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DONOHO-STARK THEOREM FOR THE QUADRATIC-PHASE FOURIER INTEGRAL OPERATORS

EL MEHDI BERKAK, EL MEHDI LOUALID, AND RADOUAN DAHER

Abstract. In this paper, we obtain a generalization of the Donoho-Stark uncertainty principle associated with the Quadratic-Phase Fourier integral operators which is defined as a generalization of several integral transforms whose kernel has an exponential form.

У цій роботі ми отримуємо узагальнення принципу невизначеності Доного-Старка, пов'язане з квадратично-фазовим інтегральним оператором Фур'є, який визначається як узагальнення кількох інтегральних перетворень з ядрами експоненціальної форми.

1. Introduction

The uncertainty principle is a fundamental principle in mathematics and physics, and also plays an important role in signal processing. It states that a function and its Fourier transform cannot be simultaneously well concentrated, i.e if the supports of a function $f \in L^1\left(\mathbb{R}^d\right)$ and its Fourier transform $\widehat{f}$ are contained in bounded rectangles, then f vanishes almost everywhere. Donoho and Stark [5] gave qualitative uncertainty principles for the Fourier transforms, namely we say that f is ε-concentrated on a measurable set Ω if

$$\|f - \chi_\Omega f\| < \varepsilon.$$

Donoho and Stark [5] show that if f of unit L^2-norm is $\varepsilon_{\mathbb{A}}$ concentrated on a measurable set $\mathbb{A}$ and its Fourier transform $\widehat{f}$ is $\varepsilon_{\mathbb{B}}$ concentrated on a measurable set $\mathbb{B}$, then

$$|\mathbb{A}|\ |\mathbb{B}| \geq \left(1 - \varepsilon_{\mathbb{A}} - \varepsilon_{\mathbb{A}}\right)^2.$$

Here, $|\mathbb{A}|$ denote the Lebesque measure of the set $\mathbb{A}$.

There are various mathematical formulations for this principle as well as extensions to other transforms, see for example [1, 2, 7, 9, 10], we refer also to the book [6] and the surveys [3, 8] for further references. The main goal of this paper is to establish the Donoho–Stark uncertainty principle for the Quadratic-Phase Fourier integral operators newly introduced by Castro et al [4] which is defined as a generalization of several integral transforms whose kernel has an exponential form such as Fourier, fractional Fourier, and linear canonical transforms.

The remaining part of the paper is organized as follows. Section 2 is a summary of the main results in the harmonic analysis associated with Quadratic-Phase Fourier integral. In Section 3, we introduce some further notation as well as some auxiliary results which are required to prove the mains results of this paper.

2. Preliminaries

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Quadratic-Phase Fourier integral operators $\mathbb{Q}$. For more details we refer to [4].

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We denote by:

$L^p = L^p(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}}, \quad if\ 1 < p < \infty,$$

$$\|f\|_\infty = \|f\|_\infty = ess\ sup_{x\in\mathbb{R}}|f(x)|.$$

$\mathcal{D}(\mathbb{R})$ is the space of even C^∞-function on $\mathbb{R}$ with compact support.

$\mathcal{C}_0(\mathbb{R})$ the Banach space of all continuous functions on $\mathbb{R}$ that vanish at infinity, endowed with the supremum norm $\|f\|_\infty$.

Let $f \in L^1$ or $f \in L^2$, the Quadratic-Phase Fourier integral operators $\mathbb{Q}$ is defined by

$$\mathbb{Q}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iQ_{(a-e)}(x,y)} f(y)\, dy, \tag{2.1}$$

where

$$Q_{(a,b,c,d,e)}(x,y) := Q_{(a-e)}(x,y) = ax^2 + bxy + cy^2 + dx + ey, \tag{2.2}$$

for parameters $a, b, c, d, e \in \mathbb{R}$ (with $b \neq 0$).

The importance of the Quadratic-Phase Fourier integral operators $\mathbb{Q}$ lies in the fact that it generalizes many integral transforms. In fact,

- For $a = c = d = e = 0$ and $b = \pm 1$, $\mathbb{Q}$ is simply the well-known Fourier and inverse Fourier integral transforms, respectively.
- For $d = e = 0$, the kernel generated by Eq. (2.2) includes the kernel of the linear canonical transform as well as of the one of the fractional Fourier transform.

Lemma 2.1. *If $f \in L^1$ then $\mathbb{Q}(f) \in \mathcal{C}_0(\mathbb{R})$, and $\|\mathbb{Q}(f)\|_\infty \leq \|f\|_1$.*

Theorem 2.2. *If $f \in L^1$ and $\mathbb{Q}f \in L^1$, then*

$$f(x) = \frac{b}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{Q}(f)(y)\, e^{-iQ_{(a-e)}(y,x)} dy, \quad \forall x \in \mathbb{R}. \tag{2.3}$$

Theorem 2.3. *(Parseval type identity)*

- *For any f, $g \in L^2$, we have*

$$\langle \mathbb{Q}(f), \mathbb{Q}(g)\rangle = \langle f, g\rangle,$$

where $\langle ., .\rangle$ denote the usual inner product in $L^2(\mathbb{R})$ given by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\, \overline{g(x)} dx.$$

- *If $f = g$, then we have*

$$\|\mathbb{Q}(f)\|_2 = \frac{1}{\sqrt{|b|}} \|f\|_2.$$

3. Donoho-Stark uncertainty principles for Quadratic-Phase Fourier Transform

Definition 3.1. Let $f \in L^p$, we say that f is ε_E-concentrated on a measurable set E if there is a function g vanishing outside E such that

$$\|f - g\|_p \leq \varepsilon_E \|f\|_p.$$

We consider a pair of operators, namely

- The orthogonal projection operator define as follow:

$$(P_E f)(x) = \begin{cases} f(x), & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

- The frequency-limiting operator define as follow

$$\mathbb{Q}(Q_E f) = P_E \mathbb{Q}(f). \tag{3.5}$$

Using the Definition 3.1 and (3.4), we have for $f \in L_2$ that$f$ is ε_E-concentrated on E if and only if

$$\|f - P_E f\|_p \leq \varepsilon_E \|f\|_p.$$

Using again Definition 3.1 and (3.5), we get for $f \in L_2$ that $\mathbb{Q}(f)$ is ε_F-concentrated on F if and only if

$$\|\mathbb{Q}(f) - \mathbb{Q}(Q_F f)\|_2 \leq \frac{\varepsilon_F}{\sqrt{|b|}} \|f\|_2. \tag{3.6}$$

If F is a set of finite measure of $\mathbb{R}$, we put $|F| = \int_F dx$.

Lemma 3.2. *If $|F| < \infty$ and $f \in L^1 \cap L^2$, then*

$$Q_F f(x) = \frac{b}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{Q}(f)(y)\, e^{-iQ_{(a-e)}(x,y)} dy.$$

Proof. Let $f \in L^1 \cap L^2$, then by Parseval theorem and Hölder's inequality, we have

$$\begin{aligned} \|\mathbb{Q}(Q_F f)\|_1 &= \int_{\mathbb{R}} |(P_F \mathbb{Q}(f))(x)| dx, \\ &= \int_{\mathbb{R}} |(\chi_F \mathbb{Q} f)(x)| dx, \\ &\leq \left(\int_{\mathbb{R}} |\chi_F(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\mathbb{Q}(f)(x)|^2 dx \right)^{\frac{1}{2}}, \\ &\leq \frac{1}{\sqrt{|b|}} |F|^{\frac{1}{2}} \|f\|_2 < \infty, \end{aligned}$$

and

$$\|\mathbb{Q}(Q_F f)\|_2 = \frac{1}{\sqrt{|b|}} \|f\|_2 < \infty.$$

Hence, $\mathbb{Q}(Q_F f) \in L^1 \cap L^2$. On the basis of (2.3), we obtain

$$Q_F f = \frac{b}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{Q}(f)(y)\, e^{-iQ_{(a-e)}(x,y)} dy. \qquad \square$$

Lemma 3.3. *Let E and F be measurable subset of $\mathbb{R}$. If $f \in L^1 \cap L^2$, then*

$$\|\mathbb{Q}(Q_F P_F f)\|_2 \leq \frac{1}{\sqrt{2\pi}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \|f\|_2.$$

Proof. If at least one of $|E|$ and $|F|$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and F have finite measure.

Let $f \in L^1 \cap L^2$. Using (3.5), we get

$$\mathbb{Q}(Q_F P_E f) = \chi_F \mathbb{Q}(P_E f).$$

Thus,

$$\|\mathbb{Q}(Q_F P_E f)\|_2 = \left(\int_F |\mathbb{Q}(P_E f)(x)|^2 dx \right)^{\frac{1}{2}}. \tag{3.7}$$

Since

$$\mathbb{Q}(P_E f)(x) = \frac{1}{\sqrt{2\pi}} \int_E e^{iQ_{(a-e)}(x,y)} f(x)\, dx,$$

it follows from Hölder's inequality that

$$|\mathbb{Q}\left(P_E f\right)(x)| \leq \frac{1}{\sqrt{2\pi}}|E|^{\frac{1}{2}}\|f\|_2.$$

By (3.7), we get

$$\|\mathbb{Q}\left(Q_F P_E f\right)\| \leq \frac{1}{\sqrt{2\pi}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}\|f\|_2. \qquad \square$$

The following uncertainty principles are our main results.

Theorem 3.4. *Let E and F be measurable subsets of $\mathbb{R}$ and $f \in L^1 \cap L^2$. If f is ε_E-concentrated on E in L^2 and $\mathbb{Q}\left(f\right)$ is ε_F-concentrated on F in L^2, then*

$$\left(1-\left(\varepsilon_E+\varepsilon_F\right)\right)^2 \leq \frac{|b|}{2\pi}|E||F|.$$

Proof. Let $f \in L^1 \cap L^2$. Then

$$\begin{aligned}
\|\left(\mathbb{Q}f\right)-\mathbb{Q}\left(Q_F P_E f\right)\|_2 &\leq \|\mathbb{Q}\left(f\right)-\mathbb{Q}\left(Q_F f\right)\|_2+\|\mathbb{Q}\left(Q_F f\right)-\mathbb{Q}\left(Q_F P_E f\right)\|_2, \\
&\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\|\mathbb{Q}\left(Q_F\left(f-P_E f\right)\right)\|_2, \\
&\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\|P_E\mathbb{Q}\left(f-P_E f\right)\|_2, \\
&\leq \varepsilon_F\|\left(\mathbb{Q}f\right)\|_2+\left(\int_E|\mathbb{Q}\left(f-P_E f\right)(x)|^2dx\right)^{\frac{1}{2}}, \\
&\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\|\mathbb{Q}\left(f\right)-\mathbb{Q}\left(P_E f\right)\|_2.
\end{aligned}$$

On the basis of (3.6), we obtain

$$\begin{aligned}
\|\left(\mathbb{Q}f\right)-\mathbb{Q}\left(Q_F P_E f\right)\|_2 &\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\frac{\varepsilon_E}{\sqrt{|b|}}\|f\|_2, \\
&\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\varepsilon_E\|\mathbb{Q}\left(f\right)\|_2.
\end{aligned}$$

Using the triangle inequality and lemma 3.3, we get

$$\begin{aligned}
\|\mathbb{Q}\left(f\right)\|_2 &\leq \|\mathbb{Q}\left(Q_F P_E f\right)\|_2+\|\mathbb{Q}\left(f\right)-\mathbb{Q}\left(Q_F P_E f\right)\|_2, \\
&\leq \frac{1}{\sqrt{2\pi}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}\|f\|_2+\varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\varepsilon_E\|\mathbb{Q}\left(f\right)\|_2, \\
&\leq \sqrt{\frac{|b|}{2\pi}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}\|\mathbb{Q}\left(f\right)\|_2+\varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\varepsilon_E\|\mathbb{Q}\left(f\right)\|_2. \qquad \square
\end{aligned}$$

Theorem 3.5. *Let E and F be measurable subsets of and $f \in L^1 \cap L^2$. If f is ε_E-concentrated in E in L^1-norm and $\mathbb{Q}\left(f\right)$ is ε_F concentrated on F in L^2-norm, then*

$$\left(1-\varepsilon_E\right)^2\left(1-\varepsilon_E\right)^2 \leq \sqrt{|b|}|E||F|.$$

Proof. Let $f \in L^1 \cap L^2$. Since $\mathbb{Q}\left(f\right)$ is ε_F-concentrated on F in L^2-norm, then, by triangle inequality we get

$$\begin{aligned}
\|\mathbb{Q}\left(f\right)\|_2 &\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+\left(\int_F|\mathbb{Q}\left(f\right)(x)|^2dx\right)^{\frac{1}{2}} \\
&\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+|F|^{\frac{1}{2}}\|\mathbb{Q}\left(f\right)\|_\infty \\
&\leq \varepsilon_F\|\mathbb{Q}\left(f\right)\|_2+|F|^{\frac{1}{2}}\|f\|_1.
\end{aligned}$$

Thus, we obtain

$$\|\mathbb{Q}\left(f\right)\|_2 \leq \frac{|F|^{\frac{1}{2}}}{1-\varepsilon_F}\|f\|_1. \tag{3.8}$$

Since f is ε_E-concentrated on E in L^1-norm, we have

$$\begin{aligned} \|f\|_1 &\leq \varepsilon_E\|f\|_1 + \int_E |f(x)|dx, \\ &\leq \varepsilon_E\|f\|_1 + |E|^{\frac{1}{2}}\|f\|_2, \\ &\leq \varepsilon_E\|f\|_1 + |E|^{\frac{1}{2}}|b|\|\mathbb{Q}(f)\|_2. \end{aligned}$$

Hence, it follows that

$$\|f\|_1 \leq \frac{|E|^{\frac{1}{2}}}{1-\varepsilon_E}|b|\|\mathbb{Q}(f)\|_2. \tag{3.9}$$

By combining (3.8) and (3.9), we get desired result. □

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El Mehdi Berkak: berkakmehdi@gmail.com
Laboratory: Topology, Algebra, Geometry and Discrete Mathematics. Department of Mathematics and Informatics, Faculty of Sciences Aïn Chock, University of Hassan II, B.P 5366 Maarif, Casablanca, Morocco.

El Mehdi Loualid: loualid.m@ucd.ac.ma
Laboratory of Engineering Sciences for Energy, National School of Applied Sciences of El Jadida, University Of Chouaib Doukkali, El Jadida, Morocco.

Radouan Daher: rjdaher024@gmail.com
Laboratory: Topology, Algebra, Geometry and Discrete Mathematics. Department of Mathematics and Informatics, Faculty of Sciences Aïn Chock, University of Hassan II, B.P 5366 Maarif, Casablanca, Morocco.

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