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AN ALTERNATIVE DEFINITION OF THE ITÔ INTEGRAL FOR THE HILBERT-SCHMIDT-VALUED STOCHASTIC PROCESS

MHELMAR A. LABENDIA

Abstract. In this paper, using generalized Riemann approach, we give an alternative definition of the Itô integral of a Hilbert–Schmidt-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process. We also show that this integral belongs to the space of all continuous square-integrable martingales.

Використовуючи узагальнений підхід Рімана, наведено альтернативне визначення інтеграла Іто для стохастичного процесу зі значеннями в просторі операторів Гільберта–Шмідта відносно Q-вінерівського процесу, що приймає значення у гілбертовому просторі. Також показано, що цей інтеграл належить до простору всіх неперервних квадратично інтегрованих мартингалів.

1. Introduction and Preliminaries

The Lebesgue integral is perhaps the most widely used almost exclusively by professional mathematicians, see [13]. However, understanding Lebesgue integration requires an extensive study of measure theory. Moreover, the Lebesgue integral fails to integrate highly oscillatory functions. It is not difficult to show that the derivative of the function $f(x) = x^2 \sin x^{-2}$ if $x \neq 0$ and $f(0) = 0$ is not Lebesgue integrable on $[0, 1]$. However, the function f is integrable on $[0, 1]$ using the Henstock integral, see [5, Theorem 9.6]. The Henstock integral is one of the notable integrals that were introduced which in some sense solves some of the limitations of the Lebesgue integral. This integral was independently studied by Henstock and Kurzweil in the 1950s, which is also known as the Henstock-Kurzweil integral.

A real-valued function f is said to be *Henstock integrable* [12, Definition 2.2] to $A \in \mathbb{R}$ on $[a, b]$ if for every $\epsilon > 0$, there is a function $\delta(\xi) > 0$ such that whenever a division D given by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_i, \xi_2, \dots, \xi_n\}$ satisfies $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \dots, n$ we have $|\sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) - A| < \epsilon$. The Henstock integral of f on $[a, b]$ is given by A and it is uniquely determined. It can be observed that the above definition of the Henstock integral is a Riemann-type definition of integral which is more explicit and does not require a deep study of measure theory. In fact, the function f is not required to be measurable. It is known that the Henstock integral includes that improper Riemann, Lebesgue, and Newton integrals and is designed to integrate highly oscillatory functions. Since then, this integral has become of interest to numerous authors, see [5, 6, 7, 8, 12, 13, 14, 15, 16]. This approach to integration is known as the generalized Riemann approach or Henstock-Kurzweil approach which has also been used to deal with stochastic integration for real-valued cases, see [2, 18, 19, 26, 27, 28, 29].

In infinite-dimensional cases, the main motivation behind the study of infinite-dimensional stochastic integrals is the theory of stochastic partial differential equations. In some examples of stochastic PDEs, the diffusion term is required to be an unbounded operator in $L_2(U_Q, V) \setminus L(U, V)$, where U and V are separable Hilbert spaces, $Q : U \to U$ is a symmetric nonnegative trace-class operator, $L_2(U_Q, V)$ is the space of all Hilbert-Schmidt operators from $U_Q := Q^{\frac{1}{2}}(U)$ to V, and $L(U, V)$ is the space of all bounded

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linear operators from from U to V. Hence, a natural space of integrands for a stochastic integral in a separable Hilbert space driven by a U-valued Q-Wiener process is the space of progressively measurable processes with values in $L_2(U_Q, V)$.

In this paper, we define the Itô-Henstock integral, a Henstock-Kurzweil approach integral, for the Hilbert-Schmidt-valued stochastic process with respect to a Q-Wiener process and show that this integral gives an alternative definition of the Itô integral in a separable Hilbert space.

Let $L(U) := L(U, U)$, $Qu := Q(u)$ if $Q \in L(U, V)$, and $L^2(\Omega, V)$ be the space of all square-integrable random variables from Ω to V. If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an orthonormal basis (abbrev. as ONB) $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, $\{\lambda_j\} \in \ell^1$, and $\lambda_j \to 0$ as $j \to \infty$, see [21, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an eigensequence defined by Q. The subspace U_Q of U equipped with the inner product $\langle u, v\rangle_{U_Q} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_U$, where $Q^{1/2}$ is being restricted to $[\mathrm{K}erQ^{1/2}]^{\perp}$ is a separable Hilbert space with $\left\{\sqrt{\lambda_j}e_j\right\}$ as its ONB, see [3, p.90], [4, p.23]. The space $L_2(U_Q, V)$ of all Hilbert-Schmidt operators from U_Q to V is a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$, see [20, p.112]. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an arbitrary ONB, see [3, p.418], [20, p.111]. We note that $L(U, V)$ is properly contained in $L_2(U_Q, V)$, see [4, p.25]. We also note that $L_2(U_Q, V)$ contains genuinely unbounded linear operators from U to V.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by Q, and $\{B_j\}$ be a sequence of independent Brownian motions (abbrev. as BM) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. A process $\tilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j$, is called a Q-Wiener process in U, with the series converging in $L^2(\Omega, U)$. For every $u \in U$, denote $\tilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u\rangle_U$, with the series converging in $L^2(\Omega, \mathbb{R})$. Then there exists a U-valued process W, known as a U-valued Q-Wiener process, such that $\tilde{W}_t(u)(\omega) = \langle W_t(\omega), u\rangle_U$ $\mathbb{P}$-almost surely (abbrev. as $\mathbb{P}$-a.s.). It should be noted that the process W is a multi-dimentional BM, and if we assume that $\lambda_j > 0$ for all j, $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j = 1, 2, \dots$, is a sequence of real-valued BM defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, see [3, p.87].

A filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *normal* if (i) $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, and (ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$. A Q-Wiener process W_t, $t \in [0, T]$ is called a *Q-Wiener process with respect to a filtration* $\{\mathcal{F}_t\}$ if (i) W_t is adapted to $\{\mathcal{F}_t\}$, $t \in [0, T]$ and (ii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t \le T$. A U-valued Q-Wiener process $W(t)$, $t \in [0, T]$, is a Q-Wiener process with respect to a normal filtration, see [20, p.16]. From now onwards, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ shall mean a probability space equipped with a normal filtration.

A stochastic process $M : [0, T] \times \Omega \to V$ is said to be a *martingale* if (i) M is adapted; (ii) for all $t \in [0, T]$, M_t is Bochner integrable, i.e. $\mathbb{E}[\|M_t\|_V] < \infty$; and (iii) for any $0 \le s \le t \le T$, $\mathbb{E}\left[M_t \middle| \mathcal{F}_s\right] = M_s$ $\mathbb{P}$-a.s.. A martingale $M : [0, T] \times \Omega \to V$ is said to be *square-integrable* if $M_T \in L^2(\Omega, V)$. It is known [20, p.21] that the space of all continuous square-integrable martingales $\mathcal{M}_T^2 := \mathcal{M}_T^2(V)$ is a Banach space with norm $\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|M_t\|_V^2\right]\right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\|M_T\|_V^2\right]\right)^{\frac{1}{2}}$, and the Q-Wiener process $W \in \mathcal{M}_T^2(U)$.

Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process measurable as mappings from $([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{F})$ to $(L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$. Then f is said to be an *elementary process* if there is a finite sequence $\{t_j\}_{j=0}^n$, $n \in \mathbb{N}$, with $0 = t_0 < t_1 < \cdots < t_n = T$ and a finite sequence of random variables φ, $\{\varphi_j\}$, $j = 0, 1, \ldots, n-1$, such that (i) $\varphi : (\Omega, \mathcal{F}_0) \to (L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$ and $\varphi_j : (\Omega, \mathcal{F}_{t_j}) \to (L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$ are measurable with $\varphi(\omega), \varphi_j(\omega) \in L(U,V)$; and (ii) $f(t,\omega) = \varphi(\omega)\chi_{\{0\}}(t) + \sum_{j=0}^{n-1} \varphi_j(\omega)\chi_{(t_j,t_{j+1}]}(t)$. Denote by $\mathcal{E} := \mathcal{E}(U,V)$ the space of all elementary processes. We say that an elementary process f is *bounded* if there exists $M > 0$ such that $\|f(t,\omega)\|_{L_2(U_Q,V)} \leq M$ for all $(t,\omega) \in [0,T] \times \Omega$. Denote by $\Lambda_{\mathcal{E}}(U,V)$ the space of all bounded elementary processes. Then the *Itô integral* of an elementary process f with respect to W is defined by $(\mathcal{I}) \int_0^t f_s \, dW_s := \sum_{j=0}^{n-1} \varphi_j (W_{t_{j+1}\wedge t} - W_{t_j \wedge t})$ for $t \in [0,T]$. It is known [4, Proposition 2.1] that for $f \in \Lambda_{\mathcal{E}}(U,V)$,

$$\mathbb{E}\left[\left\|(\mathcal{I}) \int_0^t f_s \, dW_s\right\|_V^2\right] = \mathbb{E}\left[(\mathcal{L}) \int_0^t \|f_s\|_{L_2(U_Q,V)}^2 \, ds\right] < \infty$$

for $t \in [0,T]$. This result is known as the *Itô's isometry* on $\Lambda_{\mathcal{E}}(U,V)$. It should be noted that the integral on the right hand side of the equality is a Lebesgue integral, indicated by $(\mathcal{L})$.

Let $\Lambda_{\mathcal{I}}(U_Q,V)$ be a class of all processes $f : [0,T] \times \Omega \to L_2(U_Q,V)$ measurable as mappings from $([0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{F})$ to $(L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$ such that

i) f is adapted to $\{\mathcal{F}_t\}$; and
ii) $\mathbb{E}\left[(\mathcal{L}) \int_0^T \|f_s\|_{L_2(U_Q,V)}^2 \, ds\right] < \infty$.

Hence, $\Lambda_{\mathcal{E}}(U,V) \subset \mathcal{E} \subset \Lambda_{\mathcal{I}}(U_Q,V)$. We note that $\Lambda_{\mathcal{I}}(U_Q,V)$ equipped with the norm

$$\|f\|_{\Lambda_{\mathcal{I}}(U_Q,V)} = \sqrt{\mathbb{E}\left[(\mathcal{L}) \int_0^T \|f_s\|_{L_2(U_Q,V)}^2 \, ds\right]}$$

is a Hilbert space. From [4, Proposition 2.2], if $f \in \Lambda_{\mathcal{I}}(U_Q,V)$, then there exists a sequence $\{f^{(n)}\}$ of elements in $\Lambda_{\mathcal{E}}(U,V)$ such that

$$\left\|f^{(n)} - f\right\|_{\Lambda_{\mathcal{I}}(U_Q,V)}^2 = \mathbb{E}\left[(\mathcal{L}) \int_0^T \left\|f_t^{(n)} - f_t\right\|_{L_2(U_Q,V)}^2 \, dt\right] \to 0 \text{ as } n \to \infty.$$

This means that $\Lambda_{\mathcal{E}}(U,V)$ is dense in $\Lambda_{\mathcal{I}}(U_Q,V)$. A stochastic process $f : [0,T] \times \Omega \to L_2(U_Q,V)$ is said to be *Itô integrable* if $f \in \Lambda_{\mathcal{I}}(U_Q,V)$ and the *Itô integral* of f with respect to W is the unique isometric linear extension of the mapping $f(\cdot) \to (\mathcal{I}) \int_0^T f_s \, dW_s$ from the class of bounded elementary processes to $L^2(\Omega,V)$, to a mapping from $\Lambda_{\mathcal{I}}(U_Q,V)$ to $L^2(\Omega,V)$, such that the image of $f(t,\omega) = \varphi(\omega)\chi_{\{0\}}(t) + \sum_{j=0}^{n-1} \varphi_j(\omega)\chi_{(t_j,t_{j+1}]}(t)$ is $\sum_{j=0}^{n-1} \varphi_j (W_{t_{j+1}} - W_{t_j})$. We define the Itô integral process $(\mathcal{I}) \int_0^t$, $0 \leq t \leq T$, for

$f \in \Lambda_{\mathcal{I}}(U_Q, V)$ by $(\mathcal{I})\int_0^t f_s\ dW_s = (\mathcal{I})\int_0^T f_s\chi_{[0,t]}(s)\ dW_s$. It is also worth noting [4, Theorem 2.3] that the stochastic integral $f \to (\mathcal{I})\int_0^{(\cdot)} f_s\ dW_s$ with respect to W is an isometry between $\Lambda_{\mathcal{I}}(U_Q, V)$ and the space of continuous square-integrable martingales $\mathcal{M}_T^2(V)$,

$$\mathbb{E}\left[\left\|(\mathcal{I})\int_0^t f_s\ dW_s\right\|_V^2\right] = \mathbb{E}\left[(\mathcal{L})\int_0^t \|f_s\|^2_{L_2(U_Q,V)}\ ds\right] < \infty$$

for $t \in [0, T]$.

2. Itô-Henstock Integrals

In this section, we shall introduce the Itô-Henstock integral, which will give a direct definition of the integral for adapted stochastic processes. This approach to stochastic integration does not require extending an isometry from the space of elementary processes to the space of continuous square-integrable martingales.

From now onwards, assume that U and V are separable Hilberts spaces, $Q : U \to U$ is a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by Q, and $W_t := \sum_{j=1}^{\infty}\sqrt{\lambda_j}B_t^{(j)}e_j$ is a U-valued Q-Wiener process. A stochastic process $f : [0, T] \times \Omega \to L_2(U_Q, V)$ means a process measurable as mappings from $([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F})$ to $(L_2(U_Q, V), \mathcal{B}(L_2(U_Q, V)))$. The given closed interval $[0, T]$ is nondegenerate, i.e. $0 < T$ and can be replaced with any closed interval $[a, b]$. If no confusion arises, we may write $(D)\sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection D.

Definition 2.1. Let δ be a positive function defined on $[0, T)$. A finite collection $D = \{([\xi_i, v_i], \xi_i)\}_{i=1}^n$ of interval-point pairs is a

i) *δ-fine belated full division* of $[0, T]$ if $\{[\xi_i, v_i]\}_{i=1}^n$ is a collection of non-overlapping intervals on $[0, T]$ such that $\bigcup_{i=1}^n[\xi_i, v_i] = [0, T]$ and each $[\xi_i, v_i]$ is δ-fine belated, that is, $[\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$.

ii) *δ-fine belated partial division* of $[0, T]$ if $\{[\xi_i, v_i]\}_{i=1}^n$ is a collection of non-overlapping intervals on $[0, T]$ and each $[\xi_i, v_i]$ is δ-fine belated.

We note that each ξ_i in Definition 2.1 does not necessarily belong to $[\xi_i, v_i]$. The term *partial division* is used in Definition 2.1 (ii), to emphasize that the finite collection of non-overlapping intervals of $[0, T]$ may not cover the entire interval $[0, T]$. Using the Vitali covering theorem, the following concept can be defined.

Definition 2.2. Given $\eta > 0$, a given δ-fine belated partial division $D = \{([\xi, v], \xi)\}$ of $[0, T]$ is said to be a *(δ, η)-fine belated partial division* of $[0, T]$ if it fails to cover $[0, T]$ by at most length η, that is,

$$\left|T - (D)\sum(v - \xi)\right| \le \eta.$$

Before we define the Itô-Henstock integral, we shall consider first the following lemma.

Lemma 2.3. *Let $f : [0, T] \times \Omega \to L_2(U_Q, V)$ be an adapted process. For $0 \le \xi \le v \le T$,* $\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j) \in L^2(\Omega, V)$.

Proof. Let $m, n \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned}
& \mathbb{E}\left[\left\|\sum_{j=m}^{n}(B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] \\
= & \sum_{j=m}^{n}\mathbb{E}\left[\mathbb{E}\left[(B_v^{(j)} - B_\xi^{(j)})^2 \left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V \middle| \mathcal{F}_\xi\right]\right] \\
& + \sum_{j\neq j'}\mathbb{E}\left[\mathbb{E}\left[(B_v^{(j)} - B_\xi^{(j)})(B_v^{(j')} - B_\xi^{(j')}) \left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V \middle| \mathcal{F}_\xi\right]\right] \\
= & \sum_{j=m}^{n}\mathbb{E}\left[\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V \mathbb{E}\left[(B_v^{(j)} - B_\xi^{(j)})^2 \middle| \mathcal{F}_\xi\right]\right] \\
& + \sum_{j\neq j'}\mathbb{E}\left[\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V \mathbb{E}\left[(B_v^{(j)} - B_\xi^{(j)})(B_v^{(j')} - B_\xi^{(j')}) \middle| \mathcal{F}_\xi\right]\right] \\
= & (v-\xi)\sum_{j=m}^{n}\mathbb{E}\left[\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right] = (v-\xi)\sum_{j=m}^{n}\mathbb{E}\left[\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right].
\end{aligned}$$

Let $S_k = \sum_{j=1}^{k}\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2$. By the Beppo Levi's lemma [22, p.370],

$$\lim_{k\to\infty}\mathbb{E}\left[\sum_{j=1}^{k}\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] = \mathbb{E}\left[\lim_{k\to\infty}\sum_{j=1}^{k}\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] < \infty.$$

Hence,

$$\sum_{j=1}^{\infty}\mathbb{E}\left[\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] = \mathbb{E}\left[\sum_{j=1}^{\infty}\left\|f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] = \mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right] < \infty.$$

It follows that $M_n := \sum_{j=1}^{n}(B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j}e_j)$ is Cauchy in $L^2(\Omega, V)$, hence convergent in $L^2(\Omega, V)$. Thus, $\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j}e_j) \in L^2(\Omega, V)$. □

Definition 2.4. Let $f : [0, T] \times \Omega \to L_2(U_Q, V)$ be an adapted process. Then f is said to be *Itô-Henstock integrable*, or $\mathcal{IH}$-integrable, on $[0, T]$ with respect to W if there exists $A \in L^2(\Omega, V)$ such that for every $\epsilon > 0$, there is a positive function δ on $[0, T)$ and a number $\eta > 0$ such that for any (δ, η)-fine belated partial division $D = \{([\xi_i, v_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have

$$\mathbb{E}\left[\|S(f, D, \delta, \eta) - A\|_V^2\right] < \epsilon,$$

where

$$\begin{aligned}
S(f, D, \delta, \eta) & := (D)\sum\left\{\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j}e_j)\right\} \\
& := \sum_{i=1}^{n}\left\{\sum_{j=1}^{\infty}(B_{v_i}^{(j)} - B_{\xi_i}^{(j)}) f_{\xi_i}(\sqrt{\lambda_j}e_j)\right\}.
\end{aligned}$$

In this case, f is $\mathcal{IH}$-integrable to A on $[0,T]$ and A is called the $\mathcal{IH}$-integral of f which will be denoted by $(\mathcal{IH})\int_0^T f_t\, dW_t$ or $(\mathcal{IH})\int_0^T f\, dW$. We shall denote $(\mathcal{IH})\int_0^0 f\, dW$ by the zero random variable $\mathbf{0}$ from Ω to V and denote by $\Lambda_{\mathcal{IH}}(U_Q,V)$, the collection of all Itô-Henstock integrable processes on $[0,T]$. We have to note here that if the process f is $L(U,V)$-valued, then for f to be Itô-Henstock integrable on $[0,T]$, we need to replace the Riemann sum $S(f,D,\delta,\eta)$ with $(D)\sum f_\xi(W_v - W_\xi) := \sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i})$. In this case, Definition 2.4 is the same with [11, Definition 3.3]. Denote by $\Lambda_{\mathcal{IH}}(U,V)$, the collection of all $L(U,V)$-valued Itô-Henstock integrable processes on $[0,T]$. In view of Lemma 2.3, $S(f,D,\delta,\eta)\in L^2(\Omega,V)$.

We remark that since $\mathbb{P}(W_t\in U_Q)=0$, $t>0$, if $\dim U_Q=\infty$ [1, Theorem 2.4.7], the Riemann sum $(D)\sum f_\xi(W_v-W_\xi)$ in [11, Definition 3.3] does not make sense pathwise as an application of an operator $f_\xi\in L_2(U_Q,V)$ to a vector in a Hilbert space.

Before giving an example, we shall consider first the following simple form of a stochastic differential equation, see [24, Definition 4.13].

Definition 2.5. Let U and V be separable Hilbert spaces. For $t\in[0,T]$, consider the stochastic differential equation of the form

$$\begin{cases} dX_t = A\,dt + F(X_t)\,dW_t \\ X_0=\phi \end{cases} \tag{2.1}$$

where $A\in V$, $F:V\to L(U,V)$, and $X_0=\phi$ is a V-valued random variable which is $\mathcal{F}_0$-measurable. A stochastic process $X:[0,T]\times\Omega\to V$ defined on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ is a solution of (2.1) if for each $t\in[0,T]$,

$$X_t=\phi+(\mathcal{R})\int_0^t A\,ds+(\mathcal{IH})\int_0^t F(X_s)\,dW_s$$

for almost all $\omega\in\Omega$, where the first integral is in Riemann sense.

If U is a separable Hilbert space, $V=\mathbb{R}$, and $\{\lambda_j,e_j\}_{j=1}^\infty$ is an eigensequence defined by $Q\in L(U)$, following the same argument as in [24, Example 4.14], we can show that $f(W_t)$ defined by $f(x)=\|x\|_U^2$ is a solution of

$$\begin{cases} dX_t = \operatorname{tr} Q\,dt + 2\langle W_t,\cdot\rangle_U\; dW_t \\ X_0=0, \end{cases} \tag{2.2}$$

where $\langle W_t,\cdot\rangle_U(x)=\langle W_t,x\rangle_U$, for all $x\in U$. Using Definition 2.4, we can show that

$$(\mathcal{IH})\int_0^t \langle W_x,\cdot\rangle_U\; dW_s=\frac{1}{2}\left(\|W_t\|_U^2 - t\cdot\operatorname{tr} Q\right),$$

see [23, Example 1] for analogous proof.

The proofs of the following basic properties of the $\mathcal{IH}$-integral are standard in Henstock-Kurzweil integration, hence omitted.

1) The $\mathcal{IH}$-integral is uniquely determined.
2) Let $f,g\in\Lambda_{\mathcal{IH}}(U_Q,V)$ and let $\alpha,\beta\in\mathbb{R}$. Then $\alpha f+\beta g\in\Lambda_{\mathcal{IH}}(U_Q,V)$ and

$$(\mathcal{IH})\int_0^T(\alpha f+\beta g)\;dW=\alpha\cdot(\mathcal{IH})\int_0^T f\;dW+\beta\cdot(\mathcal{IH})\int_0^T g\;dW.$$

3) *Cauchy criterion.* $f \in \Lambda_{\mathcal{IH}}(U_Q, V)$ if and only if for every $\epsilon > 0$, there exist a positive function δ on $[0,T)$ and a number $\eta > 0$ such that for any two (δ,η)-fine belated partial divisions D_1 and D_2 of $[0,T]$, we have

$$\mathbb{E}\left[\|S(f,D_1,\delta,\eta) - S(f,D_2,\delta,\eta)\|_V^2\right] < \epsilon.$$

4) If f is $\mathcal{IH}$-integrable on $[0,T]$, then f is $\mathcal{IH}$-integrable on $[c,d] \subset [0,T]$.
5) If f is $\mathcal{IH}$-integrable on $[0,c]$ and $[c,T]$ where $c \in (0,T)$, then f is $\mathcal{IH}$-integrable on $[0,T]$ and

$$(\mathcal{IH})\int_0^T f\ dW = (\mathcal{IH})\int_0^c f\ dW + (\mathcal{IH})\int_c^T f\ dW.$$

6) *Sequential definition.* $f \in \Lambda_{\mathcal{IH}}(U_Q, V)$ if and only if there exist $A \in L^2(\Omega, V)$, a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T)$, and a decreasing sequence of positive numbers η_n such that for any (δ_n,η_n)-fine belated partial division D_n of $[0,T]$, we have

$$\mathbb{E}\left[\|S(f,D_n,\delta_n,\eta_n) - A\|_V^2\right] \to 0 \text{ as } n \to \infty.$$

In this case, $A := (\mathcal{IH})\displaystyle\int_0^T f_t\ dW_t$.

7) *Saks-Henstock lemma.* Let f be $\mathcal{IH}$-integrable on $[0,T]$ and

$$F[u,v] := (\mathcal{IH})\int_u^v f\ dW$$

for any $[u,v] \subset [0,T]$. Then for every $\epsilon > 0$, there exists a positive function δ on $[0,T)$ such that for any δ-fine belated partial division $D = \{([\xi,v],\xi)\}$ of $[0,T]$, we have

$$\mathbb{E}\left[\left\|(D)\sum\left\{\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j) - F[\xi,v]\right\}\right\|_V^2\right] < \epsilon.$$

Next, we show that the Itô-Henstock integral is a continuous square integrable martingale. First, we remark that following the same argument as in [9, Lemma 3.7], we can directly show that the $\mathcal{IH}$-integral satisfies the $AC^2[0,T]$-property [10, Definition 3.9], that is for every $\epsilon > 0$, there exists an $\eta > 0$ such that for any finite collection $D = \{[\xi,v]\}$ of non-overlapping subintervals of $[0,T]$ with $(D)\sum(v-\xi) < \eta$, we have $\mathbb{E}\left[\left\|(D)\sum(\mathcal{IH})\displaystyle\int_\xi^v f\,dW\right\|_V^2\right] < \epsilon.$

Theorem 2.6. *Let $f \in \Lambda_{\mathcal{IH}}(U_Q, V)$ and define $F_t := (\mathcal{IH})\displaystyle\int_0^t f\,dW$. Then $F \in \mathcal{M}_T^2(V)$.*

Proof. By the definition of $\mathcal{IH}$-integral, F is square integrable. First, we show that F is continuous. Let $\epsilon > 0$. Since F is $AC^2[0,T]$, there exists an $\eta > 0$ such that for any finite collection $D = \{[\xi,v]\}$ of non-overlapping subintervals of $[0,T]$ with $(D)\sum(v-\xi) < \eta$, we have $\mathbb{E}\left[\left\|(D)\sum(\mathcal{IH})\displaystyle\int_\xi^v f\,dW\right\|_V^2\right] < \epsilon^3$. Let $t \in [0,T]$ and let $t' \leq t$ such that

$|t-t'|<\eta$. Then $\mathbb{E}\left[\left\|(\mathcal{IH})\int_{t'}^{t} f_s\,dW_s\right\|_V^2\right] < \epsilon^3$. By Chebychev's inequality,

$$\mathbb{P}\left\{\omega\in\Omega : \left\|(\mathcal{IH})\int_{t'}^{t} f_s\,dW_s\right\|_V \geq \epsilon\right\} \leq \frac{1}{\epsilon^2}\mathbb{E}\left[\left\|(\mathcal{IH})\int_{t'}^{t} f_s\,dW_s\right\|_V^2\right] < \epsilon.$$

Hence, F is $\mathbb{P}$-a.s. continuous on $[0,T]$.

Next, we show that F is adapted. We note that f is also $\mathcal{IH}$-integrable on $[0,t]$. From the sequential definition of $\mathcal{IH}$-integral, there exist a decreasing sequence $\{\delta_n\}$ of positive functions on $[0,T)$ and a decreasing sequence of positive numbers η_n such that for any (δ_n,η_n)-fine belated partial division D_n of $[0,T]$, we have

$$\mathbb{E}\left[\left\|S(f,D_n,\delta_n,\eta_n) - (\mathcal{IH})\int_0^t f\,dW\right\|_V^2\right] < \epsilon.$$

Note that for $0\leq\xi\leq v\leq t$, $M_n := \sum_{j=1}^{n}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)$ is $\mathcal{F}_t$-measurable. Let $M := \sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)$. We will show that M is $\mathcal{F}_t$-measurable. By Chebychev's inequality and using the fact that the filtration is normal, we have

$$\{\omega\in\Omega : M(\omega)\in B_r(v)\} = \bigcup_{n,N\in\mathbb{N}}\left[\bigcap_{k\geq N}\left\{\omega\in\Omega : M_k(\omega)\in B_{r-\frac{1}{n}}(v)\right\}\right],$$

where $B_r(v)$ is an open ball in $\mathcal{B}(V)$ centered at $v\in V$ and with radius $r>0$. This means that M is $\mathcal{F}_t$-measurable. It follows that $S(f,D_n,\delta_n,\eta_n)$ is also $\mathcal{F}_t$-measurable. Following the same argument above, F_t is $\mathcal{F}_t$-measurable.

Lastly, we show that F is a martingale. Let $0\leq s\leq t\leq T$ and $A\in\mathcal{F}_s$. Note that

$$\begin{aligned}
&\int_A \lim_{n\to\infty}(D_n)\sum_{\xi>s}\left(\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right)\\
&= \lim_{n\to\infty}(D_n)\sum_{\xi>s}\left(\sum_{j=1}^{\infty}\mathbb{E}\left[f_\xi(\sqrt{\lambda_j}e_j)\mathbb{E}\left[(B_v^{(j)} - B_\xi^{(j)})\middle|\mathcal{F}_\xi\right]\middle|\mathcal{F}_s\right]\right)\\
&= \lim_{n\to\infty}(D_n)\sum_{\xi>s}\left(\sum_{j=1}^{\infty}\mathbb{E}\left[f_\xi(\sqrt{\lambda_j}e_j)\mathbb{E}\left[B_v^{(j)} - B_\xi^{(j)}\right]\middle|\mathcal{F}_s\right]\right)\\
&= 0.
\end{aligned}$$

By the sequential definition of $\mathcal{IH}$-integral, we have

$$\begin{aligned}
\int_A F[0,t] &= \int_A \lim_{n\to\infty} S(f,D_n,\delta_n,\eta_n)\\
&= \int_A \lim_{n\to\infty}(D_n)\sum_{\xi>s}\left(\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right)\\
&\quad + \int_A \lim_{n\to\infty}(D_n)\sum_{\xi\leq s}\left(\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right)\\
&= \int_A F[0,s].
\end{aligned}$$

Thus, F is a martingale. Accordingly, $F \in \mathcal{M}_T^2(V)$. □

Before proving the Itô-isometry, we shall consider first the McShane belated integral [25, p. 379], which is equivalent [25, p. 380] to the Lebesgue integral for real-valued functions. We also need to note that the Lebesgue integral is also equivalent [16, Corollary 14] to the Itô-belated integral introduced in [17, p. 51]. A real-valued function f is Lebesgue integrable on $[0,T]$ to $A := (\mathcal{L})\int_0^T f \in \mathbb{R}$ if for every $\epsilon > 0$, there exist a positive function δ on $[0,T)$ and $\eta > 0$ such that for every (δ,η)-fine belated partial division $D = \{([\xi,v],\xi)\}$ of $[0,T]$, we have $\left|(D)\sum f(\xi)(v-\xi) - A\right| < \epsilon$. Denote by $\Lambda_{\mathcal{L}}$, the collection of all Lebesgue integrable functions on $[0,T]$.

Theorem 2.7. [25, Theorem 5] *A function $f : [0,T] \to \mathbb{R}$ is Lebesgue integrable to $A \in \mathbb{R}$ if and only if there exist a decreasing sequence of positive functions $\{\delta_n\}$ on $[0,T)$ and a decreasing sequence of positive constants $\{\eta_n\}$ such that*

$$\lim_{n\to\infty}\left|(D_n)\sum f(\xi^{(n)})(v^{(n)} - \xi^{(n)}) - A\right| = 0,$$

where D_n is any (δ_n,η_n)-fine belated partial division of $[0,T]$.

Lemma 2.8. *Let $f : [0,T]\times\Omega \to L_2(U_Q,V)$ be adapted processes and $[\xi,v] \subseteq [0,T]$. Then*

$$\mathbb{E}\left[\sum_{j\neq j'=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})(B_v^{(j')} - B_\xi^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\right] = 0.$$

Proof. Observe that

$$\mathbb{E}\left[\sum_{j\neq j'=1}^{k}(B_v^{(j)} - B_\xi^{(j)})(B_v^{(j')} - B_\xi^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\right] = 0$$

for every $k \in \mathbb{N}$. Thus, the conclusion follows. □

Lemma 2.9. *Let $f : [0,T]\times\Omega \to L_2(U_Q,V)$ be adapted processes and $[\xi,v] \subseteq [0,T]$. Then*

$$\mathbb{E}\left[\left\|\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right] = (v-\xi)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right].$$

Proof. Applying the Monotone Convergence Theorem (MCT) to

$$\sum_{j=1}^{k}(B_v^{(j)} - B_\xi^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V,$$

we have

$$\begin{aligned}&\sum_{j=1}^{\infty}\mathbb{E}\left[(B_v^{(j)} - B_\xi^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right]\\ =\;& \mathbb{E}\left[\sum_{j=1}^{\infty}(B_v^{(j)} - B_\xi^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right].\end{aligned}$$

Using Lemma 2.8, we have

$$\begin{aligned}
&(v-\xi)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right] \\
= \;& \mathbb{E}\left[\sum_{j=1}^{\infty}(B_v^{(j)}-B_\xi^{(j)})^2\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_j}e_j)\right\rangle_V\right] \\
&+\mathbb{E}\left[\sum_{j\neq j'=1}^{\infty}(B_v^{(j)}-B_\xi^{(j)})(B_v^{(j')}-B_\xi^{(j')})\left\langle f_\xi(\sqrt{\lambda_j}e_j), f_\xi(\sqrt{\lambda_{j'}}e_{j'})\right\rangle_V\right] \\
= \;& \mathbb{E}\left[\left\|\sum_{j=1}^{\infty}(B_v^{(j)}-B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\|_V^2\right].
\end{aligned}$$

□

Following the same argument above, we can easily show that for any finite collection $D=\{[\xi,v]\}$ of non-overlapping subintervals of $[0,T]$, we have

$$\mathbb{E}\left[\left\|(D)\sum\left\{\sum_{j=1}^{\infty}(B_v^{(j)}-B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)\right\}\right\|_V^2\right]=(D)\sum(v-\xi)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right].$$

Theorem 2.10 (Itô-isometry)**.** *Let $f\in\Lambda_{\mathcal{IH}}(U_Q,V)$. Then $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\in\Lambda_{\mathcal{L}}$ and*

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t\; dW_t\right\|_V^2\right]=(\mathcal{L})\int_0^T\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\; dt<\infty.$$

Proof. By the sequential definition of $\mathcal{IH}$-integral, there exist a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T)$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any (δ_n,η_n)-fine belated partial division $D_n=\{((\xi_i^{(n)},v_i^{(n)}],\xi_i^{(n)})\}_{i=1}^{p(n)}$ of $[0,T]$, we have

$$\lim_{n\to\infty}S(f,D_n,\delta_n,\eta_n)=(\mathcal{IH})\int_0^T f_t\; dW_t \qquad \text{in } L^2(\Omega,V).$$

Using Lemma 2.9, we have

$$\lim_{n\to\infty}\sum_{i=1}^{p(n)}\left(v_i^{(n)}-\xi_i^{(n)}\right)\mathbb{E}\left[\left\|f_{\xi_i^{(n)}}\right\|^2_{L_2(U_Q,V)}\right]=\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t\; dW_t\right\|_V^2\right].$$

Since the above equality holds for any (δ_n,η_n)-fine belated partial division of $[0,T]$, by Theorem 2.7, $\mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)}\right]$ is Lebesgue integrable on $[0,T]$ and

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t\; dW_t\right\|_V^2\right]=(\mathcal{L})\int_0^T\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\; dt<\infty.$$

This completes the proof. □

In view of Theorem 2.10, if $f\in\Lambda_{\mathcal{IH}}(U_Q,V)$, then $f\in\Lambda_{\mathcal{I}}(U_Q,V)$. To prove the converse, we shall consider first the next result, which is a version of convergence theorem for the $\mathcal{IH}$-integral. The proof is analogous to [9, Theorem 3.12].

Theorem 2.11. . *Let $\{f^{(n)}\}$ be a sequence of elements in $\Lambda_{\mathcal{IH}}(U,V)$ and $f:[0,T]\times\Omega\to L_2(U_Q,V)$ be an adapted process such that*

i) *for each* $n \in \mathbb{N}$, $\mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right] \in \Lambda_{\mathcal{L}}$ *and*

$$(\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right] \to 0 \qquad as \quad n \to \infty;$$

ii) $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t\right\}$ *converges in* $L^2(\Omega, V)$.

Then $f \in \Lambda_{\mathcal{IH}}(U_Q, V)$ *and*

$$\lim_{n\to\infty}(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t = (\mathcal{IH})\int_0^T f_t\ dW_t \qquad in\ L^2(\Omega, V).$$

Proof. Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, there exist a positive function $\delta_1^{(n)}$ on $[0,T)$ and $\eta_1^{(n)} > 0$ such that for any $(\delta_1^{(n)}, \eta_1^{(n)})$-fine belated partial division $D_1^{(n)}$ of $[0,T]$, we have

$$\mathbb{E}\left[\left\|(D_1^{(n)})\sum f_\xi^{(n)}(W_v - W_\xi) - (\mathcal{IH})\int_0^T f_t^{(n)} dW_t\right\|^2_V\right] < \epsilon.$$

Also, for each $n \in \mathbb{N}$, there exist a positive function $\delta_2^{(n)}$ on $[0,T)$ and $\eta_2^{(n)} > 0$ such that for any $(\delta_2^{(n)}, \eta_2^{(n)})$-fine belated partial division $D_2^{(n)}$ of $[0,T]$, we have

$$\left|(D_2^{(n)})\sum \mathbb{E}\left[\left\|f_\xi^{(n)} - f_\xi\right\|^2_{L_2(U_Q,V)}\right](v-\xi) - (\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right]\right| < \epsilon.$$

Since $(\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right] \to 0$ as $n \to \infty$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\left|(\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right]\right| < \epsilon$. Let $A := \lim_{n\to\infty}(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t$ in $L^2(\Omega, V)$. Then there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t - A\right\|^2_V\right] < \epsilon.$$

Take $N > \max\{N_1, N_2\}$ and choose $\delta(\xi) = \min\{\delta_1^{(N)}(\xi), \delta_2^{(N)}(\xi)\}$ and $\eta = \min\{\eta_1^{(N)}, \eta_2^{(N)}\}$. Let $D = \{([\xi, v], \xi)\}$ be a (δ, η)-fine belated partial division of $[0,T]$. Then by Lemma 2.9,

$$\begin{aligned}
&\mathbb{E}\left[\left\|(D)\sum\left\{\sum_{j=1}^\infty (B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j}e_j) - f_\xi^N(W_v - W_\xi\right\}\right\|^2_V\right] \\
\leq\ &\left|(D)\sum \mathbb{E}\left[\left\|f_\xi - f_\xi^{(N)}\right\|^2_{L_2(U_Q,V)}\right](v-\xi) - (\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_\xi - f_\xi^{(N)}\right\|^2_{L_2(U_Q,V)}\right]\right| \\
&\quad + \left|(\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_\xi - f_\xi^{(N)}\right\|^2_{L_2(U_Q,V)}\right]\right| \\
<\ &2\epsilon.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\left[\|S(f,D,\delta,\eta)-A\|_V^2\right] \\
\leq\ & 4\mathbb{E}\left[\left\|(D)\sum\left\{\sum_{j=1}^{\infty}(B_v^{(j)}-B_\xi^{(j)})f_\xi(\sqrt{\lambda_j}e_j)-f_\xi^N(W_v-W_\xi\right\}\right\|_V^2\right] \\
&+4\mathbb{E}\left[\left\|(D)\sum f_\xi^{(N)}(W_v-W_\xi)-(\mathcal{IH})\int_0^T f_t^{(N)}\,dW_t\right\|_V^2\right] \\
&+2\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t^{(N)}\,dW_t-A\right\|_V^2\right] \\
<\ & 14\epsilon.
\end{aligned}
$$

Thus, $f\in\Lambda_{\mathcal{IH}}(U_Q,V)$ and $(\mathcal{IH})\displaystyle\int_0^T f_t\,dW_t=A$. □

It is not difficult to show that if $f\in\Lambda_{\mathcal{E}}(U,V)$, then $f\in\Lambda_{\mathcal{IH}}(U,V)$ and $(\mathcal{IH})\displaystyle\int_0^T f_t\,dW_t=(\mathcal{I})\int_0^T f_t\,dW_t$. One may refer to [9, Theorem 3.14] for an analogous proof.

Theorem 2.12. *If $f\in\Lambda_{\mathcal{I}}(U_Q,V)$, then $f\in\Lambda_{\mathcal{IH}}$ and*

$$(\mathcal{IH})\int_0^T f_t\,dW_t=(\mathcal{I})\int_0^T f_t\,dW_t.$$

Proof. Let $f\in\Lambda_{\mathcal{I}}(U_Q,V)$ and let $\epsilon>0$. Then there exists a sequence $\{f^{(n)}\}$ of elements in $\Lambda_{\mathcal{E}}(U,V)$ such that

$$\left\|f^{(n)}-f\right\|_{\Lambda_{\mathcal{I}}(U_Q,V)}^2=\mathbb{E}\left[(\mathcal{L})\int_0^T\left\|f_t^{(n)}-f_t\right\|_{L_2(U_Q,V)}^2\,dt\right]\to 0 \text{ as } n\to\infty.$$

Using Fubini-Tonelli theorem,

$$\mathbb{E}\left[\left\|(\mathcal{I})\int_0^T(f_t^{(n)}-f_t)\,dW_t\right\|_V^2\right]=(\mathcal{L})\int_0^T\mathbb{E}\left[\left\|f_t^{(n)}-f_t\right\|_{L_2(U_Q,V)}^2\right]\,dt\to 0$$

as $n\to\infty$. Since $f^{(n)}\in\Lambda_{\mathcal{IH}}(U,V)$, $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\,dW_t\right\}_{m\in\mathbb{N}}$ converges in $L^2(\Omega,V)$. By Theorem 2.11, $f\in\Lambda_{\mathcal{IH}}$ and in $L^2(\Omega,V)$

$$
\begin{aligned}
(\mathcal{IH})\int_0^T f_t\,dW_t &= \lim_{n\to\infty}(\mathcal{IH})\int_0^T f_t^{(n)}\,dW_t \\
&= \lim_{n\to\infty}(\mathcal{I})\int_0^T f_t^{(n)}\,dW_t \\
&= (\mathcal{I})\int_0^T f_t\,dW_t.
\end{aligned}
$$

This completes the proof. □

From [4, Theorem 2.3], we can then conclude that the quadratic variation process of the process $(\mathcal{IH})\int_0^t f_s\,dW_s$ and the increasing process related to $\left\|(\mathcal{IH})\int_0^t f_s\,dW_s\right\|_V^2$ are respectively given by

$$\left\langle\!\!\left\langle (\mathcal{IH})\int_0^{\cdot} f_s\,dW_s \right\rangle\!\!\right\rangle_t = (\mathcal{L})\int_0^t (f_s Q^{\frac{1}{2}})(f_s Q^{\frac{1}{2}})^*\,ds$$

and

$$\left\langle (\mathcal{IH})\int_0^{\cdot} f_s\,dW_s \right\rangle_t = (\mathcal{L})\int_0^t \|f_s\|^2_{L_2(U_Q,V)}\,ds.$$

3. Conclusion and Recommendation

In this paper, we define a Henstock-Kurzweil approach integral, the $\mathcal{IH}$-integral, which is a continuous square-integrable martingale. We then formulate a version of convergence theorem which is necessary to give an alternative definition of the Itô integral for the Hilbert-Schmidt-valued stochastic process driven by a Hilbert space-valued Q-Wiener process. A worthwhile direction for further investigation is to use Henstock-Kurzweil approach to deal with stochastic PDEs.

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Mhelmar A. Labendia: mhelmar.labendia@g.msuiit.edu.ph
Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra and Analysis - Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines

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