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ON A CLASS OF FILTERS IN THE WATSON FOURIER WAVELET SETTING

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Abstract. In this paper, using the theory of harmonic analysis related to the Watson-Fourier transform, we study a linear time invariant filter. Also, we show that this linear time invariant filter can be expressed in the form of Watson Fourier wavelet transform. Finally, the Fredholm integral equation is defined and we give a solution of this integral equation. Next, an application of the linear time invariant filter is given in the theory of the aforesaid integral equation.

У даній роботі, використовуючи пов'язаний з перетворення Уотсона-Фур'є гармонічний аналіз, вивчено лінійний інваріантний за часом фільтр. Показано, що цей лінійний інваріантний за часом фільтр може бути представлений у вигляді вейвлет-перетворення Уотсона Фур'є. Також визначено інтегральне рівняння Фредгольма і наведено розв'язок цього інтегрального рівняння. Надано застосування лінійного інваріантного за часом фільтра до теорії вищезгаданого інтегрального рівняння.

1. Introduction

The classical theory of filtering is developed for discrete and continuous signals. The relation of the filtering theory with the Fourier analysis is of paramount importance and it was exploited in many different parts in [1]. An important class of filters is given by linear, time invariant filters. These are linear applications that transform a signal into another signal, such that the application commutes with time shifts. In [1] the authors have studied the notion of a linear time invariant filter associated with the Fourier transform, they have shown that the output signal from a linear time invariant filter of a sinusoidal input signal is also sinusoidal with the same frequency. Using the aforesaid result, they have exploited the harmonic analysis associated with the Fourier transform to express the linear time invariant filter in terms of convolution.

In the present paper, motivated by the above results, a linear time invariant filter associated with the Watson Fourier transform and Watson Fourier convolution is investigated.

The remaining part of the paper is organized as follows. Section 2 is a summary of the main results in the harmonic analysis associated with the Watson Fourier transform. In Section 3, we will introduce and study a linear time invariant filter associated with the Watson Fourier transform and the Watson Fourier convolution. We will prove that the output from a linear time invariant filter of a symmetrical Fourier kernel is also a symmetrical Fourier kernel function with the same frequency. Next, we will show that this linear time invariant filter can be expressed in a form of a Watson Fourier wavelet transform. In Section 4, an application of a linear time invariant filter is given in the theory of the Fredholm integral equation.

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2. Preliminaries

In this section, we shall collect some results and definitions from the theory of harmonic analysis associated with the Watson Fourier transform which generalizes many integral transforms, e.g., $\mathcal{Y}$ transform, Hankel transform, G- and H-transforms. For more details we refer to [2, 3, 5, 6].

We denote by:

$L^p = L^p(0,\infty)$ the class of measurable functions f on $(0,\infty)$ for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}}, \quad if\ p < \infty,$$
$$\|f\|_\infty = \|f\|_\infty = ess\ sup_{x\in(0,\infty)}|f(x)|.$$

$D(\mathbb{R})$ the space of test functions with bounded support in $\mathbb{R}$.

From [4], we recall the definitions of Mellin transform and its inverse.

Definition 2.1. Let $f(t)$ be a function defined on the positive real axis $0 < t < \infty$. The Mellin transformation $\mathcal{M}$ is the operation mapping of the function f into the function $\mathcal{F}$ defined on the complex plane by the relation:

$$\mathcal{M}[f : s] = \mathcal{F}(s) = \int_0^\infty t^{s-1} f(t)dt. \tag{2.1}$$

The function $\mathcal{F}(s)$ is called the Mellin transform of $f(t)$.

Definition 2.2. The inversion formula for Mellin transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} t^{-s}\mathcal{F}(s)ds. \tag{2.2}$$

where the integration is along a vertical line through $Re(s) = a$.

Let $(\mathcal{M}\mathcal{K})(s)$ satisfy the following condition:

$$(\mathcal{M}\mathcal{K})(s)(\mathcal{M}\mathcal{K})(1-s) = 1,$$

and from (2.2), let $\mathcal{K}(x)$ be the inverse Mellin transform of $(\mathcal{M}\mathcal{K})(s)$.

Definition 2.3. The Watson Fourier transform of a function $f \in L^1$ is formally defined by

$$(\mathcal{W}f)(x) = \int_0^\infty \mathcal{K}(xt)f(t)dt, \tag{2.3}$$

where $\mathcal{K}(x)$ is called a symmetric Fourier kernel.

Definition 2.4. The inversion formula of Watson Fourier transform is defined as

$$f(t) = \int_0^\infty \mathcal{K}(xt)(\mathcal{W}f)(x)dx, \tag{2.4}$$

From [3], we define the basic function

$$\mathcal{D}(t,x,z) = \int_0^\infty \mathcal{K}(ts)\mathcal{K}(xs)\mathcal{K}(zs)ds. \tag{2.5}$$

The above integral is convergent under the assumption $\mathcal{K} \in L^1 \cap L^\infty$ and we assume that $\mathcal{K}(0) = 1$ and $\mathcal{D}(x,y,z) > 0, \quad \forall x,y,z \in (0,\infty)$.

The inversion of (2.5) is formally given by

$$\mathcal{K}(xt)\mathcal{K}(yt) = \int_0^\infty \mathcal{D}(x,y,z)\mathcal{K}(zt)dz. \tag{2.6}$$

Taking $t = 0$ in (2.6), we get

$$\int_0^\infty \mathcal{D}(x, y, z)dz = 1. \tag{2.7}$$

Definition 2.5. Let $f \in L^1 \cap L^2$, the Watson Fourier translation is defined by

$$\tau_t f(x) = f(t, x) = \int_0^\infty f(z)\mathcal{D}(t, x, z)dz, \quad 0 < t,\ x < \infty, \tag{2.8}$$

Definition 2.6. Let $f, g \in L^1 \cap L^2$. The Watson Fourier convolution of f and g is defined by

$$f \star g = \int_0^\infty \tau_t f(y)g(y)dy = \int_0^\infty f(t, y)g(y)dy. \tag{2.9}$$

The Watson Fourier wavelet is defined, in [5, 6], for $\psi \in L^p$ by

$$\psi_{b,a}(x) = a^{-1}\psi\left(\frac{b}{a}, \frac{x}{a}\right) = \psi_a(b, x) = a^{-1}\int_0^\infty \psi(z)\mathcal{D}(\frac{b}{a}, \frac{x}{a}, z)dz,\ b \geq 0,\ a > O, \tag{2.10}$$

Definition 2.7. The Watson Fourier wavelet transform is defined by [5, 6]

$$(\mathcal{W}_\psi \phi)(b, a) = \int_0^\infty \phi(x)\overline{\psi_{b,a}(x)}dx = a^{-1}\int_0^\infty\int_0^\infty \phi(x)\overline{\psi(z)}\mathcal{D}(\frac{b}{a}, \frac{x}{a}, z)dzdx \tag{2.11}$$

Lemma 2.8 ([5])**.** *Let $\phi, \psi \in L^1$. Then*

$$(\mathcal{W}_\psi \phi)(b, a) = (\phi \star \overline{\psi_a})(b). \tag{2.12}$$

3. Linear time invariant filter

In this section, we define and study the linear time invariant filter associated with the Watson Fourier transform.

A filter L is described as "linear time invariant" if it has the following three properties:

(i) Time invariance:

If a time delay can be applied either before or after filtering, it yields the same result. That is,

$$\mathrm{L}\left[\tau_a f\right](t) = \mathrm{L}\left[f_a\right](t) = \tau_a \mathrm{L}\left[f\right](t) = \mathrm{L}\left[f\right](t, a),$$

where τ_a is given by (2.8).

(ii) Superposition invariance,

$$\mathrm{L}\left[f + g\right] = \mathrm{L}\left[f\right] + \mathrm{L}\left[g\right].$$

(iii) Scale invariance,

$$\mathrm{L}\left[cf\right] = c\mathrm{L}\left[f\right].$$

The next lemma shows that the symmetrical Fourier kernel $\mathcal{K}(.)$ is an eigenvector of the filter L.

Lemma 3.1. *Let* L *be a linear time invariant filter and λ any fixed nonnegative real number. Then there exists a function $\varphi \in L^1$ such that*

$$\mathrm{L}\left[\mathcal{K}(t\lambda)\right] = \mathcal{K}(t\lambda)\mathcal{W}(\varphi)(\lambda). \tag{3.13}$$

Proof. Let $\varphi^\lambda(t) = \mathrm{L}(\mathcal{K}(t\lambda))$. Since L is time invariant, we get

$$\mathrm{L}\left[\tau_\xi \mathcal{K}(t\lambda)\right] = \mathrm{L}\left[\mathcal{K}(\lambda\xi)\mathcal{K}(t\lambda)\right] = \varphi^\lambda(t, \xi) \tag{3.14}$$

for each $\xi \in \mathbb{R}^+$. Since L is linear, we also have

$$\begin{aligned} \mathrm{L}\left[\mathcal{K}(\lambda\xi)\mathcal{K}(t\lambda))\right] &= \mathcal{K}(\lambda\xi)\mathrm{L}\left[\mathcal{K}(t\lambda)\right], \\ &= \mathcal{K}(\lambda\xi)\varphi^\lambda(t). \end{aligned} \tag{3.15}$$

Thus, by virtue of (3.14) and (3.15), we conclude that

$$\varphi^{\lambda}(t,\xi) = \mathcal{K}(\lambda\xi)\varphi^{\lambda}(t). \tag{3.16}$$

Taking $t = 0$ in (3.16), we get

$$\varphi^{\lambda}(0,\xi) = \mathcal{K}(\lambda\xi)\varphi^{\lambda}(0),$$

hence

$$\varphi^{\lambda}(\xi) = \mathcal{K}(\lambda\xi)\varphi^{\lambda}(0).$$

Since ξ is arbitrary, we may set $\xi = t$, then

$$\varphi^{\lambda}(t) = \mathcal{K}(t\lambda)\varphi^{\lambda}(0).$$

Letting $\varphi^{\lambda}(0) = \mathcal{W}(\varphi)(\lambda) \in L^1$ and $\varphi \in L^1$, we have

$$\mathrm{L}\left[\mathcal{K}(t\lambda)\right] = \varphi^{\lambda}(t) = \mathcal{K}(t\lambda)\mathcal{W}(\varphi)(\lambda)$$

which completes the proof. □

Thus, we can conclude that the symmetric Fourier kernel $\mathcal{K}(.)$ is an eigenvector of the filter L with the corresponding eigenvalue $(\mathcal{W}\varphi)(\lambda)$.

Application of a linear, time invariant filter L to a signal f is equivalent to taking the convolution of f with a signal φ, called the impulse response of the filter or the kernel of the filter. It is the objective of the following theorem.

Theorem 3.2. *Let* L *be a linear, time invariant transformation on the space of signals that are piecewise continuous functions. Then there exists an integrable function, φ, such that*

$$\mathrm{L}(f) = f \star \varphi. \tag{3.17}$$

Proof. From the inversion formula of the Watson Fourier transform (2.4), we have

$$f(t) = \int_0^{\infty} \mathcal{K}(t\lambda)(\mathcal{W}f)(\lambda)d\lambda,$$

Then, we apply L to both sides,

$$(\mathrm{L}f)(t) = \mathrm{L}\left[\int_0^{\infty} \mathcal{K}(t\lambda)(\mathcal{W}f)(\lambda)d\lambda\right].$$

The integral on the right-hand can be approximated by a Riemann sum with a uniform sampling partition with interval length $\Delta(\lambda)$,

$$\mathrm{L}\left[\int_0^{\infty} \mathcal{K}(t\lambda)(\mathcal{W}f)(\lambda)d\lambda\right] \approx \mathrm{L}\left[\sum_j \mathcal{K}(t\lambda_j)(\mathcal{W}f)(\lambda_j)\Delta(\lambda)\right],$$

Since L is linear, we can distribute L across the sum:

$$\mathrm{L}\left[\sum_j \mathcal{K}(t\lambda_j)(\mathcal{W}f)(\lambda_j)\Delta(\lambda)\right] = \sum_j (\mathcal{W}f)(\lambda_j)\mathrm{L}(\mathcal{K}(t\lambda_j))\Delta(\lambda).$$

The Riemann sum on the right-hand side of the above expression becomes an integral and so, from Lemma 3.1, we obtain

$$\begin{aligned}
(\mathrm{L}f)(t) &= \int_0^\infty (\mathcal{W}f)(\lambda)\mathrm{L}\left[\mathcal{K}(t\lambda)\right]d\lambda,\\
&= \int_0^\infty (\mathcal{W}f)(\lambda)\mathcal{W}(\varphi)(\lambda)\mathcal{K}(t\lambda)d\lambda,\\
&= \mathcal{W}^{-1}\left[(\mathcal{W}f)(\lambda)\mathcal{W}(\varphi)(\lambda)\right](t),\\
&= \mathcal{W}^{-1}\left[\mathcal{W}(f\star\varphi)(\lambda)\right](t),\\
&= (f\star\varphi)(t).
\end{aligned}$$

which completes the proof. □

Example 3.3. Let h be a function that has finite support. For a signal f, let

$$(\mathrm{L}f)(t) = (h\star f)(t) = \int_0^\infty f(x)\tau_t h(x)dx. \tag{3.18}$$

Then L is a linear time invariant operator.

Indeed

$$\begin{aligned}
\tau_a(\mathrm{L}f)(t) &= (\mathrm{L}f)(t,a)\\
&= \int_0^\infty \left(\int_0^\infty h(z,x)f(x)dx\right)\mathcal{D}(t,a,z)dz,\\
&= \int_0^\infty \left[\int_0^\infty \left(\int_0^\infty h(y)\mathcal{D}(z,x,y)d\mu(y)\right)f(x)dx\right]\mathcal{D}(t,a,z)dz,\\
&= \int_0^\infty \left[\int_0^\infty \left(\int_0^\infty f(x)\mathcal{D}(z,x,y)d\mu(x)\right)h(y)dy\right]\mathcal{D}(t,a,z)dz,\\
&= \int_0^\infty \left[\int_0^\infty f(z,y)h(y)dy\right]\mathcal{D}(t,a,z)dz,\\
&= \int_0^\infty \left[\int_0^\infty f(z,y)\mathcal{D}(t,a,z)dz\right]h(y)dy,\\
&= \int_0^\infty f(t,a,y)h(y)dy,\\
&= \int_0^\infty f_a(t,y)h(y)dy,\\
&= (\mathrm{L}f_a)(t),\\
&= (\mathrm{L}\tau_a f)(t).
\end{aligned}$$

The following theorem shows that a linear, time invariant filter can be expressed as a Watson Fourier wavelet transform.

Theorem 3.4. *Let $f \in L^1$ and $\psi_a \in L^1$. Then the Watson Fourier wavelet transform can be expressed as*

$$(\mathcal{W}_\psi f)(t,a) = \mathrm{L}(f)(t), \tag{3.19}$$

where L *is a linear time invariant filter*

Proof. Using (2.12), we have

$$\begin{aligned}\mathcal{W}_\psi(f)(t,a) &= (f\star\overline{\psi}_a)(t),\\ &= \mathcal{W}^{-1}\left[(\mathcal{W}f)(\lambda)\mathcal{W}(\overline{\psi}_a)(\lambda)\right](t),\\ &= \int_0^\infty (\mathcal{W}f)(\lambda)\times\mathcal{W}(\overline{\psi}_a)(\lambda)\mathcal{K}(t\lambda)d\lambda,\\ &= \int_0^\infty \left[\mathcal{K}(t\lambda)\mathcal{W}(\overline{\psi}_a)(\lambda)\right](\mathcal{W}f)(\lambda)d\lambda.\end{aligned}$$

By virtue of Lemma 3.1 and the inversion formula, we have

$$\begin{aligned}\mathcal{W}_\psi(f)(t,a) &= \int_0^\infty \mathrm{L}\left[\mathcal{K}(t\lambda)\right](\mathcal{W}f)(\lambda)d\lambda,\\ &\simeq \sum_j \mathrm{L}\left[\mathcal{K}(t\lambda_j)\right](\mathcal{W}f)(\lambda_j)\Delta(\lambda),\\ &\simeq \mathrm{L}\left[\sum_j \mathcal{K}(t\lambda_j)(\mathcal{W}f)(\lambda_j)\Delta(\lambda)\right],\\ &= \mathrm{L}\left[\int_0^\infty \mathcal{K}(t\lambda)(\mathcal{W}f)(\lambda)d\lambda\right],\\ &= \mathrm{L}(f)(t).\end{aligned}$$

□

4. Fredholm integral equation

In this section, we apply a linear time invariant filter associated with the Watson Fourier transform to the theory of Fredholm integral equation type.

The Fredholm type integral equation is defined by

$$\int_0^\infty f(t)\overline{\varphi}(x,t)dt+\lambda f(x)=u(x), \tag{4.20}$$

where $\varphi(x)$ and $u(x)$ are given functions and λ is a known parameter.

Using (2.9), equation (4.20) can be written as

$$(f\star\overline{\varphi})(x)+\lambda f(x)=u(x). \tag{4.21}$$

Theorem 4.1. *Let $f\in L^1$ and $\varphi\in L^1$. Then a solution of equation (4.20) is*

$$f(x)=\int_0^\infty \mathcal{K}(x\xi)\frac{(\mathcal{W}u)(\xi)}{(\mathcal{W}\overline{\varphi})(\xi)+\lambda}d\xi. \tag{4.22}$$

Proof. Application of the Watson Fourier transform defined by (2.3) to (4.21) gives

$$(\mathcal{W}f)(\xi)\times(\mathcal{W}\overline{\varphi})(\xi)+\lambda(\mathcal{W}f)(\xi)=(\mathcal{W}u)(\xi),$$

Hence,

$$(\mathcal{W}f)(\xi)=\frac{(\mathcal{W}u)(\xi)}{(\mathcal{W}\overline{\varphi})(\xi)+\lambda}.$$

The inverse of the Watson Fourier transform leads to a formal solution

$$f(x)=\int_0^\infty \mathcal{K}(x\xi)\frac{(\mathcal{W}u)(\xi)}{(\mathcal{W}\overline{\varphi})(\xi)+\lambda}d\xi.$$

□

Theorem 4.2. *Fredholm type equation (4.20) can be written as a linear time invariant filter,*

$$(\mathrm{L}f)(t) + \lambda f(t) = u(t). \tag{4.23}$$

Proof. From (4.20), we have the Fredholm type integral equation

$$\int_0^\infty f(x)\overline{\varphi}(x,t)dx + \lambda f(t) = u(t).$$

Let $\varphi(x,t) = a^{-1}\psi(\frac{x}{a}, \frac{t}{a})$. Then we have

$$\int_0^\infty f(x)a^{-1}\overline{\psi}(\frac{x}{a}, \frac{t}{a})dx + \lambda f(t) = u(t).$$

In virtue of (2.10) we conclude that

$$\int_0^\infty f(x)\overline{\psi_{t,a}}(x)dx + \lambda f(t) = u(t).$$

From the definition of Watson Fourier wavelet transform (2.3), we get

$$\mathcal{W}_\psi(f)(t,a) + \lambda f(t) = u(t).$$

Hence, from (3.19), we get

$$(\mathrm{L}f)(t) + \lambda f(t) = u(t).$$

□

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