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ON LOCATION OF THE SPECTRUM OF AN OPERATOR WITH A HILBERT-SCHMIDT RESOLVENT IN THE LEFT HALF-PLANE

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Abstract. Let $\mathcal{H}$ be a separable Hilbert space, and A be a linear operator on $\mathcal{H}$ with a Hilbert-Schmidt resolvent and a bounded imaginary Hermitian component. Assuming that the spectrum of A lies in the open left half-plane we suggest the conditions that provide the location of the spectrum of a bounded perturbation of A in the open left half-plane.

Нехай $\mathcal{H}$ — сепарабельний гільбертовий простір, а A — лінійний оператор на $\mathcal{H}$ з резольвентою Гільберта-Шмідта та обмеженою уявниою компонентою. Припускаючи, що спектр A лежить у відкритої лівої півплощині, запропоновано пропонуємо умови, які забезпечують розташування спектру обмеженого збурення A в відкритій лівій півплощині.

1. Introduction and statement of the main result

Let $\mathcal{H}$ be a separable complex Hilbert space with a scalar product $\langle .,. \rangle$, the norm $\|.\| = \sqrt{\langle .,. \rangle}$ and unit operator I. By $\mathcal{L}(\mathcal{H})$ we denote the algebra of all bounded linear operators in $\mathcal{H}$. For a linear operator T, $D(T)$ is the domain, $\sigma(T)$ denotes the spectrum, T^{-1} is the inverse operator, $R_z(T) = (T - zI)^{-1}$ $(z \notin \sigma(T))$ is the resolvent, T^* is the adjoint operator. If $T \in \mathcal{L}(\mathcal{H})$, then $\|T\|$ is its operator norm. By $\mathcal{S}_p$ $(1 \le p < \infty)$ we denote the Schatten - von Neumann ideal of compact operators K with the finite norm $N_p(K) := [\text{trace}\,(KK^*)^{p/2}]^{1/p}$. In particular, $\mathcal{S}_2$ is the Hilbert-Schmidt ideal.

Let A and $\tilde{A}$ be closed linear operators on $\mathcal{H}$, and

$$\alpha(A) := \sup \mathrm{Re}\ \sigma(A) < 0. \tag{1}$$

In this paper, assuming that

$$D(\tilde{A}) = D(A) \text{ and } q := \|A - \tilde{A}\| < \infty, \tag{2}$$

we suggest conditions that provide the inequality

$$\alpha(\tilde{A}) < 0, \tag{3}$$

Conditions (3) play an important role in various applications. In particular, if (3) holds, then the corresponding differential equation is asymptotically stable for a wide class of operators, cf. [4, 17]. The study of spectrum perturbations is a well-developed subject. For the classical results see [13], the recent investigations can be found, in particular, in the papers [1, 7] and references given therein. At the same time, to the best of our knowledge the conditions that provide inequality (3) for non-selfadjoint operators were almost not investigated in the available literature. Here we can point only the paper [9, Section 3] which deals with relative bounded perturbations of operators having Schatten-von Neumann resolvents and Schatten-von Neumann Hermitian components. Below we do not assume that A has a Schatten-von Neumann Hermitian component. In addition, the approach in the present paper is absolutely different from the approach of paper [9].

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Let $A = A_R + iA_I$, where A_R and A_I are self-adjoint operators (the Hermitian components of A). It is supposed that

$$A_I \in \mathcal{L}(\mathcal{H}) \text{ and } A_R^{-1} \in \mathcal{S}_2. \tag{4}$$

Denote by a_k the eigenvalues of A with their multiplicities taken into account and enumerated in the non-decreasing order of their absolute values: $|a_k| \le |a_{k+1}|$ $(k = 1, 2, ...)$. Below we check that conditions (1) and (4) imply that $A^{-1} \in \mathcal{S}_2$ and

$$\gamma(A) := \sum_{k=1}^{\infty} \frac{1}{(\text{Re } a_k)^2} < \infty. \tag{5}$$

Put

$$\zeta(A) := \frac{1}{|\alpha(A)|}\left(\frac{2\|A_I\|}{\pi|\alpha(A)|} + 1\right) \exp\,[32\gamma(A)\|A_I\|^2].$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let A_R be upper-bounded, and conditions (1), (2) and (4) hold. In addition, let*

$$2q\zeta(A) < 1. \tag{6}$$

Then (3) is valid.

The proof of this theorem is divided into a series of lemmas, which are presented in the next three sections.

2. Auxiliary results

Recall the Keldysh theorem, cf. [11, Theorem V. 8.1].

Theorem 2.1. *Let $B = K_0(I + K_1)$, where $K_0 = K_0^* \in \mathcal{S}_r$ for some $r \in [1, \infty)$ and K_1 is compact. In addition, let from $Bf = 0$ $(f \in \mathcal{H})$ it follows that $f = 0$. Then B has a complete in $\mathcal{H}$ system of the root vectors.*

Lemma 2.2. *Let $A = A_R + iA_I$, where A_R and A_I are self-adjoint operators, $A_I \in \mathcal{L}(\mathcal{H})$, $A_R^{-1} \in \mathcal{S}_p$ $(1 \le p < \infty)$ and $0 \notin \sigma(A)$. Then operator A^{-1} has a complete system of root vectors and $A^{-1} \in \mathcal{S}_p$.*

Proof. Since $A = A_R + iA_I = (I + iA_I A_R^{-1})A_R$, we have $I + iA_I A_R^{-1} = AA_R^{-1}$. But

$$(AA_R^{-1})^{-1} = A_R A^{-1} = (A - iA_I)A^{-1} = I - iA_I A^{-1}.$$

Since A_I is bounded, operator $I + i^{-1}A_I A_R$ is boundedly invertible. We have

$$A^{-1} = A_R^{-1}(I + iA_I A_R^{-1})^{-1} \in \mathcal{S}_p.$$

In addition, $(I + iA_I A_R^{-1})^{-1} = I + K_2$, where

$$K_2 = (I + iA_I A_R^{-1})^{-1} - I = -iA_I A_R^{-1}(I + iA_I A_R^{-1})^{-1} \in \mathcal{S}_p.$$

So $A^{-1} = A_R^{-1}(I + K_2) \in \mathcal{S}_p$, and from $A^{-1}f = 0$ $(f \in \mathcal{H})$ it follows that $f = 0$. Now the Keldysh theorem implies the required result. □

Now Lemma 2.2 implies.

Corollary 2.3. *Under the conditions $0 \notin \sigma(A)$ and (4) operator A^{-1} has a complete system of root vectors and $A^{-1} \in \mathcal{S}_2$.*

Lemma 2.4. *Let an operator A on $\mathcal{H}$ have a compact resolvent, and for some $b \notin \sigma(A)$, $(A - bI)^{-1}$ have a complete system of root vectors. Then there is an orthogonal normal (Schur) basis $\{e_k\}_{k=1}^{\infty}$, in which A is representable by a triangular matrix $(a_{jk})_{1\le j\le k\le\infty}$:*

$$Ae_k = \sum_{j=1}^{k} a_{jk}e_j \text{ and } \langle Ae_k, e_k\rangle = a_k \ \ (k = 1, 2, ...), \tag{7}$$

where a_k $(k = 1, 2, ...)$ are the eigenvalues of A.

For the proof see [10, Lemma 4].

Hence, and from Lemmas 2.2 and Corollary 2.3 it follows

Corollary 2.5. *Let conditions $0 \notin \sigma(A)$ and (4) hold. Then there is an orthogonal normal basis $\{e_k\}_{k=1}^{\infty}$, such that (7) is valid, and therefore, $A = S + W$, where $Se_k = a_ke_k$ $(k = 1, 2, ...)$ is the diagonal part and W is defined by*

$$We_k = \sum_{j=1}^{k-1} e_j a_{jk} \ \ (k = 2, 3, ...).$$

Besides, $A^{-1} \in \mathcal{S}_2$, $\sigma(S) = \sigma(A) = \{a_k\}$ and by the Weyl inequalities [11, Section II.3]

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \le N_2^2(A^{-1}),$$

and consequently, $S^{-1} \in \mathcal{S}_2$.

Furthermore, put

$$S_I := (S - S^*)/2i = \sum_{j=1}^{\infty} (\text{Im } a_j)\Delta P_j$$

and $W_I = A_I - S_I = (W - W^*)/2i$. Obviously, $\langle We_k, e_k\rangle = 0$. Since $\langle Ae_k, e_k\rangle = \langle Se_k, e_k\rangle = a_k$, we have $\langle A_Ie_k, e_k\rangle = \langle S_Ie_k, e_k\rangle = \text{Im } a_k$. So $|\text{Im } a_k| \le \|A_I\|$ and

$$\|S_I\| = \sup_k |\text{Im } a_k| \le \|A_I\|. \tag{8}$$

In addition,

$$\|W_I\| \le \|A_I\| + \|S_I\| \le 2\|A_I\|.$$

Furthermore, note that $P_{k-1}WP_k = WP_k$ and $P_{k-1}W^*P_k = 0$. So $P_{k-1}W^*e_k = 0$ and

$$\|We_k\| = \|P_{k-1}We_k\| = \|P_{k-1}(W - W^*)e_k\| \le \|(W - W^*)e_k\|.$$

We thus obtain

$$\sup_k \|We_k\| \le 2\|W_Ie_k\| \le 2\|W_I\| \le 4\|A_I\|.$$

Hence,

$$N_2^2(WS^{-1}) = \sum_{k=1}^{\infty} \|WS^{-1}e_k\|^2 = \sum_{k=1}^{\infty} \frac{1}{|a_k|^2}\|We_k\|^2$$

and thus

$$N_2^2(WS^{-1}) \le \sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \sup_k \|We_k\|^2 \le 4^2\|A_I\|^2N_2^2(S^{-1}). \tag{9}$$

3. Lyapunov equation

We need the following well-known theorem, cf. [3, Theorem 5.1.3, p. 217].

Theorem 3.1. *Suppose that B is the infinitesimal generator of the C_0-semigroup $T(t)$ on a Hilbert space $\mathcal{H}$. Then $T(t)$ is exponentially stable if and only if there exists a bounded positive definite operator X, such that*

$$\langle Bz, Xz\rangle + \langle Xz, Bz\rangle = -\langle z, z\rangle \ \ (z \in D(B)). \tag{10}$$

Recall that, if B is the infinitesimal generator of an exponentially stable C_0-semigroup, then due to [3, Section 5.5.3a, equality (5.62)], for any $Q \in \mathcal{L}(\mathcal{H})$ the equation

$$\langle Bz_1, Xz_2\rangle + \langle Xz_1, Bz_2\rangle = -\langle z_1, Qz_2\rangle \tag{11}$$

has a solution $X_Q \in \mathcal{L}(\mathcal{H})$, which due to [3, Section to 5.5.3a] is representable as

$$X_Q = \int_0^\infty e^{B^*t} Q e^{Bt} dt. \tag{12}$$

For a self-adjoint operator C we write $C > 0$ ($C < 0$), if it is positive (negative) definite. Let $D(B) = D(B^*)$ and $B^*X + XB = -C^2, C > 0$ on $D(B)$ for some positive definite $X \in \mathcal{L}(\mathcal{H})$. Then

$$C^{-1}B^*XC^{-1} + C^{-1}XBC^{-1} = C^{-1}B^*CC^{-1}XC^{-1} + C^{-1}XC^{-1}CBC^{-1} = -I.$$

Or $M^*Y + YM = -I$, where $M = CBC^{-1}$ and $Y = C^{-1}XC^{-1}$.

According to Theorem 3.1 M generates an exponentially stable semigroup. Since M and B are similar, we arrive at

Corollary 3.2. *Let $D(B) = D(B^*)$ and $B^*X + XB < 0$ on $D(B)$ for some positive definite $X \in \mathcal{L}(\mathcal{H})$. Then* $\sup \mathrm{Re}\ \sigma(B) < 0$.

Lemma 3.3. *Suppose that B is a generator of an exponentially stable C_0-semigroup, and $\tilde{B}$ generates a C_0-semigroup $\tilde{T}(t)$, and let*

$$D(B) = D(\tilde{B}) \ \text{and}\ q_1 := \|B - \tilde{B}\| < \infty, \tag{13}$$

and

$$X = \int_0^\infty e^{B^*t} e^{Bt} dt. \tag{14}$$

If, in addition, $2q_1\|X\| < 1$, then $\tilde{T}(t)$ is also exponentially stable.

Proof. Due to (12) $B^*X + XB = -I$. Put $E = \tilde{B} - B$. Then

$$\tilde{B}^*X + X\tilde{B} = B^*X + XB + E^*X + XE = -I + E^*X + XE.$$

If $2q_1\|X\| < 1$, then $\tilde{B}^*X + X\tilde{B} < 0$. Now Corollary 3.2 proves the lemma. □

Now put

$$J(B) = \frac{1}{2\pi} \int_{-\infty}^\infty \|(B - isI)^{-1}\|^2 ds,$$

assuming that the integral converges. By the Parseval-Planscherel equality [2, Theorem 5.2.1, p. 351], for any $x \in \mathcal{H}$ we have

$$\langle Xx, x\rangle = \langle \int_0^\infty e^{B^*t} e^{Bt} x dt, x\rangle = \int_0^\infty \langle e^{Bt}x, e^{Bt}x\rangle dt = \int_0^\infty \|e^{Bt}x\|^2 dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty \|(B - isI)^{-1}x\|^2 ds.$$

Hence, $\|X\| \le J(B)$. Now Lemma 3.3 implies

Corollary 3.4. *Suppose that B is a generator of an exponentially stable C_0-semigroup, and $\tilde{B}$ generates a C_0-semigroup $\tilde{T}(t)$, and X be defined by (14). Let the conditions (13) and $2q_1 J(B) < 1$ hold. Then $\tilde{T}(t)$ is also exponentially stable.*

4. Proof of Theorem 1.1

Lemma 4.1. *Let $A = A_R + iA_I$, where A_R, A_I are selfadjoint operators, A_R is upper-bounded and $A_I \in \mathcal{L}(\mathcal{H})$. Then A generates an analytic semigroup e^{At}. If, in addition, (1) holds, then e^{At} is exponentially stable.*

Proof. A selfadjoint operator C bounded from below is a sectorial operator, and $-C$ generates an analytic semigroup, cf. [12, Section 1.2]. Since A_R is upper bounded A_R generates an analytic semigroup. Due to Proposition III.1.12 from [5] A generates an analytic semigroup, since $A_I \in \mathcal{L}(\mathcal{H})$. By Theorem 1.3.4 [12], under condition (1) it is exponentially stable. □

Once more applying Proposition III.1.12 from [5], we arrive at the following result.

Corollary 4.2. *If A satisfies the hypothesis of the previous lemma and conditions (2) hold, then $\tilde{A}$ generates an analytic semigroup.*

Put

$$\phi(x) := \sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k!}} \ \ (x \geq 0).$$

Lemma 4.3. *Let the conditions (4) and $0 \notin \sigma(A)$ hold. Then*

$$\|(A - zI)^{-1}\| \leq \frac{1}{d(A,z)} \phi(4\|A_I\| N_2((S - zI)^{-1}) \ \ (z \notin \sigma(A)),$$

where $d(A,z) = \inf_k |a_k - z|$.

Proof. Making use of Corollary 2.5, we have

$$A = S + W - zI = (I - B_z)(S - zI),$$

where $B_z := -W(S - Iz)^{-1}$ $(z \notin \sigma(S) = \sigma(A))$. According to Corollary 2.5, $B_z \in \mathcal{S}_2$. In addition, B_z is the triangular compact matrix with the zero diagonal, and therefore it is the limit of nilpotent matrices in the operator norm. Hence, due to [11, Theorem I.4.1], B_z is quasi-nilpotent, and we can write

$$(I - B_z)^{-1} = \sum_{k=0}^{\infty} B_z^k$$

and $(A - zI)^{-1} = (S - zI)^{-1}(I - B_z)^{-1}$. So

$$\|(Iz - A)^{-1}\| \leq \|(S - Iz)^{-1}\| \|(I - B_z)^{-1}\|. \tag{15}$$

Making use of [8, Corollary 7.4], we get

$$\|B_z^k\| \leq \frac{N_2^k(B_z)}{(k!)^{1/2}} \ \ (k = 1, 2, ...).$$

Thus,

$$\|(I - B_z)^{-1}\| \leq \sum_{k=0}^{\infty} \frac{N_2^k(B_z)}{(k!)^{1/2}}. \tag{16}$$

By (9),

$$N_2^2(B_z) = N_2^2(W(S - zI)^{-1}) = \sum_{k=1}^{\infty} \|W(S - z)^{-1} e_k\|^2 = \sum_{k=1}^{\infty} \frac{1}{|a_k - z|^2} \|We_k\|^2$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{|a_k - z|^2} \sup_k \|We_k\|^2 \leq 4^2 \|A_I\|^2 N_2^2((S - zI)^{-1}).$$

But

$$\|(S - zI)^{-1}\| = \frac{1}{d(A, z)},$$

Now (15) and (16) imply the required result. □

By the Schwarz inequality,

$$\phi(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{1/2}} = \sum_{k=0}^{\infty} \frac{(\sqrt{2}x)^k}{(\sqrt{2})^k (k!)^{1/2}} \leq (\sum_{j=0}^{\infty} \frac{2^j x^{2j}}{j!} \sum_{k=0}^{\infty} \frac{1}{2^k})^{1/2} = \sqrt{2} e^{x^2} \quad (x \geq 0).$$

Now Lemma 4.3 implies

Corollary 4.4. *Let the conditions* $0 \notin \sigma(A)$ *and* (4) *hold. Then*

$$\|(A - zI)^{-1}\| \leq \frac{\sqrt{2}}{d(A, z)} \exp\,[16 \|A_I\|^2 N_2^2((S - zI)^{-1})] \quad (z \notin \sigma(A)).$$

Lemma 4.5. *If conditions* (1) *and* (4) *hold, then* (5) *is valid*

Proof. Let Re $a_k = b_k$ and Im $a_k = c_k$. Then by (8)

$$b_k^2 \geq |a_k|^2 - c_k^2 \geq ||a_k|^2 - \|A_I\|^2|.$$

Since $S^{-1} \in \mathcal{S}_2$ and

$$\lim_{k \to \infty} \frac{||a_k|^2 - \|A_I\|^2|}{|a_k|^2} = 1,$$

condition (5) is fulfilled. □

Lemma 4.6. *Under the hypothesis of Theorem 1.1 one has*

$$J(A) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(A - iy)^{-1}\|^2 dy \leq \zeta(A).$$

Proof. For any real y one has

$$|a_k - iy|^2 \geq (\text{Re } a_k)^2 + (\text{Im } a_k - y)^2 \geq (\text{Re } a_k)^2$$

and

$$N_2^2((S - iy)^{-1}) = \sum_{k=1}^{\infty} \frac{1}{|a_k - iy|^2} \leq \gamma(A), \tag{17}$$

Hence, by Corollary 4.4 we have

$$\|(A - iyI)^{-1}\|^2 \leq \frac{w(A)}{d^2(A, iy)} \leq \frac{w(A)}{\alpha^2(A)} \quad (y \in \mathbb{R}),$$

where

$$w(A) := 2 \exp\,[32 \gamma(A) \|A_I\|^2].$$

If, in addition, $|y| \geq \|A_I\|$, then $|a_k - iy|^2 \geq (\text{Re } a_k)^2 + (|y| - \|A_I\|)^2$ and $d^2(A, iy) \geq \alpha^2(A) + (|y| - \|A_I\|)^2$. Hence, by Corollary 4.4,

$$\|(A - iy)^{-1}\|^2 \leq \frac{w(A)}{\alpha^2(A) + (|y| - \|A_I\|)^2} \quad (|y| \geq \|A_I\|).$$

We thus can write

$$J_1 := \int_{-\|A_I\|}^{\|A_I\|} \|(A - iy)^{-1}\|^2 dy \leq \frac{2\|A_I\| w(A)}{\alpha^2(A)}$$

and

$$J_2 := \int_{\|A_I\|}^{\infty} \|(A-iy)^{-1}\|^2 dy \le \int_{\|A_I\|}^{\infty} \frac{w(A)dy}{\alpha^2(A)+(y-\|A_I\|)^2}.$$

Since,

$$\int_b^{\infty} \frac{dy}{a^2+(y-b)^2} = \int_0^{\infty} \frac{dx}{a^2+x^2} = \frac{1}{a}\int_0^{\infty} \frac{dx_1}{1+x_1^2} = \frac{\pi}{2a} \quad (a,b>0),$$

we get $J_2 \le \frac{w(A)\pi}{2|\alpha(A)|}$. Similarly,

$$J_3 := \int_{-\infty}^{-\|A_I\|} \|(A-iy)^{-1}\|^2 dy \le \frac{w(A)\pi}{2|\alpha(A)|}.$$

Consequently,

$$J(A) = \frac{1}{2\pi}(J_1+J_2+J_3) \le \frac{w(A)}{2|\alpha(A)|}(\frac{2\|A_I\|}{\pi|\alpha(A)|}+1) =$$

$$= \exp\,[32\gamma(A)\|A_I\|^2]\frac{1}{|\alpha(A)|}(\frac{2\|A_I\|}{\pi|\alpha(A)|}+1) = \zeta(A),$$

as claimed. □

Proof of Theorem 1.1 The assertion of the theorem follows from Corollary 3.4 and Lemma 4.6.

5. Example

Let $\mathcal{H} = L^2([0,1];\mathbb{C}^n)$ be the Hilbert space of n-vector valued functions defined on $[0,1]$ with the scalar product

$$\langle v,w\rangle = \int_0^1 (v(x),w(x))_n dx \quad (v,w \in L^2([0,1];\mathbb{C}^n)),$$

where $(\cdot,\cdot)_n$ is the scalar product in $\mathbb{C}^n$. Let $M(x)$ be a twice continuously differentiable $n\times n$-matrix valued function defined on $[0,1]$. For the brevity put $L^2([0,1];\mathbb{C}^n) = L^2_n$ and consider the operator

$$(\tilde{A}f)(x) = f''(x) + M(x)f(x) \quad (0\le x\le 1) \tag{18}$$

with

$$D(\tilde{A}) = \{h \in L^2_n : h'' \in L^2_n, h(0)=h(1)=0\}. \tag{19}$$

Take

$$(Af)(x) = f''(x) + M_0 f(x) \quad (0\le x\le 1)$$

with a constant matrix M_0 and $D(A) = D(\tilde{A})$. For example, one can take $M_0 = M(0)$ or $M_0 = \int_0^1 M(x)dx$. Let $\lambda_j(M_0)$ $(j=1,...,n)$ be the eigenvalues of M_0. Then $-\pi^2k^2 + \lambda_j(M_0)$ $(j=1,...,n;\ k=1,2,...)$ are the eigenvalues of A. We have $\alpha(A) = \alpha_M$, where $\alpha_M := -\pi^2 + \max_k \operatorname{Re}\lambda_k(M_0)$, and $q = q_M$, where $q_M = \|M(x)-M_0\|_n$. Here $\|.\|_n$ is the spectral norm (the the operator norm in $\mathbb{C}^n$ with respect to the Euclidean vector norm).

In addition, assuming that $\alpha_M < 0$, we have $\gamma(A) = \gamma_M$, where

$$\gamma_M := \sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1}{[\pi^2k^2 - \operatorname{Re}\lambda_j(M_0)]^2} < \infty.$$

Moreover, $\|A_I\| = \|M_{0I}\|_n$, where $M_{0I} = (M_0 - M_0^*)/2i$. Therefore, $\zeta(A) = \zeta_M$, where

$$\zeta_M := \frac{1}{|\alpha_M|}\left(\frac{2\|M_{0I}\|_n}{\pi|\alpha_M|}+1\right)\exp\,[32\gamma_M\|M_{0I}\|_n].$$

Now Theorem 1.1 implies

Corollary 5.1. *Let $\tilde{A}$ be defined by (18), (19). If $\alpha_M < 0$ and $2q_M\zeta_M < 1$, then (3) holds.*

For the recent results on the spectra of differential operators see, for instance, the works [6, 14, 15, 16] and the references which are given therein.

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