin \ell_\infty(\Gamma_{a.s}), \text{ for } \gamma \in \Lambda.$$

Let $\eta = \{\eta_k\}$ be a complex uncertain sequence such that $\eta_k(\gamma) = N^{\frac{1}{p_k}}\zeta_k(\gamma), \ \forall \ \gamma \in \Gamma$. Therefore, $\eta \in [\ell_\infty(p)]_{\Gamma_{a.s}}$, but

$$(A\eta)_n = A_n\eta(\gamma) = \sum_{k=1}^{\infty} a_{nk} N^{\frac{1}{p_k}} \zeta_k(\gamma) \notin \ell_\infty(\Gamma_{a.s}).$$

This is a contradiction to our hypothesis that $A \in \left([\ell_\infty(p)]_{\Gamma_{a.s}}, \ell_\infty(\Gamma_{a.s})\right)$.

For the sufficient part, let $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}}$ be finite and $\zeta = \{\zeta_k\} \in [\ell_\infty(p)]_{\Gamma_{a.s}}$. We consider a natural number N such that

$$N > \max\left\{1, \sup_{k\in\mathbb{N}} |\zeta_k(\gamma)|^{\frac{1}{p_k}}\right\}, \quad \text{for } \gamma \in \Lambda.$$

Then,

$$\sup_{k\in\mathbb{N}} |(A\zeta)_n| \le \sup_{k\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| N^{\frac{1}{p_k}} < \infty, \quad \forall \ \gamma \in \Lambda.$$

Therefore, $A\zeta \in \ell_\infty(\Gamma_{a.s})$. Hence the theorem is proved. □

In the following section we prove the famous Silverman-Toeplitz theorem and Kojima-Schur theorem considering complex uncertain sequences as application of matrix transformation.

4. Silverman-Toeplitz and Kojima-Schur Theorems

Theorem 4.1 (Silverman-Topelitz Theorem)**.** *A bounded linear operator $A : c(\Gamma_{a.s}) \to c(\Gamma_{a.s})$ preserves the limit if and only if the following conditions are satisfied:*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.
(ii) $a_{nk} \to 0$, *as $n \to \infty$, while k is fixed.*
(iii) $\sum_{n=1}^{\infty} a_{nk} = 1$, *for fixed k.*

Proof. Consider an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $A : c(\Gamma_{a.s}) \to c(\Gamma_{a.s})$, a bounded linear operator which preserves limit. Let k be a fixed positive integer and the complex uncertain variables ζ_n be defined as follows:

$$\zeta_n(\gamma) = \begin{cases} 1 & \text{if } n = k; \\ 0 & \text{otherwise;} \end{cases}$$

and let $\zeta(\gamma) = 0$, for all $\gamma \in \Gamma$. Then, $\lim_{n\to\infty}\{||\zeta_n - \zeta||\} = 0$, the norm operator considered here is due to Chen et al. [14]. Hence, the complex uncertain sequence $\{\zeta_n\}$ is convergent with respect to almost surely and it converges to zero, for any fixed k. Thus, $\sum_{n=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$, (by our hypothesis), which implies $\lim_{n\to\infty} a_{nk} = 0$, uniformly for all k. Thus, the condition (ii) is proved.

For the necessity of (iii), let $\{\zeta_n\}$ be a complex uncertain sequence such that $\zeta_n = 1, \ \forall \ n \in \mathbb{N}$ and let $\zeta = 1$. Then, $||\zeta_n - \zeta|| = 0$, for all $n \in \mathbb{N}$. Thus the sequence $\{\zeta_n\}$ converges to ζ with respect to almost surely. Consequently, the transformed sequence $(A\zeta_k)_n = A_n(\zeta_k)$ where $A_n(\zeta_k) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ also converges to $\zeta = 1$. Therefore, $\sum_{n=1}^{\infty} a_{nk}\zeta_k(\gamma) \longrightarrow 1$, as $n \longrightarrow \infty$, for fixed k, which implies

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1, \qquad \text{as } \zeta_k(\gamma) = 1.$$

Now, $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists for each n and tends to ζ, whenever $\{\zeta_k\}$ converges to ζ with respect to almost surely. Then by corollary 3.5, we can say that $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

For sufficiency, let the three conditions holds true and the complex uncertain sequence $\{\zeta_n\}$ converges with respect to almost surely to ζ.

Now,

$$\sum_{n=1}^{\infty} a_{nk}\zeta_k(\gamma) = \sum_{n=1}^{\infty} a_{nk}(\zeta_k(\gamma) - \zeta(\gamma)) + \zeta(\gamma)\sum_{n=1}^{\infty} a_{nk}.$$

Using condition (i) and the fact that $\zeta_n \to \zeta$ with respect to almost surely, we have the first term of the right hand side of the above equation is zero. Again by condition (iii), the second term of the right hand side tends to ζ. Therefore, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = \zeta$. Hence $A \in (c(\Gamma_{a.s}), c(\Gamma_{a.s}))$ and it keeps the limit preserved. □

Theorem 4.2 (Kojima-Schur Theorem)**.** *The linear operator $A : c(\Gamma_{a.s}) \to c(\Gamma_{a.s})$ is bounded operator if and only if the following conditions are satisfied.*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}|$ *is finite;*
(ii) *for each $p \in \mathbb{N}$, there exists $a_p = \lim_n \sum_{k=p}^{\infty} a_{nk}$.*

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space and $A : c(\Gamma_{a.s}) \to c(\Gamma_{a.s})$ be a bounded linear operator. Suppose the complex uncertain sequence $\{\zeta_n\} \in c(\Gamma_{a.s})$ converges to ζ.
Then, by the theorem 4.1, $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and thus (i) is proved.
Consider the complex uncertain sequence $\{\zeta_n\}$ in such a way that

$$\zeta_n(\gamma) = \begin{cases} 0 & n < p; \\ 1 & otherwise; \end{cases}$$

for some finite p and $\zeta(\gamma) = 1,\ \forall \gamma \in \Gamma$. Then, $\lim_{n\to\infty} ||\zeta_n - \zeta|| = 0$ and so $\{\zeta_n\} \in c(\Gamma_{a.s})$, which converges to $\zeta = 1$ with respect to almost surely. Thus $\lim_{n\to\infty} A\zeta_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\zeta_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = a_p$.

Conversely, let conditions (i) and (ii) hold true and $\{\zeta_n\} \in c(\Gamma_{a.s})$ converges to ζ. Then,

$$\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = \sum_{k=1}^{\infty}(\zeta_k(\gamma) - \zeta(\gamma)) + \zeta(\gamma)\sum_{k=1}^{\infty} a_{nk} = S_{\Gamma_n} + \zeta\sum_{k=1}^{\infty} a_{nk},$$

where $S_{\Gamma_n} = \sum_{k=1}^{\infty}(\zeta_k(\gamma) - \zeta(\gamma))$. Now, by condition (i), $\zeta(\gamma)\sum_{k=1}^{\infty} a_{nk}$ tends to $\zeta(\gamma)a_1$. Suppose,

$$b_k = \lim_{n\to\infty} a_{nk} = \lim_{n\to\infty}\left(\sum_{j=k}^{\infty} a_{nj} - \sum_{j=k+1}^{\infty} a_{nj}\right) = a_k - a_{k+1},$$

for each k. So,

$$\sum_{k\to\infty} |b_k| = \sum_{k\to\infty} |\lim_{n\to\infty} a_{nk}| \le \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty \qquad \text{(by condition (i))}.$$

Again,

$$\sum_{k=1}^{\infty}(a_{nk} - b_k)||\zeta_k(\gamma) - \zeta(\gamma)|| = \sum_{k=1}^{\infty}(a_{nk} - b_k)||\zeta_k(\gamma) - \zeta(\gamma)||.$$

Since $\{\zeta_n(\gamma)\}$ converges to ζ almost surely, so S_{Γ_n} tends to $\sum_{k=1}^{\infty} b_k(\zeta_k(\gamma) - \zeta(\gamma))$. Hence $A \in (c(\Gamma_{a.s}), c(\Gamma_{a.s.}))$. □

5. Conclusion

In this article, we made an initial study of matrix transformation of complex uncertain sequences by applying the notion of convergent uncertain series. Application of matrix transformation is shown by establishing Silverman-Toeplitz theorem and Kojima-Schur theorem considering convergent complex uncertain sequences. This study can be extended for further generalization in this direction.

References

[1] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Statistical convergence of complex uncertain triple sequence*, Comm. Stat. Theo. Meth., (2021), doi: 10.1080/03610926.2020.1871016.

[2] B. Das, B. C. Tripathy, P. Debnath, J. Nath and B. Bhattacharya, *Almost convergence of complex uncertain triple sequences*, Proc. Nat. Aca. Sci. Ind. Sec. A Phy. Sci., **91** (2021), No. 2, 245-256.

[3] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Characterization of statistical convergence of complex uncertain double sequence*, Anal. Math. Phys., **10** (2020), No. 4, 1-20.

[4] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Study of Matrix Transformation of Uniformly Almost Surely Convergent Complex Uncertain Sequences*, Filomat, **34** (2020), No. 14, 4907-4922.

[5] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Almost Convergent Complex Uncertain Sequences Double Sequences*, Filomat, **35** (2021), No. 1, 61-78.

[6] B. Das, P. Debnath and B. C. Tripathy, *Characterization of Matrix Classes Transforming Between Almost Sure Convergent Sequences of Complex Uncertain Variables,* Journal of Uncertain Systems, **14** (2021), No. 3, 2150019.

[7] B. Liu, *Uncertainty Theory*, Springer-Verlag, Berlin, 2007.

[8] C. You, *On the Convergence of Uncertain Sequences*, Math. Comp. Model., **49** (2009), 482–487.

[9] I. J. Maddox, *Matrix Transformation of (C,-1), Summable Series*, Proc. Kon. Ned. Aka. Van. Wet., **A 68** (1965), 129–132.

[10] I. J. Maddox, *Matrix Transformation in a Banach Space*, Proc. Kon. Ned. Aka. Van. Wet., **A 69** (1966), 25–29.

[11] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, Cambridge, 1970

[12] P. K. Nath and B. C. Tripathy, *Convergent Complex Uncertain Sequences Defined by Orlicz Function*, Anal. Univ. Crai. Math. Comp. Sci. Ser., **46** (2019), no. 1, 139–149.

[13] S. Saha, B. C. Tripathy and S. Roy, *Some Results on p-distance and Sequence of Complex Uncertain Variables*, Commu. Kor. Math. Soc., **35** (2020), no. 3, 907–916.

[14] X. Chen, Y. Ning and X. Wang, *Convergence of Complex Uncertain Sequences*, J. Int. Fuzzy Syst., **30** (2016), 3357–3366.

[15] Z. Peng, *Complex Uncertain Variable* [Ph D Thesis]. Tsinghua University, Tsinghua, 2012

Birojit Das: dasbirojit@gmail.com
Department of Mathematics, NIT Agartala, India, 799046

Binod Chandra Tripathy: tripathybc@rediffmail.com
Department of Mathematics, Tripura University, India, 799022

Piyali Debnath: piyalimaths@gmail.com
Department of Mathematics, NIT Agartala, India, 799046

Received 05/03/2021; Revised 13/10/2021