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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

ON THE NUMBER OF NODAL DOMAINS ON A RECTANGLE WITH A SLIT

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Abstract. In spectral geometry, one is interested in estimating the number of nodal domains of eigenfunctions of the Laplacian on planar domains. Well-known classical results due to Courant and Pleijel establish upper bounds, implying that the n-th eigenfunction has at most n nodal domains and that indeed only a finite number of eigenfunctions attain this maximal value. Surprisingly, however, a seemingly simpler question remains largely open. Namely, does there always exist a subsequence of eigenfunctions with an unbounded number of nodal domains? It is the aim of this note to investigate this question in the context of a rectangular domain with a slit.

В спектральній геометрії цікавим є оцінка кількості вузловіх областей власних функцій лапласіана в плоских областях. Відомі класичні результати Куранта і Плейеля встановлюють верхні межі, з яких випливає, що n-та власна функція має не більше ніж n вузлових областей, і лише скінченна кількість власних функцій досягають цього максимального значення. Однак, більш просте питання ще залишається відкритим. А саме, чи завжди існує підпослідовність власних функцій з необмеженою кількість вузлових областей? Метою цієї роботи є дослідження цього питання в контексті прямокутної області з прорізом.

1. Introduction

Currently, there is an interesting open problem in spectral geometry related to the number of nodal domains of eigenfunctions of the Laplacian on bounded domains Ω in $\mathbb{R}^2$. To introduce the problem, let $-\Delta^D$ denote the Dirichlet Laplacian and $-\Delta^N$ the Neumann Laplacian which are defined through their associated quadratic forms,

$$q_D[\varphi] = \int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x \ , \quad \mathcal{D}(q_D) := H_0^1(\Omega) \ ,$$

and

$$q_N[\varphi] = \int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x \ , \quad \mathcal{D}(q_N) := H^1(\Omega) \ .$$

The operators $-\Delta^D$ and $-\Delta^N$ are then constructed according to the first representation theorem for quadratic forms [3]. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ for an arbitrary bounded domain $\Omega \subset \mathbb{R}^2$, the spectrum of $-\Delta^D$ is purely discrete. Concerning $-\Delta^N$ the situation is more complex since $H^1(\Omega)$ is not necessarily compactly embedded in $L^2(\Omega)$. The embedding is compact, however, as long as Ω has a Lipschitz boundary; a more general criterion is established in [1] (see also [2]). In this paper, we will consider a domain $\Omega \subset \mathbb{R}^2$ which is indeed *not* Lipschitz but for which $H^1(\Omega)$ is nevertheless compactly embedded in $L^2(\Omega)$ and which implies that $-\Delta^N$ also has purely discrete spectrum.

We will now formulate the problem for the Dirichlet Laplacian only, the generalization to the Neumann case being obvious. Assume that $-\Delta^D$ is defined over $L^2(\Omega)$ for an arbitrary bounded domain $\Omega \subset \mathbb{R}^2$ and has eigenvalues $(E_n)_{n\in\mathbb{N}}$ with associated

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eigenfunctions $(\varphi_n)_{n\in\mathbb{N}} \in H_0^1(\Omega)$, forming an orthonormal basis of $L^2(\Omega)$. It is well-known that $\varphi_n \in C^\infty(\Omega)$ and hence one may define the set

$$\mathcal{E}(\varphi_n) := \{x \in \Omega : \ \varphi_n(x) \neq 0\}$$

whose connected components are called the *nodal domains* of the eigenfunction φ_n. In this paper we are interested in the number of such nodal domains which we denote as $\mathcal{N}(\varphi_n)$. There are two classical results in spectral geometry concerning the sequence $(\mathcal{N}(\varphi_n))_{n\in\mathbb{N}}$: a theorem due to Courant states that $\mathcal{N}(\varphi_n) \leq n$ for all $n \in \mathbb{N}$ and about thirty years later Pleijel showed that actually

$$\limsup_{n\to\infty} \frac{\mathcal{N}(\varphi_n)}{n} \leq \left(\frac{2}{j}\right)^2 ,$$

where $j \sim 2.4$ is the smallest positive zero of the Bessel function J_0 [10]. Recently, the result of Pleijel has been slightly improved by Bourgain [5] and Steinerberger [12], showing that the constant $(2/j)^2$ is not optimal. Indeed, a conjecture of Polterovich [11] states the following:

Conjecture 1.1. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions to either $-\Delta^D$ or $-\Delta^N$ on a bounded domain $\Omega \subset \mathbb{R}^2$ with regular boundary. Then*

$$\limsup_{n\to\infty} \frac{\mathcal{N}(\varphi_n)}{n} \leq \frac{2}{\pi} . \tag{1.1}$$

Conjecture 1.1 refers to the asymptotic behaviour of the number of nodal domains. On the other hand, it is generally expected and supported by numerics that $\mathcal{N}(\varphi_n) \to \infty$ as $n \to \infty$ at least for a subsequence of eigenfunctions. Generically, one may even suspect that $\mathcal{N}(\varphi_n) \sim E_n$ for a subsequence [6]. However, to prove that – for a given bounded domain $\Omega \subset \mathbb{R}^2$ – there does not exist a constant $M > 0$ such that

$$\mathcal{N}(\varphi_n) \leq M \tag{1.2}$$

for all $n \in \mathbb{N}$ has turned out to be a (surprisingly) difficult problem and only little is known, see [13, 8, 9] and references therein (we remark that T. Hoffmann-Ostenhof also asked the question as to whether there exists a Schrödinger operator for which one has $\limsup_{n\to\infty} \mathcal{N}(\varphi_n) < \infty$).

It is the aim of this paper to investigate the (non-) existence of a bound as in (1.2) and the Conjecture 1.1 for a perturbed rectangular domain $\Omega \subset \mathbb{R}^2$ which does not – a priori – allow for a separation of variables.

2. Results

Consider the rectangle $\Omega_R := (-a\pi/2, +a\pi/2) \times (-b\pi/2, +b\pi/2)$ with $\frac{a^2}{b^2} \notin \mathbb{Q}$ (this irrationality condition shall be assumed throughout the paper). The Dirichlet Laplacian on Ω_R has purely discrete spectrum and its eigenvalues are given by

$$\lambda_{n_1 n_2} = \frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} , \quad n_1, n_2 \in \mathbb{N} .$$

The irrationality condition implies that the eigenvalues have multiplicity one and the associated (unnormalized) eigenfunctions are given by

$$\varphi_{n_1 n_2}(x_1, x_2) = \sin\left(\frac{n_1}{a}x_1\right) \sin\left(\frac{n_2}{b}x_2\right) .$$

Note that, for convenience, we prefer to neglect normalizing factors in the following. Each such eigenfunction $\varphi_{n_1 n_2}$ has exactly $n_1 n_2$ nodal domains and hence the number of nodal domains goes to infinity as the energy goes to infinity (meaning that $\lambda_{n_1 n_2} \to \infty$). In other words, for Ω_R there does not exist a bound as in (1.2). Furthermore, as shown in [4], (1.1) holds for Ω_R.

In a next step we introduce the perturbed rectangular domain $\widehat{\Omega}_R(\varepsilon)$ (namely, a rectangle with a slit) which is obtained from Ω_R by adding a segment to $\partial\Omega_R$. More explicitly, we require that

$$\partial\widehat{\Omega}_R(\varepsilon) = \partial\Omega_R \cup \{(x,y) \in \mathbb{R}^2 : x = 0 \text{ and } -\varepsilon \leq y \leq +\varepsilon\}$$

with $0 < \varepsilon < +b\pi/2$. On $\widehat{\Omega}_R(\varepsilon)$ we then introduce the Dirichlet and Neumann Laplacian as described above. Although $\widehat{\Omega}_R(\varepsilon)$ is not a Lipschitz domain, one nevertheless has the following result.

Proposition 2.1. *On $L^2(\widehat{\Omega}_R(\varepsilon))$, both $-\Delta^D$ and $-\Delta^N$ have purely discrete spectrum.*

Proof. As described in the introduction, for the Dirichlet Laplacian $-\Delta^D$ there is nothing to prove.

Regarding the Neumann Laplacian $-\Delta^N$, one begins with a bounded sequence in $(\varphi_n)_{n\in\mathbb{N}} \subset H^1(\widehat{\Omega}_R(\varepsilon))$ and restricts each function to the right rectangle $\Omega_1 := (0, +a\pi/2) \times (-b\pi/2, +b\pi/2)$; this yields a bounded sequence in $H^1(\Omega_1)$ and since Ω_1 is Lipschitz, there exists a convergent subsequence in $L^2(\Omega_1)$. We then pick the corresponding subsequence $(\varphi_{n_k})_{k\in\mathbb{N}} \subset H^1(\widehat{\Omega}_R(\varepsilon))$ whose restriction to Ω_1 is exactly subsequence contructed before and restrict this sequence to the left rectangle $\Omega_2 := (-a\pi/2, 0) \times (-b\pi/2, +b\pi/2)$. As before we conclude the existence of a subsequence that converges in $L^2(\Omega_2)$ and consequently we obtain a subsequence of $(\varphi_n)_{n\in\mathbb{N}} \subset H^1(\widehat{\Omega}_R(\varepsilon))$ that converges in $L^2(\widehat{\Omega}_R(\varepsilon))$ which proves the statement. □

We know, by the irrationality condition $\frac{a^2}{b^2} \notin \mathbb{Q}$, that the eigenvalues of the Dirichlet Laplacian $-\Delta^D$ have multiplicity one when defined over the rectangle Ω_R. The following result of Hillairet and Judge tells us that the eigenvalues of $-\Delta^D$ defined on $\widehat{\Omega}_R(\varepsilon)$ are, for all $\varepsilon \in (0, \varepsilon_0)$ up to a countable set and some $\varepsilon_0 > 0$, again non-degenerate.

Proposition 2.2. *[Propositions 4.2 and 7.1, [7]] There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ up to a countable set, the spectrum of $-\Delta^D$ on $\widehat{\Omega}_R(\varepsilon)$ is simple.*

Now, in a first step we investigate the existence of a bound on the number of nodal domains as in (1.2). From an intuitive point of view, it seems plausible that such a bound $M > 0$ does not exist – at least for ε small enough – since one does not have a bound for Ω_R. On the other hand, however, the number of nodal domains is a somewhat global property and hence a proof of the non-existence of such a bound $M > 0$ seems desirable.

Theorem 2.1. *Let $0 < \varepsilon < +b\pi/2$ be given. Then, for $-\Delta^D$ and $-\Delta^N$ there exists an orthonormal basis of eigenfunctions that contains a subsequence of eigenfunctions for which the number of nodal domains is unbounded.*

In particular, for some $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$ up to a countable set, the associated unique (up to phase factors) orthonormal basis of eigenfunctions of $-\Delta^D$ contains a subsequence of eigenfunctions with an unbounded number of nodal domains.

Proof. We prove the statements for the Dirichlet Laplacian $-\Delta^D$ and mention how to proceed in the Neumann case. One starts on the rectangle $\Omega_1 := (0, +a\pi/2) \times (-b\pi/2, +b\pi/2)$ and picks the Dirichlet eigenfunctions given by, $n_1, n_2 \in \mathbb{N}$,

$$\varphi_{n_1 n_2}(x_1, x_2) = \sin\left(\frac{2n_1}{a}x_1\right) \sin\left(\frac{n_2}{b}x_2\right)$$

which are extended onto $\widehat{\Omega}_R(\varepsilon)$ by reflecting each such function across $x = 0$ and multiplying the reflection by minus one. This immediately yields a set of eigenfunctions for $-\Delta^D$ on $L^2(\widehat{\Omega}_R(\varepsilon))$ since they satisfy Dirichlet boundary conditions on the segment $\{(x,y) \in \mathbb{R}^2 : x = 0 \text{ and } -\varepsilon \leq y \leq +\varepsilon\}$. Furthermore, the number of nodal domains of the function $\varphi_{n_1 n_2}$ on Ω_1 is $n_1 n_2$ and hence the number of nodal domains on Ω is given by $2n_1 n_2$ which proves the statement. The second part of the statement is a direct consequence of Proposition 2.2.

Finally, regarding the first part of the statement, the procedure is the same in the Neumann case, except that one starts with the eigenfunctions of the Neumann Laplacian on Ω_1 and reflects them across $x = 0$ with a positive sign. □

In a final result we investigate Conjecture 1.1 in the context of $\widehat{\Omega}_R(\varepsilon)$ and establish a lower bound.

Theorem 2.2. *For $0 < \varepsilon < +b\pi/2$ consider the Dirichlet Laplacian $-\Delta^D$ on $L^2(\widehat{\Omega}_R(\varepsilon))$. Then, there exists an orthonormal basis of eigenfunctions such that*

$$\frac{2}{\pi} \leq \limsup_{n\to\infty} \frac{\mathcal{N}(\varphi_n)}{n} \ .$$

Proof. The starting point is the set of eigenfunctions $\{\hat{\varphi}_{n_1 n_2}\}$ constructed in the proof of Theorem 2.1. We recall that $\hat{\varphi}_{n_1 n_2}$ is constructed via a reflection (with a negative sign) of $\varphi_{n_1 n_2} \in L^2(\Omega_1)$, $\Omega_1 := (0, +a\pi/2) \times (-b\pi/2, +b\pi/2)$.

Setting $\lambda := \lambda_{n_1 n_2}$, one obtains

$$\begin{aligned} k(n_1, n_2) &: = \#\{(\tilde{n}_1, \tilde{n}_2) : \lambda_{\tilde{n}_1 \tilde{n}_2} < \lambda\} \\ &= \frac{\pi ab}{8}\lambda + o(\lambda) \end{aligned}$$

as $\lambda \to \infty$ where $\lambda_{n_1 n_2} = \frac{4n_1^2}{a^2} + \frac{n_2^2}{b^2}$, $n_1, n_2 \in \mathbb{N}$; see also [4].

Let $\nu(n_1, n_2) \in \mathbb{N}$ denote the order of the eigenfunction $\hat{\varphi}_{n_1 n_2}$. Weyl's law readily implies

$$\frac{k(n_1, n_2)}{\nu(n_1, n_2)} \longrightarrow \frac{1}{2}$$

as $\lambda \to \infty$. Now, taking into account that $\hat{\varphi}_{n_1 n_2}$ has $2n_1 n_2$ nodal domains and choosing a subsequence such that $n_1 \to \infty$ and $\frac{n_2}{n_1} \to \frac{2b}{a}$, we get

$$\frac{\mathcal{N}\left(\hat{\varphi}_{n_1 n_2}\right)}{\nu(n_1, n_2)} \longrightarrow \frac{2}{\pi}$$

along this subsequence. This proves the statement. □

Of course, in view of Conjecture 1.1 and Theorem 2.2, it would be highly interesting to derive an upper bound on $\limsup_{n\to\infty} \frac{\mathcal{N}(\varphi_n)}{n}$. In order to do this, one would need to derive suitable bounds on the number of nodal domains on the rectangle Ω_1 with mixed Dirichlet and Neumann boundary conditions.

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