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EXISTENCE RESULTS FOR SECOND-ORDER NEUTRAL STOCHASTIC EQUATIONS DRIVEN BY ROSENBLATT PROCESS

RAKIA AHMED YAHIA, ABBES BENCHAABANE, AND HALIM ZEGHDOUDI

Abstract. In this paper we consider a class of second-order impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and a standard Brownian motion in a Hilbert space. We prove an existence and uniqueness result under non-Lipschitz condition which is weaker than Lipschitz one and we establish some conditions ensuring the controllability for the mild solution by means of the Banach fixed point principle. At the end we provide a practical example in order to illustrate the viability of our result.

Розглянуто клас імпульсних стохастичних функціонально-диференціальних рівнянь другого порядку, які керуються процесом Розенблата і стандартним броунівський рухом у ґільбертовому просторі одночасно за умови, яка є слабкішою за умови Ліпшица. Також встановлено умови керованості для помірного розв'язку за допомоги принципу Банаха про нерухому точку. Наведено приклад з практики, що ілюструє отримані результати.

1. Introduction

In this paper, we are interested in the second-order neutral stochastic differential equations driven by Brownian motion and an independent Rosenblatt process of the type

$$\begin{cases} d\left(x^{'}(t) - h(t, x(t))\right) = Ax(t)dt + f\left(t, x(t)\right)dt + g(t, x(t)dw(t) + \sigma(t)dZ_H(t), \\ x(0) = x_0, \quad x^{'}(0) = x_{00}, \quad t \in [0, T] \end{cases} \tag{1.1}$$

where $x(\cdot)$ takes values in the separable Hilbert space X, $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on X. Let Q_K be a positive, self-adjoint and trace class operator on K, and let $\mathcal{L}_2(K, X)$ be the space of all Q_K -Hilbert-Schmidt operators acting between K and X equipped with the Hilbert-Schmidt norm $\|.\|_{\mathcal{L}_2}$, and w be a Q_K-Wiener process on Hilbert space K. Let Q be a positive, self-adjoint and trace class operator on Y and let $\mathcal{L}_2^0(Y, X)$ be the space of all Q -Hilbert-Schmidt operators acting between Y and X equipped with the Hilbert-Schmidt norm $\|.\|_{\mathcal{L}_2^0}$. Let Z_H be a Q-Rosenblatt process on a Hilbert space Y, the process w and Z_H are independent and h, f, g and σ are given functions to be specified later. Let $(\Omega, \mathcal{F}_T, P)$ be the complete probability space with the natural filtration $\{\mathcal{F}_t \mid t \in [0, T]\}$ generated by random variables $\{Z_H(s), \ w(s), \ s \in [0, T]\}$, let x_0 and x_{00} be $\mathcal{F}_0$-measurable X-valued random variables independent of w and Z_H.

We define the following classes of functions: let $\mathcal{L}_2(\Omega, \mathcal{F}_T, X)$ be the Hilbert space of all $\mathcal{F}_T$-measurable, square integrable variables with values in X, $\mathcal{L}_2^{\mathcal{F}}([0, T], X)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$-adapted processes with values in X, $C\left([0, T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X)\right)$ be a Banach space of continuous maps satisfying the condition $\sup_{t\in[0,T]} \mathbf{E} \|x(t)\|^2 < \infty$ and Δ_2^T be the closed subspace of $C\left([0, T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X)\right)$ consisting of measurable and $\mathcal{F}_t$-adapted processes $x(t)$, then Δ_2^T is a Banach space with the

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norm defined by

$$\|x\|_{\Delta_2^T} = \left(\sup_{t\in[0,T]} \mathbf{E}\,\|x(t)\|^2\right)^{\frac{1}{2}}.$$

Recently, there has been a growing interest in stochastic functional differential equations driven by fractional Brownian motion (hereafter, fBm). The reader is referred to the works ([2, 9]). Also in general, it is the Gaussian process and the calculus for it is much easier than other ones. However, in concrete situations where the Gaussianity is not plausible for the model, one can employ the Rosenblatt process. The theory of Rosenblatt process has been developed accordingly owing to its nice properties, namely self-similarity, stationarity of the increments, long-range dependence, etc. (see [14, 19, 11]). The Rosenblatt processes can also be inputs in models where self-similarity is observed in empirical data which appears to be non-Gaussian. There exists a consistent literature that focuses on different theoretical aspects of the Rosenblatt processes ([10, 8, 12, 15]). Some special kind of dynamical systems require mixed process to model its dynamic ([1, 17]).

Motivated by some recent works ([21, 7, 5, 3]), this paper is concerned to prove the existence and uniqueness of mild solution for system (1.1) under non-Lipschitz conditions, which are more general than the Lipschitz and linear growth see ([13, 4]). Further, controllability problem is discussed for system (1.1), it should be mentioned that existence and uniqueness of solutions for second-order neutral stochastic differential equations driven by a Wiener process and an independent Rosenblatt process under non-Lipschitz conditions has not been investigated yet. The rest of this paper is organized as follows, in Section 2, we will introduced some notations, basic concepts, and basic results about Rosenblatt process, Wiener integral with respect to it over Hilbert spaces. In Sections 3 and 4, we will prove our main result. In Section 5, we give an example to illustrate the efficiency of the obtained result.

2. Rosenblatt process

Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z_H(t),\ t\in[0,T]\}$ the one-dimensional Rosenblatt process with parameter $H\in(\frac{1}{2},1)$. By Tudor [20], it is well known that Z_H has the following integral representation:

$$Z_H(t) = d(H)\int_0^t\int_0^t\Big[\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\Big]dB(y_1)dB(y_2), \tag{2.2}$$

where

$$\begin{cases} B=\{B(t):t\in[0,T]\} \text{ is a Wiener process, } H'=\frac{H+1}{2},\\ d(H)=\frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}} \text{ is a normalizing constant,}\end{cases}$$

and $K^H(t,s)$ is the kernel given by

$$\begin{cases} K^H(t,s)=c_H s^{1/2-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}du, \text{ for } t>s,\\ K^H(t,s)=0, \text{ for } t\le s,\end{cases}$$

where

$$\begin{cases} c_H=\sqrt{\frac{H(2H-1)}{\mathcal{B}(2-2H,H-1/2)}}\\ \text{and } \mathcal{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \ \text{ is the Beta function.}\end{cases}$$

The covariance of the Rosenblatt process $\{Z_H(t),\ t\in[0,T]\}$ satisfies

$$R_H(s,t):=E(Z_H(t)Z_H(s))=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}) \qquad \text{for every } s,t\ge 0.$$

Let X and Y be two real, separable Hilbert spaces. Let $Q\in\mathcal{L}(Y,X)$ be an operator defined by $Qe_n=\lambda_n e_n$ with finite trace $trQ=\sum_{n=1}^\infty \lambda_n<\infty$, where $\lambda_n\ge 0$ $(n=1,2...)$

are non-negative real numbers and $\{e_n\}$ $(n = 1, 2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional Q-Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(t), \tag{2.3}$$

where $(z_n)_{n\geq 0}$ is a family of real independent Rosenblatt processes. Note that the series (2.3) is convergent in $L^2(\Omega)$ for every $t \in [0, T]$, since

$$E\left|Z_Q(t)\right|^2 = \sum_{n=1}^{\infty} \lambda_n E\left(z_n(t)\right)^2 = t^{2H} \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Note also that Z_Q has covariance function in the sense that

$$E\left\langle Z_Q(t), x\right\rangle \left\langle Z_Q(s), y\right\rangle = R(s,t)\left\langle Q(x), y\right\rangle \qquad \text{for all } x,\ y \in Y \text{ and } t,\ s \in [0, T].$$

In order to define Wiener integrals with respect to the Q-Rosenblatt process. Let $\phi(s) : s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$, such that

$$\sum_{n=1}^{\infty} \left\| K^* \phi Q^{1/2} e_n \right\|_{\mathcal{L}_2^0}^2 < \infty.$$

The Wiener integral with respect to Z_Q is defined by

$$\begin{aligned} \int_0^t \phi(s) dZ_Q(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dz_n(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \int_0^t K_H^* \left(\phi e_n\right)(y_1, y_2) dB(y_1) dB(y_2). \end{aligned} \tag{2.4}$$

Now, we end this subsection by stating the following fundamental inequality which was proved in [16].

Lemma 2.1. *If $\phi : [0, T] \to \mathcal{L}_2^0(Y, X)$ satisfies $\int_0^T \|\phi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$, then the above sum in (2.4) is well defined as a X-valued random variable and we have*

$$E\left\| \int_0^t \phi(s) dZ_H(s) \right\|^2 \leq 2H t^{2H-1} \int_0^t \|\phi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

2.1. **Cosine Family.** Now let use recall some facts about cosine families of operators (see [6]).

Definition 2.2. The strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ is the one-parameter family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X, X)$ satisfying

(1) $C(0) = I$,
(2) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in X$,
(3) $C(t + s) + C(t - s) = 2C(t)C(s)$, for all $t,\ s \in \mathbb{R}$.

Definition 2.3. The strongly continuous sine family $\{S(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X, X)$ associated with $\{C(t)\}_{t\in\mathbb{R}}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \qquad t \in \mathbb{R},\ x \in X.$$

Definition 2.4. The infinitesimal generator $A : X \to X$ of $\{C(t)\}_{t\in\mathbb{R}}$ is given by

$$Ax = \frac{d^2}{dt^2} C(t)|_{t=0}, \qquad \text{for all } x \in D(A),$$

with

$$D(A) = \left\{x \in X : C(.)x \in C^2(\mathbb{R}, X)\right\}.$$

The infinitesimal generator A is a closed and densely defined operator on X.

Proposition 2.5. *Suppose that A is the infinitesimal generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ with the corresponding sine family $\{S(t)\}_{t\in\mathbb{R}}$. Then, the following holds true:*

(1) *There exist a constants $M_A \geq 1$ and $\lambda \geq 0$ such that*

$$\|C(t)\| \leq M_A e^{\lambda|t|} \qquad \text{and, hence, } \|S(t)\| \leq M_A e^{\lambda|t|}.$$

(2) *For any $x \in X$ and all $0 \leq s \leq r < \infty$,*

$$\int_s^r S(t)xdt \in D(A) \quad \text{and} \quad A\int_s^r S(t)xdt = [C(r) - C(s)]\,x.$$

(3) *There exists a constant $\beta \geq 1$ such that, for all $0 \leq s \leq r < \infty$,*

$$\|S(r) - S(s)\| \leq \beta \left|\int_s^r e^{\lambda|\theta|} d\theta\right|.$$

Remark 2.6. The uniform boundedness principle, together with Proposition (2.5), implies that both $\{C_t\}_{t\in[0,T]}$ and $\{S(t)\}_{t\in[0,T]}$ are uniformly bounded, i.e., there exists a positive constant $M = M_A e^{\lambda|T|}$ such that

$$\|C(t)\| \leq M \text{ and } \|S(t)\| \leq M. \tag{2.5}$$

3. Existence and uniqueness of mild solution

In this section, we study the existence and uniqueness of mild solution for (1.1). To do this, we first present the definition of mild solutions for the system (1.1).

Definition 3.1. A stochastic process $x \in \Delta_2^T$ is a mild solution of (1.1) if it satisfies the following integral equation:

$$\begin{aligned} x(t) \;=\; & C(t)x_0 + S(t)(x_{00} - h(0, x_0)) + \int_0^t C(t-s)h(s, x(s))ds \qquad (3.6)\\ & + \int_0^t S(t-s)f(s, x(s)ds + \int_0^t S(t-s)g(s, x(s))dw(s) \\ & + \int_0^t S(t-s)\sigma(s)dZ_H(s), \quad P - a.s. \end{aligned}$$

We assume the following non-Lipschitz condition:

(H1) A is the infinitesimal generator of the strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ on X.

(H2) The function $\sigma : [0,T] \to \mathcal{L}_2^0(Y, X)$ is bounded, that is there exists a positive constant L such $\|\sigma(t)\|^2_{\mathcal{L}_2^0} \leq L$ uniformly in $t \in [0,T]$.

(H3) The functions $h, f : [0,T] \times X \to X$, $g : [0,T] \times X \to \mathcal{L}_2$ are measurable and continuous in x for each fixed $t \in [0,T]$ and there exists a function $G : [0,T] \times [0,+\infty) \to [0,+\infty)$, $(t, v) \to G(t, v)$ such that

$$\mathbf{E}\,\|h(t,x)\|^2 + \mathbf{E}\,\|f(t,x)\|^2 + \mathbf{E}\,\|g(t,x)\|^2_{\mathcal{L}_2} \leq G(t, \mathbf{E}\,\|x\|^2) \tag{3.7}$$

for all $t \in [0,T]$ and all $x \in \mathcal{L}^2(\Omega, \mathcal{F}_T, X)$.

(H4) $G(t, v)$ is locally integrable in t for each fixed $v \in [0,+\infty)$ and is continuous non-decreasing in v for each fixed $t \in [0,T]$ and for all $\lambda > 0$, $v_0 \geq 0$ the integral equation $v(t) = v_0 + \lambda \int_0^t G(s, v(s))ds$ has a global solution on $[0,T]$.

(H5) There exists a function $K : [0,T] \times [0,+\infty) \to [0,+\infty)$ such that

$$\begin{array}{l} \mathbf{E}\left\|h(t,x)-h(t,y)\right\|^2+\mathbf{E}\left\|f(t,x)-f(t,y)\right\|^2 \leq K(t,\mathbf{E}\left\|x-y\right\|^2), \\ \mathbf{E}\left\|g(t,x)-g(t,y)\right\|^2_{\mathcal{L}_2} \leq K(t,\mathbf{E}\left\|x-y\right\|^2) \end{array} \tag{3.8}$$

for all $t \in [0,T]$ and all $x, y \in \mathcal{L}^2(\Omega, \mathcal{F}_T, X)$,

(H6) $K(t,v)$ is locally integrable in t for each fixed $v \in [0,+\infty)$ and continuous non-decreasing in v for each fixed $t \in [0,T]$. Moreover, $K(t,0) = 0$, and if a non-negative continuous function $z(t),\ t \in [0,T]$, satisfies

$$\left\{ \begin{array}{l} z(t) \leq \sigma \int_0^t K(s,z(s))ds, \quad t \in [0,T], \\ z(0) = 0 \end{array} \right. \tag{3.9}$$

for some $\sigma > 0$, then $z(t) = 0$ for all $t \in [0,T]$.

Remark 3.2. (1) If $K(t,u) = Lu,\ u \geq 0,\ L > 0$, condition (H5) implies global Lipschitz condition.

(2) If K is concave with respect to the second variable for each fixed $t \geq 0$ and

$$\|f(t,x)-f(t,y)\|^2 + \|g(t,x)-B(t,y)\|^2_{L_2^0} \leq K(t, \|x-y\|^2)$$

for all $x, y \in H$ and $t \geq 0$, by Jensens inequality, (3.8) is satisfied.

(3) Let $K(t,u) = \eta(t)\vartheta(u),\ t \geq 0, u \geq 0$, where $\eta(t) \geq 0$ is locally integrable and $\vartheta : [0,+\infty) \to [0,+\infty)$ is a continuous, monotone non-decreasing and concave function with $\vartheta(0) = 0,\ \vartheta(u) > 0$ for $u > 0$ and $\int_{0^+} 1/\vartheta(u)du = \infty$.

Now let us give some examples of the function ϑ, see [18], let $\epsilon > 0$ be sufficiently small, define

$$\begin{array}{l} \vartheta_1(u) = \left\{ \begin{array}{ll} u\log(u^{-1}), & 0 \leq u \leq \epsilon, \\ \epsilon\log(\epsilon^{-1}) + \vartheta_1'(\epsilon_-)(u-\epsilon), & u > \epsilon, \end{array} \right. \\ \vartheta_2(u) = \left\{ \begin{array}{ll} u\log(u^{-1})\log\log(u^{-1}), & 0 \leq u \leq \epsilon, \\ \epsilon\log(\epsilon^{-1})\log\log(\epsilon^{-1}) + \vartheta_2'(\epsilon_-)(u-\epsilon), & u > \epsilon. \end{array} \right. \end{array} \tag{3.10}$$

Theorem 3.3. *Assume that the conditions (H1)-(H6) hold. Then there exists a unique solution of (1.1) in Δ_2^T.*

The proof of this theorem is based on the Picard type approximate method. Let us construct a sequence of stochastic processes $\{x_n\}_{n\geq 0}$ as follows:

$$\left\{ \begin{array}{rcl} x_0(t) & = & C(t)x_0 + S(t)(x_{00} - h(0,x_0)), \\ x_{n+1}(t) & = & C(t)x_0 + S(t)(x_{00} - h(0,x_0)) + \int_0^t C(t-s)h(s,x_n(s))ds \\ & & + \int_0^t S(t-s)f(s,x_n(s))ds + \int_0^t S(t-s)g(s,x_n(s))dw(s) \\ & & + \int_0^t S(t-s)\sigma(s)dZ_H(s). \end{array} \right. \tag{3.11}$$

Lemma 3.4. *Under the conditions (H1)-(H5) the sequence $\{x_n\}_{n\geq 0}$ is uniformly bounded in Δ_2^T, i.e., $\mathbf{E}\left[\sup_{s\in[0,T]}\|x_n(s)\|^2\right] \leq C$, where C is a constant.*

Proof. We have

$$\begin{aligned} \mathbf{E}\left\|x_{n+1}(t)\right\|^2 \leq & 6\mathbf{E}\left\|C(t)x_0\right\|^2 + 6\mathbf{E}\left\|S(t)(x_{00}-h(0,x_0))\right\|^2 \\ & + 6\mathbf{E}\left\|\int_0^t C(t-s)h(s,x_n(s))ds\right\|^2 + 6\mathbf{E}\left\|\int_0^t S(t-s)f(s,x_n(s))ds\right\|^2 \\ & + 6\mathbf{E}\left\|\int_0^t S(t-s)g(s,x_n(s))dw(s)\right\|^2 + 6\mathbf{E}\left\|\int_0^t S(t-s)\sigma(s)dZ_H(s)\right\|.^2 \end{aligned}$$

By Lemma (2.1), Cauchy-Schwartz inequality and Ito isometry theorem, we have

$$\begin{aligned}\mathbf{E}\left\|x_{n+1}(t)\right\|^2 \leq & 6M^2\mathbf{E}\left\|x_0\right\|^2+12M^2(\mathbf{E}\left\|x_{00}\right\|^2+\mathbf{E}\left\|h(0,x_0)\right\|^2)\\ &+6M^2T\mathbf{E}\int_0^t\left(\mathbf{E}\left\|h(s,x_n(s))\right\|^2+\mathbf{E}\left\|f(s,x_n(s))\right\|^2\right)ds\\ &+6M^2\int_0^t\mathbf{E}\left\|g(s,x_n(s))\right\|^2_{\mathcal{L}_2}ds+12M^2HT^{2H-1}\int_0^t\mathbf{E}\left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0}ds.\end{aligned}$$

Then, from (H1)-(H5), we obtain

$$\begin{aligned}\mathbf{E}\left\|x_{n+1}(t)\right\|^2 &\leq 6M^2\mathbf{E}\left\|x_0\right\|^2+12M^2\mathbf{E}\left\|x_{00}\right\|^2+12M^2G\left(0,\mathbf{E}\left\|x_0\right\|^2\right)\\ &\quad+6M^2T^2\left(C_h+C_f\right)\int_0^t G\left(s,\mathbf{E}\left\|x_n(s)\right\|^2\right)ds\\ &\quad+6M^2TC_g\int_0^t G\left(s,\mathbf{E}\left\|x_n(s)\right\|^2\right)ds+12M^2HT^{2H-1}TL\\ &\leq C_1+C_2\int_0^t G\left(s,\mathbf{E}\left\|x_n(s)\right\|^2\right)ds\end{aligned}$$

Hence, we get

$$\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+1}(s)\right\|^2\right]\leq C_1+C_2\int_0^t G\left(s,\mathbf{E}\left[\sup_{r\in[0,s]}\left\|x_n(r)\right\|^2\right]\right)ds, \tag{3.12}$$

where

$$\begin{cases}C_1=6M^2\left(\mathbf{E}\left\|x_0\right\|^2+2\mathbf{E}\left\|x_{00}\right\|^2+2G\left(0,\mathbf{E}\left\|x_0\right\|^2\right)+2HT^{2H-1}TL\right),\\ C_2=6M^2T\left(T\left(C_h+C_f\right)+C_g\right).\end{cases}$$

Therefore, from (H4) and inequality (3.12), there is a $v(t),\ t\in[0,T]$, satisfying

$$v(t)=C_1+C_2\int_0^t G(s,v(s))ds.$$

We shall show, by induction, for $n=0,1,2,\ldots,$

$$\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_n(s)\right\|^2\right]\leq v(t),\qquad \forall t\in[0,T]. \tag{3.13}$$

By using the induction argument

$$\begin{aligned}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_0(s)\right\|^2\right] &= \mathbf{E}\left[\sup_{s\in[0,t]}\left\|C(s)x_0+S(s)(x_{00}-h(0,x_0))\right\|^2\right]\\ &\leq M^2\left(\mathbf{E}\left\|x_0\right\|^2+2\mathbf{E}\left\|x_{00}\right\|^2+2G\left(0,\mathbf{E}\left\|x_0\right\|^2\right)\right)\leq C_1\leq v(t)\end{aligned}$$

for all $t\in[0,T]$. Let use assume that (3.13) is true for some $n\in\mathbb{N}$, then by (3.12), the assumption of the mathematical induction and the non-decreasing property of G in v, we have

$$v(t)-\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_n(s)\right\|^2\right]\geq C_2\int_0^t\left(G\left(s,v(s)\right)-G\left(s,\mathbf{E}\left[\sup_{r\in[0,s]}\left\|x_n(r)\right\|^2\right]\right)\right)ds\geq 0$$

for all $t\in[0,T]$. By induction, we obtain for any $n\in\mathbb{N}$

$$\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_n(s)\right\|^2\right]\leq v(t)\leq v(T)<\infty.$$

□

Proof of Theorem 3.3. Step 1: Existence: By an argument similar to that in Lemma 3.4, we have

$$\begin{aligned}
\mathbf{E}\left\|x_{n+m}(t)-x_n(t)\right\|^2 &\leq 3\mathbf{E}\left\|\int_0^t C(t-s)\left(h(s,x_{n+m-1}(s))-h(s,x_{n-1}(s))\right)ds\right\|^2\\
&\quad+3\mathbf{E}\left\|\int_0^t S(t-s)\left(f(s,x_{n+m-1}(s))-f(s,x_{n-1}(s))\right)ds\right\|^2\\
&\quad+3\mathbf{E}\left\|\int_0^t S(t-s)\left(f(s,x_{n+m-1}(s))-f(s,x_{n-1}(s))\right)dw(s)\right\|^2\\
&\leq 2TMC_h\int_0^t \mathbf{E}\left\|h(s,x_{n+m-1}(s))-h(s,x_{n-1}(s))\right\|^2 ds\\
&\quad+2TMC_f\int_0^t \mathbf{E}\left\|f(s,x_{n+m-1}(s))-f(s,x_{n-1}(s))\right\|^2 ds\\
&\quad+2M\int_0^t \mathbf{E}\left\|g(s,x_{n+m-1}(s))-g(s,x_{n-1}(s))\right\|^2 ds.
\end{aligned}$$

Thus, we obtain

$$\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]\leq C_3\int_0^t K\left(s,\mathbf{E}\left[\sup_{r\in[0,s]}\left\|x_{n+m-1}(s)-x_{n-1}(s)\right\|^2\right]\right)ds, \tag{3.14}$$

where $C_3=2M\left(T(C_h+C_f)+C_g\right)$.

It follows from Lemma (3.4) that $\sup_{n,m}\left\|x_{n+m-1}-x_{n-1}\right\|^2<\infty$. Therefore, we can apply Fatou's Lemma to (3.14),

$$\begin{aligned}
&\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]\\
&\qquad\leq C_3\int_0^t K\left(s,\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{r\in[0,s]}\left\|x_{n+m-1}-x_{n-1}\right\|^2\right]\right)ds, \qquad (3.15)
\end{aligned}$$

Set

$$z(t):=\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right].$$

Then the above inequality (3.15) can be rewritten as

$$z(t)\leq C_3\int_0^t K\left(s,z(s)\right)ds,$$

By (H6) we immediately get that

$$z(t)=\lim\sup_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]=0 \qquad \text{for all } t\in[0,T],$$

which implies

$$\lim_{n,m\to\infty}\mathbf{E}\left[\sup_{s\in[0,t]}\left\|x_{n+m}(s)-x_n(s)\right\|^2\right]=0 \qquad \text{for all } t\in[0,T].$$

So $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the Banach space Δ_2^T. Denote the limit by $x(t)$. Now letting $n\to\infty$ in the equation (3.11), we obtain (3.6). In other words, we have shown the existence of a mild solution in Δ_2^T.

Step 2 : Uniqueness. Assume that x_1 and $x_2 \in \Delta_2^T$ are mild solutions of (1.1). Analogously as in the proof of (3.14), we obtain for any $t \in [0,T]$

$$\mathbf{E}\left[\sup_{s\in[0,t]} \|x_1(s)-x_2(s)\|^2\right] \le C_3 \int_0^t K\left(s, \mathbf{E}\left[\sup_{r\in[0,s]} \|x_1(s)-x_2(s)\|^2\right]\right) ds. \tag{3.16}$$

Due to hypothesis (H6) we get that $\mathbf{E}\left[\sup_{s\in[0,T]} \|x_1(s)-x_2(s)\|^2\right] = 0$, *i.e.*, $x_1 = x_2$. □

4. Controllability result

In this section we state and prove the controllability for second-order neutral stochastic equation driven by Brownian motion and an independent Rosenblatt process of the form

$$\begin{cases} d\left(x'(t) - h(t,x(t))\right) &= Ax(t)dt + Bu(t)dt + f\left(t,x(t)\right)dt \\ & \quad + g(t,x(t)dw(t) + \sigma(t)dZ_H(t), \\ x(0) = x_0, \quad x'(0) = x_{00}, \quad t \in [0,T] \end{cases} \tag{4.17}$$

where h, f, g, σ, A are the same as in the Eq.(1.1), $B : U \to X$ is a given mapping and the control function u takes values in $U_{ad} = L^2([0,T],U)$, the Hilbert space of admissible control functions for separable Hilbert space U.

Definition 4.1. A stochastic process $x \in \Delta_2^T$ is a mild solution of (4.17) if for each $u \in U_{ad}$ it satisfies the following integral equation

$$\begin{aligned} x(t) &= C(t)x_0 + S(t)(x_{00} - h(0,x_0)) + \int_0^t C(t-s)h(s,x(s))ds \\ &\quad + \int_0^t S(t-s)\left(Bu(s) + f(s,x(s)\right)ds + \int_0^t S(t-s)g(s,x(s))dw(s) \\ &\quad + \int_0^t S(t-s)\sigma(s)dZ_H(s), \quad P-a.s. \end{aligned}$$

Definition 4.2. The system (4.17) is said to be controllable on the interval $[0,T]$, if for every initial function $x(0) = x_0$, $x'(0) = x_{00}$ and desired final state $x_1 \in X$, there exists a stochastic control $u \in U_{ad}$ such that the mild solution of the system (4.17) corresponding to this control satisfies $x(T) = x_1$.

The following are the additional assumptions in this section.

(H7) The functions h, $f : [0,T]\times X \to X$ and $g : [0,T]\times X \to \mathcal{L}_2(K,X)$ satisfy the linear growth and Lipschitz conditions, that there exist positive constants C_h, C_f and C_g such that for $x, y \in X$ and $t \in [0,T]$

$$\begin{aligned} &\|h(t,x)\|^2 \le C_h(1+\|x\|^2), && \|h(t,x)-h(t,y)\|^2 \le C_h\|x-y\|^2, \\ &\|f(t,x)\|^2 \le C_f(1+\|x\|^2), && \|f(t,x)-f(t,y)\|^2 \le C_f\|x-y\|^2, \\ &\|g(t,x)\|_{\mathcal{L}_2}^2 \le C_g(1+\|x\|^2), && \|g(t,x)-g(t,y)\|_{\mathcal{L}_2}^2 \le C_g\|x-y\|^2. \end{aligned}$$

(H8) The linear operator $\Gamma : L^2([0,T],U) \to \mathcal{L}_2(\Omega,\mathcal{F}_T,X)$, is defined by

$$\Gamma u = \int_0^T S(T-s)Bu(s)ds$$

has a bounded inverse, Γ^{-1}, which takes values in $L^2([0,T],U)/\ker\Gamma$ and there exist positive constants M_B, M_Γ such that $\|B\|^2 \le M_B$ and $\left\|\Gamma^{-1}\right\|^2 \le M_\Gamma$.

Now, we describe the controllability result as follows and give its proof.

Theorem 4.3. *Under assumptions (H1)-(H2) and (H7)-(H8), the system* (4.17) *is controllable on* $[0, T]$.

Proof. Using the hypothesis (H8), for an arbitrary $x_T \in \mathcal{L}_2(\Omega, \mathcal{F}_T, X)$, we define the stochastic control

$$\begin{aligned} u_x(t) &= \Gamma^{-1}\left\{ x_T - C(T)x_0 - S(T)(x_{00} - h(0, x_0)) - \int_0^T C(T-s)h(s, x(s))ds \right. \\ &\quad - \int_0^T S(T-s)f(s, x(s))ds - \int_0^T S(T-s)g(s, x(s))dw(s) \qquad (4.18) \\ &\quad \left. - \int_0^T S(T-s)\sigma(s)dZ_H(s) \right\}(t). \end{aligned}$$

Define the operator $\Psi : \Delta_2^T \to \Delta_2^T$ by

$$\begin{aligned} (\Psi x)(t) &= C(t)x_0 + S(t)(x_{00} - h(0, x_0)) + \int_0^t C(t-s)h(s, x(s))ds \\ &\int_0^t S(t-s)\left(Bu_x(s) + f(s, x(s))\right)ds + \int_0^t S(t-s)g(s, x(s))dw(s) \qquad (4.19) \\ &+ \int_0^t S(t-s)\sigma(s)dZ_H(s) \end{aligned}$$

Now, we show that the operator Ψ has a fixed point in Δ_2^T which is a mild solution of the system (4.17). Substituting (4.18) in (4.19) we find that $(\Psi x)(T) = x_T$, indicating that the control u_x steers the system from x_0 to x_T in finite time T, which further implies that the system (4.17) is controllable. We divide the proof into three steps.

Step 1: For any $x \in \Delta_2^T$, $(\Psi x)(t)$ is continuous on the interval $[0, T]$ in L^2-sense. Let $0 \le t_1 \le t_2 \le T$. Then for any fixed $x \in \Delta_2^T$

$$\begin{aligned} \mathbf{E}\big\|(\Psi x)(t_2) - (\Psi x)(t_1)\big\|^2 &\le 5\mathbf{E}\big\|\left(C(t_2) - C(t_1)\right)x_0 + \left(S(t_2) - S(t_1)\right)\left(x_{00} - h(0, x_0)\right)\big\|^2 \\ &+ 5\mathbf{E}\left\|\int_0^{t_2}\left[C(t_2 - s)h(s, x(s)) + S(t_2 - s)f(s, x(s))\right]ds \right. \\ &\left. - \int_0^{t_1}\left[C(t_1 - s)h(s, x(s)) + S(t_1 - s)f(s, x(s))\right]ds\right\|^2 \\ &+ 5\mathbf{E}\left\|\int_0^{t_2} S(t_2 - s)f(s, x(s))dw(s) - \int_0^{t_1} S(t_1 - s)g(s, x(s))dw(s)\right\|^2 \\ &+ 5\mathbf{E}\left\|\int_0^{t_2} S(t_2 - s)\sigma(s)dZ_H(s) - \int_0^{t_1} S(t_1 - s)\sigma(s)dZ_H(s)\right\|^2 \\ &+ 5\mathbf{E}\left\|\int_0^{t_2} S(t_2 - s)Bu_x(s)ds - \int_0^{t_1} S(t_1 - s)Bu_x(s)ds\right\|^2 \\ &= 5\sum_{1 \le i = 5}\mathbf{E}\left\|D_i\right\|^2 \end{aligned}$$

By the strong continuity of $C(t)$ and $S(t)$, we have

$$\lim_{t_2 - t_1 \to 0}\left(C(t_2) - C(t_1)\right)x_0 + \left(S(t_2) - S(t_1)\right)\left(x_{00} - h(0, x_0)\right) = 0.$$

From property (2.5), we have

$$\|(C(t_2)-C(t_1))\,x_0+(S(t_2)-S(t_1))\,(x_{00}-h(0,x_0))\| \le \mathbf{2}M\,\|x_0\|+\mathbf{2}M\,\|x_{00}-h(0,x_0)\|$$

Thus we conclude by the Lebesgue's dominated convergence theorem that

$$\lim_{t_2-t_1\to 0}\mathbf{E}\,\|D_1\|^2=0.$$

For the second term D_2, we have

$$\begin{aligned}\|D_2\| &\le \Big\|\int_0^{t_1}\big[(C(t_2-s)-C(t_1-s))\,h(s,x(s))\\ &\qquad +(S(t_2-s)-S(t_1-s))\,f(s,x(s))\big]ds\Big\|\\ &\quad +\Big\|\int_{t_1}^{t_2}\left[C(t_2-s)h(s,x(s))+S(t_2-s)f(s,x(s))\right]ds\Big\|\\ &\le D_{21}+D_{22}.\end{aligned}$$

By the Holder inequality

$$\begin{aligned}\mathbf{E}\,\|D_{21}\|^2 \le t_1\mathbf{E}\int_0^{t_1}\big\|&(C(t_2-s)-C(t_1-s))\,h(s,x(s))\\ &+(S(t_2-s)-S(t_1-s))\,f(s,x(s))\big\|^2ds.\end{aligned}$$

By the strong continuity of $C(t)$ and $S(t)$, we have

$$\lim_{t_2-t_1\to 0}(C(t_2-s)-C(t_1-s))\,h(s,x(s))+(S(t_2-s)-S(t_1-s))\,f(s,x(s))=0.$$

By using property (2.5) and conditions (H7), we obtain

$$\begin{aligned}\|(C(t_2-s)-C(t_1-s))\,h(s,x(s))+(S(t_2-s)-S(t_1-s))\,f(s,x(s))\|\\ \le 2M\left(\|h(s,x(s))\|+\|f(s,x(s))\|\right).\end{aligned}$$

Then we conclude by the Lebesgue's dominated convergence theorem that

$$\lim_{t_2-t_1\to 0}\mathbf{E}\,\|D_{21}\|^2=0.$$

By property (2.5), condition (H7) and the Holder inequality, we get

$$\begin{aligned}\mathbf{E}\,\|D_{22}\|^2 &\le 2M^2(t_2-t_1)\int_{t_1}^{t_2}\mathbf{E}\left(\|h(s,x(s))\|^2+\|f(s,x(s))\|^2\right)ds\\ &\le 2M^2(C_h+C_f)(t_2-t_1)\int_0^T\left(\mathbf{E}\,\|x(s)\|^2+1\right)ds.\end{aligned}$$

Thus,

$$\lim_{t_2-t_1\to 0}\mathbf{E}\,\|D_{22}\|^2=0.$$

Now, for the term D_3, we have

$$\begin{aligned}\|D_3\| &\le \Big\|\int_0^{t_1}(S(t_2-s)-S(t_1-s))\,g(s,x(s))dw(s)\Big\|\\ &\quad +\Big\|\int_{t_1}^{t_2}S(t_2-s)g(s,x(s))dw(s)\Big\|\\ &\le \|D_{31}\|+\|D_{32}\|.\end{aligned}$$

By Ito isometry theorem, we have

$$\begin{aligned}\|D_{31}\| &\leq \left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)g(s,x(s))dw(s)\right\|^2\\ &\leq \int_0^{t_1}\left\|\left(S(t_2-s)-S(t_1-s)\right)g(s,x(s))\right\|_{\mathcal{L}_2}^2 ds.\end{aligned}$$

Since, by the strong continuity of $S(t)$, we have

$$\lim_{t_2-t_1\to 0}\left\|\left(S(t_2-s)-S(t_1-s)\right)g(s,x(s))\right\|_{\mathcal{L}_2}^2=0.$$

Moreover,

$$\left\|\left(S(t_2-s)-S(t_1-s)\right)g(s,x(s))\right\|_{\mathcal{L}_2}^2\leq 4M^2\left\|g(s,x(s))\right\|_{\mathcal{L}_2}^2.$$

Then we conclude by the Lebesgue's dominated convergence theorem that

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{31}\right\|^2=0.$$

For the second term D_{32}, similarly one get

$$\mathbf{E}\left\|D_{32}\right\|^2\leq 2M^2\int_{t_1}^{t_2}\left\|g(s,x(s))\right\|_{\mathcal{L}_2}^2 ds$$

thus,

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{32}\right\|^2=0.$$

For D_4, it is obvious that

$$\begin{aligned}\|D_4\| &\leq \left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)dZ_H(s)\right\|\\ &\quad+\left\|\int_{t_1}^{t_2}S(t_2-s)\sigma(s)dZ_H(s)\right\|\\ &\leq \|D_{41}\|+\|D_{42}\|.\end{aligned}$$

By Lemma (2.1), we have

$$\begin{aligned}\|D_{41}\|^2 &\leq \left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)dZ_H(s)\right\|^2\\ &\leq 2Ht_1^{2H-1}\int_0^{t_1}\left\|\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)\right\|_{\mathcal{L}_2^0}^2 ds.\end{aligned}$$

We have by the strong continuity of $S(t)$

$$\lim_{t_2-t_1\to 0}\left\|\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)\right\|_{\mathcal{L}_2^0}^2=0.$$

Moreover,

$$\left\|\left(S(t_2-s)-S(t_1-s)\right)\sigma(s)\right\|_{\mathcal{L}_2^0}^2\leq 4M^2\left\|\sigma(s)\right\|_{\mathcal{L}_2^0}^2.$$

According to the Lebesgue's dominated convergence theorem, we can obtain

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{41}\right\|^2=0.$$

In a similar way, we obtain

$$\mathbf{E}\left\|D_{42}\right\|^2\leq 4M^2H\left(t_2^{2H-1}-t_1^{2H-1}\right)\int_{t_1}^{t_2}\left\|\sigma(s)\right\|_{\mathcal{L}_2^0}^2 ds.$$

Thus,

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{42}\right\|^2=0.$$

Using the Holder inequality, property (2.5), (H2), (H7) and (H8), we obtain

$$\begin{aligned}\mathbf{E}\left\|u_x(t)\right\|^2 &\leq 5M_\Gamma\Big\{\mathbf{E}\left\|x_T\right\|^2+M^2\mathbf{E}\left\|x_0^2\right\|+2M^2(\mathbf{E}\left\|x_{00}\right\|^2+\mathbf{E}\left\|h(0,x_0)\right\|^2)\\ &\quad +M^2\left(T(C_h+C_f)+C_g\right)(1+\|x\|^2_{\Delta_2^T})+2M^2HT^{2H-1}L_\sigma\Big\} \quad (4.20)\\ &\leq M_u(1+\|x\|^2_{\Delta_2^T}).\end{aligned}$$

Next, observe that

$$\begin{aligned}\mathbf{E}\left\|D_5\right\|^2 &\leq 2\mathbf{E}\left\|\int_0^{t_1}\left(S(t_2-s)-S(t_1-s)\right)Bu_x(s)ds\right\|^2\\ &\quad +2\mathbf{E}\left\|\int_{t_1}^{t_2}S(t_2-s)Bu_x(s)ds\right\|^2\\ &\leq 2\left(\mathbf{E}\left\|D_{51}\right\|^2+\mathbf{E}\left\|D_{52}\right\|^2\right).\end{aligned}$$

Use the similar procedure as before, we obtain

$$\mathbf{E}\left\|D_{51}\right\|^2\leq t_1\int_0^{t_1}\mathbf{E}\left\|\left(S(t_2-s)-S(t_1-s)\right)Bu_x(s)\right\|^2ds.$$

Combing this with the strong continuity of $S(t)$ and inequality (4.20), we obtain

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{51}\right\|^2=0.$$

For the second term D_{42} similarly one get

$$\mathbf{E}\left\|D_{52}\right\|^2\leq M^2\left\|B\right\|^2(t_2-t_1)\int_{t_1}^{t_2}\mathbf{E}\left\|u_x(s)\right\|^2ds.$$

We obtain

$$\lim_{t_2-t_1\to 0}\mathbf{E}\left\|D_{52}\right\|^2=0.$$

The above argument show that $\lim_{t_2-t_1\to 0}\mathbf{E}\left\|(\Psi x)(t_2)-(\Psi x)(t_1)\right\|^2=0$ Thus we conclude $(\Psi x)(t)$ is continuous from the right in $[0,T)$. A similar reasoning show that it is also continuous from the left in $(0,T]$.

Step 2 : The operator Ψ sends Δ_2^T into itself.
Let $x\in\Delta_2^T$, then we have

$$\begin{aligned}\mathbf{E}\left\|(\Psi x)(t)\right\|^2 &\leq 7\mathbf{E}\left\|C(t)x_0\right\|^2+7\mathbf{E}\left\|S(t)(x_{00}-h(0,x_0))\right\|^2\\ &\quad +7\mathbf{E}\left\|\int_0^t C(t-s)h(s,x(s))ds\right\|^2+7\mathbf{E}\left\|\int_0^t S(t-s)f(s,x(s))ds\right\|^2\\ &\quad +7\mathbf{E}\left\|\int_0^t S(t-s)g(s,x(s))dw(s)\right\|^2+7\mathbf{E}\left\|\int_0^t S(t-s)\sigma(s)dZ_H(s)\right\|^2\\ &\quad +7\mathbf{E}\left\|\int_0^t S(t-s)Bu_x(s)ds\right\|^2.\end{aligned}$$

By Holder inequality, Ito isometry theorem and property (2.5), we have

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t)\right\|^2 &\leq 7M^2\mathbf{E}\left\|x_0\right\|^2 + 14M^2(\mathbf{E}\left\|x_{00}\right\|^2 + \mathbf{E}\left\|h(0,x_0)\right\|^2) \\
&\quad +7M^2T\mathbf{E}\int_0^t \left(\mathbf{E}\left\|h(s,x(s))\right\|^2 + \mathbf{E}\left\|f(s,x(s))\right\|^2\right) ds \\
&\quad +7M^2\int_0^t \mathbf{E}\left\|g(s,x(s))\right\|^2_{\mathcal{L}_2} ds + 14M^2HT^{2H-1}\int_0^t \mathbf{E}\left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0} ds \\
&\quad +7M^2\left\|B\right\|^2 T\int_0^t \mathbf{E}\left\|u_x(s)\right\|^2 ds.
\end{aligned}$$

Hence, from (H2) and (H7), combined with property (2.5) and inequality (4.20), we have

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t)\right\|^2 &\leq 7M^2\mathbf{E}\left\|x_0\right\|^2 + 14M^2(\mathbf{E}\left\|x_{00}\right\|^2 + \mathbf{E}\left\|h(0,x_0)\right\|^2) \\
&\quad +7M^2T^2\left(C_h + C_f\right)\left(1 + \left\|x\right\|^2_{\Delta_2^T}\right) + 7M^2TC_g\left(1 + \left\|x\right\|^2_{\Delta_2^T}\right) \\
&\quad +14M^2HT^{2H-1}TL + 7M^2\left\|B\right\|^2 T^2 M_u\left(1 + \left\|x\right\|^2_{\Delta_2^T}\right) \\
&\leq 7M^2\left(\mathbf{E}\left\|x_0\right\|^2 + 2(\mathbf{E}\left\|x_{00}\right\|^2 + \mathbf{E}\left\|h(0,x_0)\right\|^2) + 2HT^{2H-1}TL\right) \\
&\quad +7M^2\left(\left\|B\right\|^2 T^2\left(C_h + C_f + M_u\right) + TC_g\right)\left(1 + \left\|x\right\|^2_{\Delta_2^T}\right) \\
&= c_1 + c_2\left\|x\right\|^2_{\Delta_2^T},
\end{aligned}$$

where $c_1 \geq 0$ and $c_2 \geq 0$ are suitable constants. Therefore, we obtain that $\left\|(\Psi x)\right\|^2_{\Delta_2^T} < \infty$. Since $(\Psi x)(t)$ is continuous on $[0,T]$ and so Ψ maps Δ_2^T into itself.

Step 3 : Ψ is a contraction mapping in Δ_2^T. Let $x, y \in \Delta_2^T$, then for any fixed $t \in [0,T]$ we have

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t) - (\Psi y)(t)\right\|^2 &\leq 4\mathbf{E}\left\|\int_0^t S(t-s)B\left(u_x(s) - u_y(s)\right) ds\right\|^2 \\
&\quad +4\mathbf{E}\left\|\int_0^t C(t-s)\left(h(s,x(s)) - h(s,y(s))\right) ds\right\|^2 \\
&\quad +4\mathbf{E}\left\|\int_0^t S(t-s)\left(f(s,x(s)) - f(s,y(s))\right) ds\right\|^2 \\
&\quad +4\mathbf{E}\left\|\int_0^t S(t-s)\left(g(s,x(s)) - g(s,y(s))\right) dw(s)\right\|^2
\end{aligned}$$

By property (2.5), combined with Hölder's inequality and Ito isometry theorem, we get that

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t) - (\Psi y)(t)\right\|^2 &\leq 4M^2\left\|B\right\|^2 T\int_0^t \mathbf{E}\left\|u_x(s) - u_y(s)\right\|^2 ds \\
&\quad +4M^2T\int_0^t \mathbf{E}\left\|h(s,x(s)) - h(s,y(s))\right\|^2 ds \\
&\quad +4M^2T\int_0^t \mathbf{E}\left\|f(s,x(s)) - f(s,y(s))\right\|^2 ds \\
&\quad +4M^2\int_0^t \mathbf{E}\left\|g(s,x(s)) - g(s,y(s))\right\|^2_{\mathcal{L}_2} ds.
\end{aligned}$$

From (H7), (H8) and property (2.5), combined with Hölder's inequality and Ito isometry theorem, we have

$$\begin{aligned}
\mathbf{E}\left\|u_x(s)-u_y(s)\right\|^2 &\leq 3\mathbf{E}\left\|\Gamma^{-1}\int_0^T C(T-s)\left[h(s,x(s))-h(s,y(s))\right]ds\right\|^2\\
&\quad+3\mathbf{E}\left\|\Gamma^{-1}\int_0^t S(T-s)\left[f(s,x(s))-f(s,y(s))\right]ds\right\|^2\\
&\quad+3\mathbf{E}\left\|\Gamma^{-1}\int_0^t S(T-s)\left[g(s,x(s))-g(s,y(s))\right]dw(s)\right\|^2\\
&\leq 3M_\Gamma M^2 T(C_h+C_f)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds\\
&\quad+3M_\Gamma M^2 C_g\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds\\
&\leq 3M_\Gamma M^2\left(T(C_h+C_f)+C_g\right)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds\\
&= M_\mu\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds,
\end{aligned}$$

where $M_\mu = 3M_\Gamma M^2\left(T(C_h+C_f)+C_g\right).$

Therefore,

$$\begin{aligned}
\mathbf{E}\left\|(\Psi x)(t)-(\Psi y)(t)\right\|^2 &\leq 4M^2T\left(C_h+C_f\right)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds\\
&\quad+4M^2C_g\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds\\
&\quad+4M^2\left\|B\right\|^2T^2M_\mu\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds.
\end{aligned}$$

Hence, we obtain a positive real constant $\gamma(T)$ such that

$$\mathbf{E}\left\|(\Psi x)(t)-(\Psi y)(t)\right\|^2 \leq \gamma(T)\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds,$$

where

$$\gamma(T) = 4M^2T\left(\left\|B\right\|^2 TM_\mu + T\left(C_h+C_f\right)+C_g\right).$$

Moreover,

$$\begin{aligned}
\mathbf{E}\left\|(\Psi^2 x)(t)-(\Psi^2 y)(t)\right\|^2 &\leq \gamma(T)\int_0^t \mathbf{E}\left\|(\Psi x)(s)-(\Psi y)(s)\right\|^2 ds\\
&\leq \gamma(T)t\int_0^t \gamma(T)\mathbf{E}\left\|x(s)-y(s)\right\|^2 ds\\
&= \left(\gamma(T)\right)^2 t\int_0^t \mathbf{E}\left\|x(s)-y(s)\right\|^2 ds.
\end{aligned}$$

For any natural number n, using mathematical induction, one can get

$$\begin{aligned}
\mathbf{E}\left\|(\Psi^n x)(t)-(\Psi^n y)(t)\right\|^2 &\leq \gamma(T)\int_0^t \mathbf{E}\left\|(\Psi^{n-1} x)(s)-(\Psi^{n-1} y)(s)\right\|^2 ds\\
&\leq \frac{\left(t\gamma(T)\right)^n}{n!}\left\|x-y\right\|^2_{\Delta_2^t}.
\end{aligned}$$

Then taking the supremum over $[0,T]$,

$$\|(\Psi^n x)(t)-(\Psi^n y)(t)\|^2_{\Delta_2^T} \leq \frac{(T\gamma(T))^n}{n!}\|x-y\|^2_{\Delta_2^T}.$$

For sufficiently large n we have $\frac{(T\gamma(T))^n}{n!}<1$. It follows that Ψ^n is a strict contraction mapping on Δ_2^T, so that The Banach fixed point theorem ensure that Ψ has a unique fixed point, which is a mild solution for (4.17). Which implies that the system (4.17) is controllable on $[0,T]$. □

5. Example

Consider the control system driven by the process w and Z_H to illustrate the obtained theory

$$\begin{cases} \partial\left[\frac{\partial x(t,z)}{\partial t}-h_1\left(t,x(t,z)\right)\right] = \frac{\partial^2}{\partial z^2}x(t,z)\partial t+\left(v(t,z)+f_1\left(t,x(t,z)\right)\right)\partial t \\ \qquad +g_1\left(t,x(t,z)\right)dw(t)+\sigma(t)dZ_H, \qquad t\in]0,T[, \quad z\in[0,\pi], \\ x(0,z)=x_0(z), \qquad z\in[0,\pi], \\ \frac{\partial x(0,z)}{\partial t}=x_{00}(z), \qquad z\in[0,\pi], \\ x(t,0)=x(t,\pi)=0, \qquad t\in[0,T]. \end{cases} \tag{5.21}$$

Let $X=K=Y=U=L_2[0,\pi]$ and $x_0,\ x_{00}\in L_2[0,\pi]$. Let $A\subset D(A):X\to X$ be the linear operator given by $Ay=y^{''}$, where

$$D(A)=\{y\in X\ /\ y,\ y^{'} \text{ are absolutely continuous } y^{''}\in X, \quad y(0)=y(\pi)=0\}.$$

Here $w(t)$ denotes a one dimensional standard Brownian motion and Z_H is a Rosenblatt, the processes w and Z_H are independent. Suppose $h_1,f_1,g_1:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ are continuous, satisfy Lipschitz condition and linear growth condition and are uniformly bounded.

First of all, note that there exists a complete orthonormal set $\{e_n\}_{n\geq 1}$ of eigenvectors of A with

$$e_n(z)=\sqrt{(2/\pi)}\sin nz, \qquad 0\leq z\leq\pi, \quad n=1,2,\ldots,$$

and the following properties hold:

i): If $y\in D(A)$, then

$$Ay=-\sum_{n=1}^{\infty}n^2\langle y,e_n\rangle e_n(y), \qquad y\in D(A).$$

ii): The operator $C(t)$ defined by

$$C(t)y=\sum_{n=1}^{\infty}\cos(nt)\langle y,e_n\rangle e_n, \qquad y\in X,$$

is the cosine family in X generated by $(A,D(A))$, and the associated sine family is given by

$$S(t)y=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle y,e_n\rangle e_n, \qquad y\in X.$$

It is clear that $C(\cdot)x$ and $S(\cdot)x$ are periodic functions, and $\|C(t)\|\leq 1,\ \|S(t)\|\leq 1,\ t\in\mathbb{R}$.

Now define the functions: $h,\ f:[0,T]\times X\to X$, and $g:[0,T]\times X\to\mathcal{L}_2(K,X)$ as follows:

$$h(t,x)(z)=h_1(t,x(z)),$$
$$f(t,x)(z)=f_1(t,x(z)),$$
$$g(t,x)(z)=g_1(t,x(z))$$

for $t \in [0,T],\ x \in X$ and $0 < z < \pi$. The function $\sigma : [0,T] \to \mathcal{L}_2^0(Y,X)$ is bounded.

Let $B : U \to X$ is a bounded linear operator defined by

$$Bu(t)(z) = v(t,z),\ 0 \le z \le \pi,\ u \in L^2([0,T],U).$$

The operator $\Gamma : L^2([0,T],U) \to X$ is given by

$$\Gamma u = \int_0^T S(T-s)v(s,z)ds.$$

Then Γ is a bounded linear operator but not necessarily one-to-one. Let

$$\ker(\Gamma) = \{x \in L^2([0,T],U),\ \Gamma x = 0\}$$

be the null space of Γ and $[\ker(\Gamma)]^{\perp}$ be its orthogonal complement in $L^2([0,T],U)$. Let $\widetilde{\Gamma} : [\ker(\Gamma)]^{\perp} \to Range(\Gamma)$ be the restriction of Γ to $[\ker(\Gamma)]^{\perp}$, $\widetilde{\Gamma}$ is necessarily one-to-one operator. The inverse mapping theorem says that $\widetilde{\Gamma}^{-1}$ is bounded since $[\ker(\Gamma)]^{\perp}$ and $Range(\Gamma)$ are Banach spaces. So That Γ^{-1} is bounded and takes values in $L^2([0,T],U) \perp \ker(\Gamma)$, hypothesis (H4) is satisfied. Hence, all conditions of Theorem(4.3) are satisfied, and consequently system.(5.21) is controllable on $[0,T]$.

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Rakia Ahmed Yahia: r.ahmedyahia@centre-univ-mila.dz
Laboratoire des Mathématiques et leurs interactions "MELILAB". Abdelhafid Boussouf University Centre, Mila 43000, Algeria

Abbes Benchaabane: benchaabane.abbes@univ-guelma.dz
Laboratory of Analysis and Control of Differential Equations "ACED". Univ. 8 May 1945 Guelma, Algeria

Halim Zeghdoudi:
Laboratory of Analysis and Control of Differential Equations "ACED". Univ. 8 May 1945 Guelma, Algeria

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