

## ROBIN BOUNDARY CONDITIONS FOR THE LAPLACIAN ON METRIC GRAPH COMPLETIONS

ROBERT CARLSON

ABSTRACT. A generalization of Robin boundary conditions leading to self-adjoint operators is developed for the second derivative operator on metric graphs with compact completion and totally disconnected boundary. Harmonic functions and their properties play an essential role.

Для оператора другої похідної розроблено узагальнення граничних умов Робена, що веде до самоспряжених операторів на метричних графах з компактним поповненням і повністю незв'язною границею. Істотну роль відіграють гармонічні функції і їх властивості.

### 1. INTRODUCTION

Let  $\mathcal{G}$  denote a connected, locally finite metric graph with a countable vertex set  $\mathcal{V}_{\mathcal{G}}$  and edge set  $\mathcal{E}_{\mathcal{G}}$ . Edges  $e \in \mathcal{E}_{\mathcal{G}}$  are assigned a length  $l_e$  and are identified with intervals  $[a_e, b_e]$  of length  $l_e$ . With the usual geodesic distance,  $\mathcal{G}$  becomes a metric space.

The boundary  $\partial\mathcal{G}$  of  $\mathcal{G}$  will be a subset of vertices, including all vertices with degree 1. The interior  $\mathcal{G}_{int}$  of  $\mathcal{G}$  will be the complement of the boundary vertices. As a metric space,  $\mathcal{G}$  has a completion  $\overline{\mathcal{G}}$ ; the boundary of  $\overline{\mathcal{G}}$  will be the complement of the graph interior,  $\partial\overline{\mathcal{G}} = \overline{\mathcal{G}} \setminus \mathcal{G}_{int}$ .

The boundary and function theory developed here are based on two essential assumptions:  $\overline{\mathcal{G}}$  is compact and  $\partial\overline{\mathcal{G}}$  is totally disconnected. Simple examples, with boundary homeomorphic to the Cantor set, may be constructed from homogeneous trees with decaying edge lengths. A sufficient (but not necessary) condition for these properties is that the volume, which is the sum of the edge lengths, is finite. In the finite volume cases  $\overline{\mathcal{G}}$  is [9] the end compactification of  $\mathcal{G}$ .

When  $\mathcal{G}$  is finite, various authors have characterized boundary conditions leading to a self-adjoint Laplace differential operator  $-D^2$  on  $L^2(\mathcal{G})$ . When  $\mathcal{G}$  is infinite and the edge lengths of have a positive lower bound, the existence of a unique self-adjoint extension of a 'minimal' symmetric operator  $-D^2$  is common. A useful discussion and numerous references are in [2]; a more recent source with additional information is [8].

For infinite graphs having compact completions  $\overline{\mathcal{G}}$  with totally disconnected boundary  $\partial\overline{\mathcal{G}}$ , 'minimal' symmetric operators  $-D^2$  on  $L^2(\mathcal{G})$  satisfying standard Kerckhoff conditions at interior vertices may have many distinct self-adjoint extensions. Recent works addressing related questions about self-adjoint operators include [10] and [16]. Physical models motivate a search for such extensions characterized by 'boundary conditions'. This search leads to novel problems, especially when the boundary conditions describe behavior at points in  $\partial\overline{\mathcal{G}}$  that are not vertices of  $\mathcal{G}$ . Some of these boundary conditions and corresponding operators were described in [3]. Initial domains there consisted of functions which either (i) vanished outside compact subsets of  $\mathcal{G}_{int}$ , or (ii) had derivatives vanishing outside compact subsets of  $\mathcal{G}_{int}$ . Such domains extend the classical Dirichlet or Neumann boundary conditions. The symmetric operators  $-D^2$  with these domain are nonnegative, so have self-adjoint Friedrichs extensions.

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The main goal of this work is to identify and study a suitable generalization of 'mixed' or Robin boundary conditions leading to self-adjoint Laplace operators. Consider the classical second derivative operator  $-D^2$  acting on  $L^2_{\mathbb{R}}[0, 1]$  with the boundary conditions  $f'(0) = \alpha f(0)$  and  $f'(1) = \beta f(1)$ . To satisfy these boundary conditions, start with a domain consisting of smooth functions which have the form  $c_0(\alpha x + 1)$  in some neighborhood of  $x = 0$ , and  $c_1(\beta x + 1 - \beta)$  near  $x = 1$ . This domain, which is a core for a self-adjoint operator, is defined with the aid of functions which are harmonic near the boundary. This simple example will be generalized to build domains for symmetric and self-adjoint operators  $-D^2$  on  $L^2(\mathcal{G})$ . Results describing the existence and properties of harmonic functions on  $\overline{\mathcal{G}}$  play an essential role.

The results are developed in three subsequent sections. Section 2 begins with a review of basic material on metric graphs. Some results about compact totally disconnected metric spaces such as  $\partial\overline{\mathcal{G}}$  are then presented, along with a theorem which links the totally disconnected boundary with a 'weakly connected' condition for  $\overline{\mathcal{G}}$  which appeared in [3]. Section 3 treats the existence and properties of harmonic functions on  $\overline{\mathcal{G}}$ . The introduction of energy spaces provides a new approach to solving the Dirichlet problem for metric graphs. Level sets of harmonic functions are considered; these help provide needed refinements of the existence results. Section 4 then addresses the construction of symmetric and nonnegative self-adjoint Laplace operators based on novel boundary conditions, defined with the aid of harmonic functions. The quadratic forms for these operators include boundary terms which distinguish them from the Dirichlet and Neumann cases.

## 2. GRAPHS WITH TOTALLY DISCONNECTED BOUNDARY

**2.1. Metric graphs.** Suppose  $w_1, w_2 \in \mathcal{V}_{\mathcal{G}}$ . A vertex path from  $w_1$  to  $w_2$  is a finite vertex sequence  $v_1, \dots, v_N$  with  $v_1 = w_1$ ,  $v_N = w_2$ , and  $v_n$  adjacent to  $v_{n+1}$  for  $n = 1, \dots, N - 1$ . If the edge  $e_n$  joining  $v_n$  to  $v_{n+1}$  has length  $l_n$  and  $L = \sum_{n=1}^{N-1} l_n$ , then a path  $\gamma$  from  $w_1$  to  $w_2$  (with length  $L$ ) is the function  $\gamma : [0, L] \rightarrow \mathcal{G}$  obtained by traversing the edges  $e_n$  from  $v_n$  to  $v_{n+1}$  and  $n = 1, \dots, N - 1$ .

If  $e = \{v_1, v_2\} \in \mathcal{E}_{\mathcal{G}}$  is identified with the interval  $[a, b]$  and  $x \in e$  is not a vertex, it may be useful to treat  $x$  as an added vertex adjacent to  $v_1, v_2$ . Then identify  $\{v_1, x\}$  with  $[a, x]$  and  $\{x, v_2\}$  with  $[x, b]$ . A path joining two such points  $x_1$  and  $x_2$  may be defined as above. The distance  $d(x_1, x_2)$  between points  $x_1$  and  $x_2$  in  $\mathcal{G}$  is defined as the infimum of the lengths of paths joining  $x_1$  and  $x_2$ . This metric extends continuously to  $\overline{\mathcal{G}}$ .

Points  $x_1, x_2 \in \overline{\mathcal{G}}$  can be joined by possibly infinite paths  $\gamma : [0, L] \rightarrow \mathcal{G}$  or  $\gamma : (-L, L) \rightarrow \mathcal{G}$  of finite length, defined analogously for sequences  $v_1, v_2, v_3, \dots$  or bidirectional sequences  $\dots, v_{-2}, v_{-1}, v_0, v_1, \dots$ . In this case it is assumed that  $x_2 = \lim_{t \rightarrow L} \gamma(t)$  and  $x_1 = \gamma(0)$  or  $x_1 = \lim_{t \rightarrow -L} \gamma(t)$  as appropriate.

Metric graphs may be equipped with a variety of function spaces. A function  $f : \mathcal{G} \rightarrow \mathbb{R}$  has components  $f_e : [a_e, b_e] \rightarrow \mathbb{R}$ . In this work functions are real-valued unless otherwise noted. Our basic Hilbert space is the usual Lebesgue space

$$L^2(\mathcal{G}) = \oplus_{e \in \mathcal{E}_{\mathcal{G}}} L^2[a_e, b_e],$$

with inner product

$$\langle f, g \rangle_2 = \int_{\mathcal{G}} fg = \sum_{e \in \mathcal{E}_{\mathcal{G}}} \int_{a_e}^{b_e} f_e g_e.$$

The notation  $e \sim v$  indicates that the edge  $e$  is incident on a vertex  $v$ . If  $e \sim v$ , the notation  $\partial_\nu f_e(v)$  is used to indicate the derivative of  $f_e$  at  $v$  computed in outward pointing local coordinates. That is, for this computation, the identification of  $e$  with  $[a_e, b_e]$ , identifies  $v$  with  $a_e$ .

As an initial domain for  $-D^2$ , let  $\mathcal{D}_{max}$  denote the continuous real valued functions  $f$  on  $\mathcal{G}$  which have absolutely continuous derivatives on each edge  $e$ , with  $f$  and  $f'' \in L^2(\mathcal{G})$ , and which satisfy

$$\sum_{e \sim v} \partial_\nu f_e(v) = 0 \quad (2.1)$$

at interior vertices  $v \in \mathcal{V}_{\mathcal{G}}$ .

The interior vertex condition (2.1) leads to an important integration by parts lemma.

**Lemma 2.1.** *Suppose  $\mathcal{G}$  is a finite graph with boundary  $\partial\mathcal{G}$ . If  $f, g \in \mathcal{D}_{max}$ , then*

$$\begin{aligned} \int_{\mathcal{G}} f'' g &= - \sum_{v \in \partial\mathcal{G}} \sum_{e \sim v} g(v) \partial_\nu f_e(v) - \int_{\mathcal{G}} f' g' \\ &= \sum_{v \in \partial\mathcal{G}} \sum_{e \sim v} [f(v) \partial_\nu g_e(v) - g(v) \partial_\nu f_e(v)] + \int_{\mathcal{G}} f g'' \end{aligned} \quad (2.2)$$

*Proof.* Using the identification of edges  $e$  with intervals  $[a_e, b_e]$ , integration by parts gives

$$\int_{\mathcal{G}} f'' g = \sum_{e \in \mathcal{E}_{\mathcal{G}}} \int_{a_e}^{b_e} f''_e g_e = \sum_{e \in \mathcal{E}_{\mathcal{G}}} [f'_e(b_e) g_e(b_e) - f'_e(a_e) g_e(a_e)] - \int_{\mathcal{G}} f' g'$$

Regroup the sum of boundary terms, collecting those having evaluation at the same vertex. The terms  $f'_e(b_e) g_e(b_e)$  have derivatives computed in inward pointing local coordinates, and  $g$  is continuous at each  $v$ , so

$$\int_{\mathcal{G}} f'' g = - \sum_{v \in \mathcal{V}_{\mathcal{G}}} g(v) \sum_{e \sim v} \partial_\nu f_e(v) - \int_{\mathcal{G}} f' g'$$

(2.1) implies that terms in the last sum coming from interior vertices vanish, giving the first line of (2.2). Another integration by parts produces the second line.  $\square$

**2.2. Totally disconnected boundary.** Recall that  $\bar{\mathcal{G}}$  is compact and  $\partial\bar{\mathcal{G}}$  is totally disconnected. Since  $\bar{\mathcal{G}}$  is compact it must be totally bounded, leading to the following observation.

**Proposition 2.1.**  *$\bar{\mathcal{G}}$  is compact if and only if for every  $\epsilon > 0$  there is a finite subgraph  $\mathcal{G}_0$  of  $\mathcal{G}$ , such that for every  $y \in \bar{\mathcal{G}}$  there is an  $x \in \mathcal{G}_0$  with  $d(x, y) < \epsilon$ .*

Given  $\epsilon > 0$ , it will be convenient to have a subgraph  $\mathcal{G}_\epsilon$  of  $\mathcal{G}$  containing all points  $x \in \mathcal{G}$  whose distance from  $\partial\bar{\mathcal{G}}$  is at least  $\epsilon$ . The edges  $e$  of  $\mathcal{G}_\epsilon$  are the (closed) edges of  $\mathcal{G}$  containing a point  $x$  with  $d(x, \partial\bar{\mathcal{G}}) \geq \epsilon$ . If an edge  $e$  of  $\mathcal{G}_\epsilon$  is incident in  $\mathcal{G}$  on a vertex  $v$ , then  $v$  is a vertex of  $\mathcal{G}_\epsilon$ .

**Lemma 2.2.** *If  $\bar{\mathcal{G}}$  is compact then  $\mathcal{G}_\epsilon$  is a finite graph.*

*Proof.* Arguing by contradiction, suppose  $\mathcal{G}$  has a sequence  $\{e_n, n = 1, 2, 3, \dots\}$  of distinct edges, each containing a point  $x_n$  with  $d(x_n, \partial\bar{\mathcal{G}}) \geq \epsilon$ . Since  $\bar{\mathcal{G}}$  is compact, the sequence  $\{x_n\}$  has a convergent subsequence  $\{z_k, k = 1, 2, 3, \dots\}$  with limit  $z \in \bar{\mathcal{G}}$ . Since  $d(z, \partial\bar{\mathcal{G}}) \geq \epsilon$  it must be that  $z \in \mathcal{G}$ .

Suppose  $z$  is in the edge  $e$ . Since  $\mathcal{G}$  is locally finite, there are only finitely many edges sharing a vertex with  $e$ . With only finitely many exceptions, the points  $z_k$  are outside this set of edges, so  $z$  cannot be the limit of the subsequence. Thus  $\mathcal{G}_\epsilon$  cannot have infinitely many distinct edges.  $\square$

The boundary  $\partial\bar{\mathcal{G}}$  is a closed subset of  $\bar{\mathcal{G}}$ , so it is a totally disconnected compact metric space. Some general facts about a such metric spaces will be useful. In particular, as a totally disconnected compact metric space,  $\partial\bar{\mathcal{G}}$  will have a rich collection of clopen subsets, which are both open and closed in  $\partial\bar{\mathcal{G}}$ .

A version of the next result about a totally disconnected compact metric space  $\Omega$  is in [12, p. 97]. Suppose  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are partitions of  $\Omega$ . Partition  $\mathcal{E}_2$  is a refinement of  $\mathcal{E}_1$  if each set in  $\mathcal{E}_2$  is a subset of a set in  $\mathcal{E}_1$ .

**Proposition 2.2.** *Suppose  $\Omega$  is a totally disconnected compact metric space. For any  $\epsilon > 0$ , there is a finite partition  $\mathcal{E} = \{E(n), n = 1, \dots, N\}$  of  $\Omega$  by clopen sets such that the diameter of each  $E(n) \in \mathcal{E}$  is less than  $\epsilon$ .*

*There is a sequence  $\{\mathcal{E}_j, j = 1, 2, 3, \dots\}$  of partitions of  $\Omega$  by clopen sets, with  $\mathcal{E}_{j+1}$  a refinement of  $\mathcal{E}_j$ , such that the diameter of each set in  $\mathcal{E}_j$  is less than  $1/j$ .*

The next result characterizes the compact completions  $\bar{\mathcal{G}}$  with totally disconnected boundary. In an earlier work [3] the author used an assumption that  $\bar{\mathcal{G}}$  was weakly connected.  $\bar{\mathcal{G}}$  is weakly connected if for every pair of distinct points  $x, y \in \bar{\mathcal{G}}$ , there is a finite set of points  $W = \{w_1, \dots, w_K\}$  in the graph  $\mathcal{G}$  separating  $x$  from  $y$ . That is, there are disjoint open subsets  $U_x, U_y$  of  $\bar{\mathcal{G}}$  with  $x \in U_x$  and  $y \in U_y$  such that  $\bar{\mathcal{G}} \setminus W = U_x \cup U_y$ .

**Theorem 2.1.** *If  $\bar{\mathcal{G}}$  is compact, then  $\partial\bar{\mathcal{G}}$  is totally disconnected if and only if  $\bar{\mathcal{G}}$  is weakly connected.*

*Proof.* Assume  $\bar{\mathcal{G}}$  is weakly connected. If  $x$  and  $y$  are distinct points in  $\partial\bar{\mathcal{G}}$ , they are in distinct clopen subsets of  $\partial\bar{\mathcal{G}}$  given by  $U_x \cap \partial\bar{\mathcal{G}}$  and  $U_y \cap \partial\bar{\mathcal{G}}$ , with  $[U_x \cap \partial\bar{\mathcal{G}}] \cup [U_y \cap \partial\bar{\mathcal{G}}] = \partial\bar{\mathcal{G}}$ . Thus  $x$  and  $y$  lie in distinct connected components, and  $\partial\bar{\mathcal{G}}$  is totally disconnected.

Assume now that  $\bar{\mathcal{G}}$  is compact with a totally disconnected boundary. Consider distinct points  $x, y$  in  $\bar{\mathcal{G}}$ . If either  $x$  or  $y$  is a point of  $\mathcal{G}$ , they are easily separated by the removal of a finite set of points in  $\mathcal{G}$ . Assume then that  $x$  and  $y$  belong to  $\partial\bar{\mathcal{G}}$ , but are not boundary vertices of  $\mathcal{G}$ . Then by Proposition 2.2 there are disjoint clopen sets  $E_x, E_y \subset \partial\bar{\mathcal{G}}$  with  $x \in E_x, y \in E_y$ , and  $E_x \cup E_y = \partial\bar{\mathcal{G}}$ .

For  $z \in \bar{\mathcal{G}}$ , let  $B_\epsilon(z)$  denote the open ball of radius  $\epsilon > 0$  centered at  $z$ , while the  $\epsilon$  neighborhood of a set  $E \subset \bar{\mathcal{G}}$  is  $N_\epsilon(E) = \cup_{z \in E} B_\epsilon(z)$ . Since  $E_x$  and  $E_y$  are compact and disjoint in  $\bar{\mathcal{G}}$ , the neighborhoods  $N_\epsilon(E_x)$  and  $N_\epsilon(E_y)$  are disjoint if  $\epsilon > 0$  is sufficiently small [15, p. 86].

The subgraph  $\mathcal{G}_\epsilon$ , which is finite by lemma 2.2, is now useful. The set  $\bar{\mathcal{G}} \setminus [N_\epsilon(E_x) \cup N_\epsilon(E_y)]$  is a subset of  $\mathcal{G}_\epsilon$ . Define

$$U_x = N_\epsilon(E_x) \setminus \mathcal{G}_\epsilon, \quad U_y = N_\epsilon(E_y) \setminus \mathcal{G}_\epsilon.$$

Note that  $U_x \cup U_y \cup \mathcal{G}_\epsilon = \bar{\mathcal{G}}$ . The sets  $U_x$  and  $U_y$  are still open neighborhoods of  $x, y$  respectively.

Let  $W$  be the set of vertices in  $\mathcal{G}_\epsilon$ , and let  $V$  be the complement of  $U_x$  in  $\bar{\mathcal{G}} \setminus W$ .  $V$  is open since it is the union of  $U_y$  and the collection of open edges of  $\mathcal{G}_\epsilon$ . The sets  $U_x, V$  provide the desired separation of  $x$  and  $y$  by a finite set  $W$  of points from  $\mathcal{G}$ , showing that  $\bar{\mathcal{G}}$  is weakly connected.  $\square$

An important role in the function theory of  $\bar{\mathcal{G}}$  is played by an algebra  $\mathcal{A}$  of 'eventually flat' functions.  $\mathcal{A}$  is the set of functions  $\phi : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  which are continuous on  $\bar{\mathcal{G}}$  and infinitely differentiable on the open edges of  $\mathcal{G}$ , with  $\phi' = 0$  in the complement of a finite collection of edges, and in an open neighborhood of each vertex  $v \in \mathcal{G}$ . With pointwise multiplication,  $\mathcal{A}$  is a subalgebra of the continuous functions on  $\bar{\mathcal{G}}$  which contains the constant functions. A similar class of functions and its relation to the end compactification of a graph was considered in [4].

**Lemma 2.3.** *Suppose  $\bar{\mathcal{G}}$  is compact with a totally disconnected boundary. Assume that  $E$  and  $E^c = \partial\bar{\mathcal{G}} \setminus E$  are nonempty clopen subsets of  $\partial\bar{\mathcal{G}}$ . Then there is a function  $\phi \in \mathcal{A}$  with  $\phi(x) = 1$  for  $x \in E$  and  $\phi(x) = 0$  for  $x \in E^c$ .*

*Proof.* Since  $E$  and  $E^c$  are disjoint and compact, we may choose  $\epsilon > 0$  such that  $d(x, y) > 3\epsilon$  for all  $x \in E$  and  $y \in E^c$ . Begin by taking  $\phi(x) = 1$  if  $d(x, E) \leq \epsilon$  and  $\phi(x) = 0$  if  $d(x, E^c) \leq \epsilon$ . Now  $\phi$  must be extended to  $\mathcal{G}_\epsilon$ .

Let  $e$  be a closed edge of  $\mathcal{G}_\epsilon$  which contains a point  $x$  where  $\phi$  is not yet defined. First, if  $e$  has no point  $x$  with  $\phi(x) = 1$  define  $\phi(x) = 0$  for  $x \in e$ . For the remaining edges,  $\phi^{-1}(1) \cap e$  and  $\phi^{-1}(0) \cap e$  will be separated subintervals of  $e$ , each containing an endpoint of  $e$ . Extend  $\phi$  smoothly to these edges  $e$ , with  $\phi' = 0$  in a neighborhood of the endpoints of  $e$ . Since  $\mathcal{G}_\epsilon$  is finite by lemma 2.2, the extended function  $\phi$  is in  $\mathcal{A}$ .  $\square$

Proposition 2.2 easily shows that  $\mathcal{A}$  separates points of  $\overline{\mathcal{G}}$ , so the Stone-Weierstrass theorem implies the next result.

**Theorem 2.2.** *Assume  $\overline{\mathcal{G}}$  is compact with totally disconnected boundary  $\partial\overline{\mathcal{G}}$ . Then  $\mathcal{A}$  is uniformly dense in the space of continuous functions on  $\overline{\mathcal{G}}$ , and the boundary values of  $\mathcal{A}$  are uniformly dense in the space of continuous functions on  $\partial\overline{\mathcal{G}}$ .*

### 3. FINITE ENERGY HARMONIC FUNCTIONS

**3.1. Energy spaces.** The introduction of an additional Hilbert space  $\mathbb{H}_1$  will assist in understanding the harmonic functions on  $\overline{\mathcal{G}}$ . Let  $\mu$  be a finite positive measure on  $\partial\overline{\mathcal{G}}$ . The  $\mathbb{H}_1$  inner product is

$$\langle f, g \rangle_1 = \int_{\mathcal{G}} f'g' + \int_{\partial\overline{\mathcal{G}}} fg \, d\mu. \quad (3.1)$$

The elements of  $\mathbb{H}_1$  are the functions  $f : \overline{\mathcal{G}} \rightarrow \mathbb{R}$  which are continuous on  $\overline{\mathcal{G}}$  and absolutely continuous on the edges of  $\mathcal{G}$ , with  $f' \in L^2(\mathcal{G})$ . Addition and scalar multiplication are defined pointwise as usual. Similar spaces appear in the study of resistor networks [6] and the associated operator theory [13]. The measure  $\mu$  plays a modest role in this work, but can have significance in physical modeling, as the next example illustrates.

Examples incorporating Robin boundary conditions and energy space inner products arise from models of strings coupled to springs. Suppose the string displacement from equilibrium is  $u(t, x)$ , with  $a \leq x \leq b$ . The string is attached to a spring at  $a$  with spring constant  $k_a > 0$ , and at  $b$  with constant  $k_b > 0$ . The springs are constrained to move transversely. A standard model [11, p. 30] for the system motion uses the wave equation  $\partial^2 u / \partial t^2 - \partial^2 u / \partial x^2 = 0$  with the boundary conditions

$$\frac{\partial u(t, x)}{\partial x} \Big|_{x=a} = k_a u(t, a), \quad \frac{\partial u(t, x)}{\partial x} \Big|_{x=b} = -k_b u(t, b). \quad (3.2)$$

The associated Sturm-Liouville operator  $-D^2 = -\partial^2 / \partial x^2$  with the boundary conditions  $f'(a) = k_a f(a)$  and  $f'(b) = -k_b f(b)$  has the quadratic (potential energy) form

$$\int_a^b (-D^2 f) f \, dx = -f'f \Big|_a^b + \int_a^b (f')^2 \, dx = k_b f(b)^2 + k_a f(a)^2 + \int_a^b (f')^2 \, dx, \quad (3.3)$$

which is strictly positive for  $f \neq 0$ . This is just the energy space form if the graph has a single edge and the measure  $\mu$  assigns mass  $k_a$  to  $a$  and  $k_b$  to  $b$ .

These models have a direct generalization to finite graphs  $\mathcal{G}_0$ , with continuity and (2.1) holding at interior vertices. At boundary vertices the Robin condition becomes

$$\sum_{e \sim v} \partial_\nu f_e(v) = k_v f(v), \quad (3.4)$$

with integration giving

$$\int_{\mathcal{G}_0} (-D^2 f) f = \sum_{v \in \partial\mathcal{G}_0} k_v f(v)^2 + \int_{\mathcal{G}_0} (f')^2.$$

Without the spring contributions, string vibration problems on finite graphs have been studied before [1, 5, 17].

Returning to the general case, the next result establishes that  $\mathbb{H}_1$  is complete.

**Proposition 3.1.** *Assume that  $\mu$  is a positive measure on  $\partial\overline{\mathcal{G}}$  with  $0 < \mu(\partial\overline{\mathcal{G}}) < \infty$ . A Cauchy sequence in  $\mathbb{H}_1$  converges uniformly to a continuous function on  $\overline{\mathcal{G}}$ .  $\mathbb{H}_1$  is a Hilbert space.*

*Proof.* Suppose first that  $\|f\|_1^2 = \langle f, f \rangle_1 = 0$ . Then  $f = 0$ ,  $\mu$ -almost everywhere in  $\partial\overline{\mathcal{G}}$ . Pick  $x_0 \in \partial\overline{\mathcal{G}}$  with  $f(x_0) = 0$ . For  $x \in \overline{\mathcal{G}}$ , pick a path  $\gamma$  of finite length joining  $x_0$  to  $x$ . Since  $f$  is absolutely continuous and  $f' = 0$  as an element of  $L^2(\mathcal{G})$ ,

$$f(x) = \int_{\gamma} f'(t) dt = 0.$$

The form (3.1) is thus positive definite, and so defines an inner product.

Suppose  $\{f_n\}$  is a Cauchy sequence in  $\mathbb{H}_1$ , with

$$\int_{\mathcal{G}} (f'_n - f'_m)^2 + \int_{\partial\overline{\mathcal{G}}} (f_n - f_m)^2 d\mu \rightarrow 0, \quad m, n \rightarrow \infty.$$

Given  $\epsilon > 0$  and  $m, n$  large, there are points  $x_0 \in \partial\overline{\mathcal{G}}$  with  $|f_n(x_0) - f_m(x_0)| < \epsilon$ . For  $x \in \overline{\mathcal{G}}$ , integration over a path  $\gamma$  from  $x_0$  to  $x$  with length at most  $2 * \text{diam}(\mathcal{G})$  gives

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x_0) - f_m(x_0)| + \left| \int_{x_0}^x f'_n(t) - f'_m(t) dt \right| \\ &\leq \epsilon + 2\text{diam}(\mathcal{G})^{1/2} \|f_n - f_m\|_1. \end{aligned}$$

Thus  $\{f_n\}$  is a uniformly convergent sequence of continuous functions on  $\overline{\mathcal{G}}$ , with a continuous limit  $f$ .

Since  $L^2(\mathcal{G})$  is complete, the sequence  $\{f'_n\}$  converges in  $L^2(\mathcal{G})$  to a function  $g$ . Integration again gives

$$f_n(x) - f_n(x_0) = \int_{\gamma} f'_n(t) dt.$$

The  $L^2(\mathcal{G})$  convergence of  $f'_n$  to  $g$  implies  $L^1$  convergence on the path from  $x_0$  to  $x$ , so

$$f(x) - f(x_0) = \int_{x_0}^x g(t) dt.$$

Thus  $f$  is absolutely continuous [19, p. 110],  $g(x) = f'(x)$  almost everywhere, and  $\mathbb{H}_1$  is complete.  $\square$

**3.2. Harmonic functions.** A continuous function  $f : \overline{\mathcal{G}} \rightarrow \mathbb{R}$  is harmonic if (i)  $D^2f = 0$  on each edge, so  $f$  is piecewise linear, and (ii)  $f$  satisfies the standard vertex conditions (2.1) at each interior vertex. Say that a harmonic function  $f$  has finite energy if  $f \in \mathbb{H}_1$ . Let  $H_{fin}$  denote the set of finite energy harmonic functions on  $\overline{\mathcal{G}}$ .

Harmonic functions satisfy the mean value property at interior vertices. Assume that  $v$  is an interior vertex with  $N$  incident edges  $e = [a_e, b_e]$  pointing away from  $v$  so that each  $a_e$  is identified with  $v$ . Suppose that  $f$  is a function which is continuous at  $v$ , and whose restriction  $f_e$  to  $e$  is linear. The identity  $f_e(a_e) = f_e(x) - (x - a_e)f'_e$  holds for  $a_e \leq x \leq b_e$ . If  $x \leq \min_{e \sim v} (b_e)$  and  $x - a_e$  has the same value on each edge, then

$$f(v) = \frac{1}{N} \sum_{n=1}^N f_e(x) - \frac{x - a_e}{N} \sum_{e \sim v} \partial_\nu f_e,$$

from which the next result is obtained.

**Lemma 3.1.** *If  $f$  is linear on the edges incident on  $v$  and continuous at  $v$ , then  $f(v)$  is the mean value of the equidistant edge values  $f_e(x)$  if and only if  $\sum_{n=1}^N \partial_\nu f_e = 0$ .*

Recall that  $\bar{\mathcal{G}}$  is path connected and compact. The mean value property for a harmonic function  $f$  on a metric graph means that  $f$  has an interior maximum or minimum on  $\bar{\mathcal{G}}$  if and only if  $f$  is constant. Moreover,  $f$  has a maximum and minimum, which must occur on the  $\partial\bar{\mathcal{G}}$ .

**3.3. The Dirichlet problem.** Let  $\mathcal{D}_{min}$  denote the continuous functions  $f : \bar{\mathcal{G}} \rightarrow \mathbb{R}$ , with compact support in the interior of  $\mathcal{G}$ , which are infinitely differentiable on the (closed) edges of  $\mathcal{G}$ , and which satisfy (2.1) at each interior vertex.  $\mathcal{D}_{min}$  is dense in  $L^2(\mathcal{G})$ , but the situation is different in  $\mathbb{H}_1$ .

**Proposition 3.2.** *Functions in the  $\mathbb{H}_1$  closure of  $\mathcal{D}_{min}$  vanish on  $\partial\bar{\mathcal{G}}$ . The orthogonal complement of  $\mathcal{D}_{min}$  in  $\mathbb{H}_1$  is the set  $H_{fin}$  of finite energy harmonic functions.*

*Proof.* By Proposition 3.1, convergence in  $\mathbb{H}_1$  implies uniform convergence, so functions in the closure of  $\mathcal{D}_{min}$  vanish on  $\partial\bar{\mathcal{G}}$ .

First suppose that  $f \in \mathcal{D}_{min}$  and  $g \in H_{fin}$ . Since  $f$  vanishes outside a finite graph, the integration by parts formula lemma 2.1 yields  $\langle g, f \rangle_1 = - \int_{\mathcal{G}} f g'' = 0$ . The finite energy harmonic functions are thus orthogonal to  $\mathcal{D}_{min}$  in  $\mathbb{H}_1$ .

Suppose  $g \in \mathbb{H}_1$  and for all  $f \in \mathcal{D}_{min}$

$$\langle g, f \rangle_1 = \int_{\mathcal{G}} g' f' + \int_{\partial\bar{\mathcal{G}}} g f \, d\mu = 0.$$

The boundary integral is zero, so will not play a role.

Each edge  $e \in \mathcal{E}_{\mathcal{G}}$  is identified with an interval  $[a, b]$ . Consider the functions  $f \in \mathcal{D}_{min}$  with support in  $(a, b)$ . For such  $f$ , integration by parts gives

$$0 = \int_a^b g' f' = - \int_a^b g f''.$$

As a function in  $L^2[a, b]$  the restriction of  $g$  to  $[a, b]$  is orthogonal to all such  $f''$ , which implies [7, p. 1291] that  $g'' = 0$  on each edge.

Recall that functions in  $\mathbb{H}_1$  are continuous on  $\bar{\mathcal{G}}$  by definition. Returning to more general  $f \in \mathcal{D}_{min}$ , suppose that  $f$  is a nonzero constant in a small neighborhood of an interior vertex  $v$ , and the support of  $f$  lies in the union of the edges incident on  $v$ . Since  $\langle g, f \rangle_1 = 0$ , summing over the edges incident on  $v$  gives

$$0 = \sum_{e \sim v} \int_e f' g' = -f(v) \sum_e \partial_\nu g_e(v) - \sum_{e \sim v} \int_e f g''.$$

But  $g'' = 0$  on each edge, so  $g$  satisfies the vertex conditions (2.1). That is,  $g \in H_{fin}$ .  $\square$

**Corollary 1.** *Suppose  $f \in \mathbb{H}_1$ . Among all  $g \in \mathbb{H}_1$  which agree with  $f$  on  $\partial\bar{\mathcal{G}}$ , a unique harmonic function minimizes  $\|g\|_1$ .*

*If  $f \in \mathbb{H}_1$  vanishes on  $\partial\bar{\mathcal{G}}$ , then  $\langle f, h \rangle_1 = 0$  for all harmonic functions  $h \in \mathbb{H}_1$ .*

*Proof.* Write  $g = g_1 + g_2$  with  $g_1$  in the  $\mathbb{H}_1$  closure of  $\mathcal{D}_{min}$  and  $g_2 \in H_{fin}$ . As noted in the proof of lemma 3.1, a function in the  $\mathbb{H}_1$  closure of  $\mathcal{D}_{min}$  must vanish on  $\partial\bar{\mathcal{G}}$ . Thus  $g_2$  agrees with  $f$  on  $\partial\bar{\mathcal{G}}$ . Since

$$\|g\|_1^2 = \|g_1\|_1^2 + \|g_2\|_1^2,$$

$g_2$  the desired minimizer.

Similarly, if  $f \in \mathbb{H}_1$  vanishes on  $\partial\bar{\mathcal{G}}$  and  $f = f_1 \oplus f_2$ , with  $f_1$  in the  $\mathbb{H}_1$  closure of  $\mathcal{D}_{min}$  and  $f_2 \in H_{fin}$ , the harmonic part  $f_2$  vanishes on  $\partial\bar{\mathcal{G}}$ , forcing  $f_2 = 0$ .  $\square$

The Dirichlet problem for  $\bar{\mathcal{G}}$  can now be solved. This approach emphasizes  $H_{fin}$ , an aspect not discussed in the proof in [3]. Use  $1_E$  to denote the characteristic function of a set  $E$ ; in this case  $E \subset \partial\bar{\mathcal{G}}$ . The existence of a rich collection of partitions  $\mathcal{E}$  is a consequence of Proposition 2.2.

**Theorem 3.1.** *Suppose  $\mathcal{E} = \{E(n), n = 1, \dots, N\}$  is a finite partition of  $\partial\bar{\mathcal{G}}$  by clopen sets. For any function  $F = \sum_{n=1}^N c_n 1_{E(n)}$ , which is a linear combination of the characteristic functions of the sets  $E(n)$ , there is a unique  $f \in H_{fin}$  with  $f = F$  on  $\partial\bar{\mathcal{G}}$ .*

*Proof.* By lemma 2.3 there is a function  $g \in \mathcal{A}$  which agrees with  $F$  on  $\partial\bar{\mathcal{G}}$ . Since  $g \in \mathbb{H}_1$ , Corollary 1 implies there is an  $f \in H_{fin}$  which agrees with  $g$  on  $\partial\bar{\mathcal{G}}$ . The uniqueness of  $f$  follows immediately from the maximum principle.  $\square$

**Corollary 2.** *Suppose  $G : \partial\bar{\mathcal{G}} \rightarrow \mathbb{R}$  is continuous. Then there is a unique harmonic function  $g : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  with  $g = G$  on  $\partial\bar{\mathcal{G}}$ .*

*Proof.* Since  $\partial\bar{\mathcal{G}}$  is compact, continuous functions  $f : \partial\bar{\mathcal{G}} \rightarrow \mathbb{R}$  are uniformly continuous. Consequently, the functions  $F = \sum_{n=1}^N c_n 1_{E(n)}$  from the proof of Theorem 3.1 are uniformly dense in the continuous functions on  $\bar{\mathcal{G}}$ .

Using Theorem 3.1, pick a sequence  $f_n : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  of harmonic functions converging uniformly to  $G$  on  $\partial\bar{\mathcal{G}}$ . By the maximum principle  $f_n : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  is a uniformly Cauchy sequence, which converges uniformly to the desired harmonic function  $g$ .  $\square$

Note that not every harmonic function  $f : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  is in  $\mathbb{H}_1$ .

**Proposition 3.3.** *Suppose  $\partial\bar{\mathcal{G}}$  is not a set of isolated points. Then there are continuous functions  $F : \partial\bar{\mathcal{G}} \rightarrow \mathbb{R}$  whose harmonic extensions are not in  $\mathbb{H}_1$ .*

*Proof.* A function  $f \in H_{fin}$  will satisfy a Hölder condition. Suppose  $x, y \in \bar{\mathcal{G}}$ , and  $\gamma$  is a path with length at most  $2d(x, y)$  from  $x$  to  $y$ . Then

$$|f(y) - f(x)|^2 = \left| \int_{\gamma} f'(t) dt \right|^2 \leq \int_{\gamma} (f')^2 \int_{\gamma} 1 \leq 2d(x, y) \|f\|_1^2. \quad (3.5)$$

Suppose  $x_0$  is a limit point of  $\partial\bar{\mathcal{G}}$ . Consider the function  $F(x) = \sqrt{d(x, x_0)}$ . Since  $x^{1/2}$  is continuous on  $[0, \infty)$  but  $(x^{1/2})' = x^{-1/2}/2$ , the function  $F$  cannot satisfy the Hölder condition (3.5).  $\square$

**3.4. Level sets.** Assume  $\mathcal{G}_0$  is a connected finite graph, with  $E$  and  $E^c = \partial\mathcal{G}_0 \setminus E$  nonempty subsets of  $\partial\mathcal{G}_0$ . Suppose  $f : \mathcal{G}_0 \rightarrow \mathbb{R}$  is harmonic, with  $f(v) = 1$  for  $v \in E$  and  $f(v) > 1$  for  $v \in E^c$ . By the maximum principle,  $\partial_\nu f(v) > 0$  for  $v \in E$ . It will be helpful to have a similar result for infinite graphs  $\mathcal{G}$ .

A point  $x \in \mathcal{G}$  is a critical point for  $f$  if  $x$  is a vertex or  $f'(x) = 0$ . A number  $y \in \mathbb{R}$  is a critical value for  $f$  if  $f^{-1}(y)$  contains a critical point. Points in the range of  $f$  that are not critical values are regular values. If  $f$  is harmonic and  $f'(x) = 0$  for some  $x$  in an edge  $e$ , then  $f$  is constant on  $e$ . Since  $\mathcal{V}_{\mathcal{G}}$  and  $\mathcal{E}_{\mathcal{G}}$  are countable, a harmonic function  $f$  has a countable set of critical values.

**Lemma 3.2.** *Suppose  $f : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  is harmonic, and  $c$  is a regular value of  $f$ . Assume there is no  $x \in \partial\bar{\mathcal{G}}$  with  $f(x) = c$ . Then  $f^{-1}(c)$  is a finite set.*

*Proof.* If  $f^{-1}(c)$  were an infinite set, then by compactness there would be an infinite sequence of distinct points  $\{x_n\} \subset f^{-1}(c)$  converging to a point  $z$ , with  $f(z) = c$ . Since  $z \notin \partial\bar{\mathcal{G}}$  and  $c$  is a regular value,  $z$  is an interior point of some edge  $e$ . Since  $f$  is harmonic and  $x_n \rightarrow z$ ,  $f$  must be constant on  $e$ , contradicting the assumption that  $c$  is a regular value for  $f$ .  $\square$



**Lemma 3.3.** *Suppose that  $E$  and  $E^c = \partial\bar{\mathcal{G}} \setminus E$  are nonempty clopen subsets of  $\partial\bar{\mathcal{G}}$ . Assume that  $f : \bar{\mathcal{G}} \rightarrow \mathbb{R}$  is harmonic, with  $f(x) = C \geq 0$  for  $x \in E$ , and  $f(x) > C$  for  $x \in E^c$ . Given  $\epsilon > 0$  there is a  $t > C$  such that  $d(x, E) < \epsilon$  if  $f(x) \leq t$ .*

*Proof.* Since  $E^c$  is compact,  $y = \min_{x \in E^c} f(x) > C$ . Arguing by contradiction, assume there is a sequence  $t_n \rightarrow C$  and points  $x_n \in f^{-1}(t_n)$  with  $d(x_n, E) \geq \epsilon$ . By compactness of  $\bar{\mathcal{G}}$  the sequence  $\{x_n\}$  has a subsequential limit  $z$ , with  $f(z) = C$ . Since  $z \notin E^c$  and  $d(z, E) \geq \epsilon$ ,  $z$  must be in  $\mathcal{G}$ , contradicting the maximum principle.  $\square$

Using the hypotheses of lemma 3.3, for  $\epsilon > 0$  select a regular value  $t$ , with  $C < t < \min_{x \in E^c} f(x)$  and  $d(x, E) < \epsilon$  if  $f(x) \leq t$ . By lemma 3.2 the set  $f^{-1}(t)$  is a finite set of points interior to edges of  $\mathcal{G}$ . Add the vertices  $f^{-1}(t)$  to  $\mathcal{G}$ , subdivide the corresponding edges, and call the resulting graph  $\tilde{\mathcal{G}}$ . The graph  $\tilde{\mathcal{G}}$  is now replaced by the set  $\mathcal{G}_t = \tilde{\mathcal{G}} \cap f^{-1}[t, \infty)$ .

**Lemma 3.4.**  *$\mathcal{G}_t$  is a subgraph of  $\tilde{\mathcal{G}}$  with  $f^{-1}(t) \subset \partial\mathcal{G}_t$ . Moreover,  $\partial_\nu f(x) > 0$  when  $f(x) = t$ .*

*Proof.* Suppose  $e$  is an edge of  $\tilde{\mathcal{G}}$ , with vertices  $v_1, v_2$ . If  $x \in e$  with  $f(x) = t$ , then  $f'(x) \neq 0$  and  $\mathcal{G}_t$  contains only one of  $[v_1, x]$  or  $[x, v_2]$ . The point  $x$  is then a degree one vertex in  $\mathcal{G}_t$ , so is a boundary vertex. If  $e$  contains no point  $x$  with  $f(x) = t$ , then  $e$  is either entirely in  $\mathcal{G}_t$ , or in the complement. Thus  $\mathcal{G}_t$  is the union of closed edges of  $\tilde{\mathcal{G}}$ .

Finally,  $\partial_\nu f(x) > 0$  since  $t$  is a minimum value for  $f$  on  $\mathcal{G}_t$ .  $\square$

The volume of  $\mathcal{G}$  is its Lebesgue measure, i.e. the sum of the edge lengths. The volume of the  $\epsilon$ -neighborhood of the clopen set  $E \subset \partial\bar{\mathcal{G}}$  is the Lebesgue measure of  $N_\epsilon(E)$ .

**Proposition 3.4.** *Suppose  $\epsilon > 0$ ,  $N_\epsilon(E)$  has finite volume, and  $t$  is chosen so that  $\{f \leq t\} \subset N_\epsilon(E)$ . Then every  $x \in \mathcal{G}$  with  $f(x) = t$  can be connected to  $\partial\bar{\mathcal{G}}$  by a path  $\gamma$  in the set  $\{f \leq t\}$ , and  $\gamma$  can be chosen to have a single limit point in  $\partial\bar{\mathcal{G}}$ .*

*Proof.* If  $f(x) = t$ , then  $x$  is interior to an edge  $e(0)$ , and  $f'_{e(0)}(x) \neq 0$ . Walk in the direction of decreasing  $f$  until you hit a vertex  $v$ . If  $v$  is not a boundary vertex, then since  $\partial_\nu f_{e(0)}(v) > 0$  and (2.1) holds, there is another edge  $e(1)$  incident on  $v$  with  $\partial_\nu f_{e(1)} < 0$ . Walk along  $e(1)$ , and then continue in this fashion. Either a boundary vertex is encountered after finitely many steps, or there is a path  $\gamma$  with infinitely many distinct edges  $e(n)$ . By lemma 2.2 the distance from  $\gamma(s)$  to  $\partial\bar{\mathcal{G}}$  has limit zero as  $n \rightarrow \infty$ .

Let  $z$  be a limit point of  $\gamma$ . By Proposition 2.2 there is a sequence  $\{\mathcal{E}_j, j = 1, 2, 3, \dots\}$  of finite partitions of  $\partial\bar{\mathcal{G}}$  by clopen sets, with  $\mathcal{E}_{j+1}$  a refinement of  $\mathcal{E}_j$ , such that the diameter of each set in  $\mathcal{E}_j$  is less than  $1/j$ . Let  $z \in E_k$  with  $E_k \in \mathcal{E}_k$ .

Since  $\mathcal{E}_k$  is a finite partition by clopen sets there is a  $\delta > 0$  such that  $d(x, y) > \delta$  if  $x, y$  lie in distinct sets of  $\mathcal{E}_k$ . Since the sum of the lengths of the edges in  $\gamma$  is finite, any limit of  $\gamma$  must lie in  $E_k$ , for  $k = 1, 2, 3, \dots$ . Since the diameters of the sets  $E_k$  have limit zero, any limit point of  $\gamma$  must be  $z$ .  $\square$

Suppose  $t_1$  and  $t_2$  are as above, with  $t_2 < t_1$ . Consider the set  $\mathcal{G}_2 = [f^{-1}(t_2), f^{-1}(t_1)]$ .

**Lemma 3.5.**  *$\mathcal{G}_2$  is a finite graph containing no boundary vertices of  $\mathcal{G}$ . If  $f^{-1}(t_1)$  and  $f^{-1}(t_2)$  are considered as boundary vertices of  $\mathcal{G}_{t_1}$  and  $\mathcal{G}_{t_2}$  respectively, then*

$$\sum_{v \in f^{-1}(t_1)} \partial_\nu f = \sum_{v \in f^{-1}(t_2)} \partial_\nu f.$$

*Proof.*  $\mathcal{G}_2$  consists of a collection of closed edges as in the proof of Theorem 3.4. The values  $t_1$  and  $t_2$  are chosen to be positive, but smaller than  $f(x)$  for  $x \in E^c$ . Thus  $\mathcal{G}_2$  contains no boundary vertices of  $\mathcal{G}$ .

If  $\mathcal{G}_2$  had infinitely many distinct edges, there would be a sequence  $x_n$  from distinct edges with  $f(t_2) \leq f(x_n) \leq f(t_1)$ , but with  $z = \lim_{n \rightarrow \infty} x_n \in E$  by lemma 2.2. This would force  $f(z) = C$ , which is impossible.

Since  $\mathcal{G}_2$  contains no boundary vertices of  $\mathcal{G}$ , lemma 2.1 gives

$$0 = \int_{\mathcal{G}_2} -f'' \cdot 1 = \sum_{v \in f^{-1}(t_2)} \partial_\nu f(v) + \sum_{v \in f^{-1}(t_1)} \partial_\nu f(v).$$

But the outward pointing derivatives  $\partial_\nu f(v)$  for  $v \in f^{-1}(t_1)$  flip sign when considered as boundary vertices for  $\mathcal{G}_{t_1}$ , finishing the proof.  $\square$

#### 4. DIFFERENTIAL OPERATORS ON $L^2(\mathcal{G})$

Recall that  $\mathcal{D}_{max}$  denotes the continuous real valued functions  $f$  on  $\mathcal{G}$  which have absolutely continuous derivatives on each edge  $e$ , with  $f$  and  $f''$  in  $L^2(\mathcal{G})$ , and which satisfy (2.1) at each interior vertex.  $\mathcal{D}_{min}$  denotes the functions  $f \in \mathcal{D}_{max}$  which are infinitely differentiable on the edges of  $\mathcal{G}$ , with compact support  $supp(f)$  in the interior of  $\mathcal{G}$ .  $S_{min}$  will be the operator  $-D^2$  acting on  $L^2(\mathcal{G})$  with domain  $\mathcal{D}_{min}$ . Any symmetric extension of  $S_{min}$  will have an adjoint which is a restriction of  $S_{min}^*$ .

**Proposition 4.1.**  *$S_{min}$  is a nonnegative symmetric operator on  $L^2(\mathcal{G})$ . The adjoint  $S_{min}^*$  is the operator  $-D^2$  with domain  $\mathcal{D}_{max}$ .*

*Proof.* If  $f, g \in \mathcal{D}_{min}$  there is a finite subgraph  $\mathcal{G}_0$  containing  $supp(f) \cup supp(g)$  such that

$$\int_{\mathcal{G}} (-D^2 f)g = \int_{\mathcal{G}_0} (-D^2 f)g = \int_{\mathcal{G}_0} f'g' = \int_{\mathcal{G}} f(-D^2 g).$$

Thus  $S_{min}$  is a nonnegative and symmetric. For each edge  $e \in \mathcal{E}_{\mathcal{G}}$ , the domain  $\mathcal{D}_{min}$  includes the  $C^\infty$  functions with compact support in the interior of  $e$ . By [7, p. 1294] or [14, p. 169-171]  $S_{min}^*$  acts by  $-D^2$ , and functions in the domain of  $S_{min}^*$  have absolutely continuous derivatives on each edge  $e$ , with  $S_{min}^* f = -D^2 f \in L^2(\mathcal{G})$ .

Suppose now that  $v$  is an interior vertex with incident edges  $e_n = [v, w_n]$  for  $n = 1, \dots, e_N$ . Assume that  $f \in \mathcal{D}_{min}$  has support in  $U = \bigcup_{n=1}^N e_n$ . Choose such an  $f$  with  $f = 1$  in a neighborhood of  $v$  and with  $f(w_n)$  and  $f'(w_n)$  vanishing in a neighborhood of  $w_n$ . For  $g$  in the domain of  $S_0^*$ ,

$$0 = \int_U (-f'')g - \int_U f(-g'') = \sum_{e \sim v} [f_e(v) \partial_\nu g_e(v) - g_e(v) \partial_\nu f_e(v)] = \sum_{e \sim v} \partial_\nu g_e(v).$$

Next, for distinct  $m, n$ , choose  $f$  with  $f(v) = 0$ , but with  $\partial_\nu f_{e_m} = -1$ ,  $\partial_\nu f_{e_n} = 1$ , and  $\partial_\nu f_{e_j} = 0$  for  $j \neq m, n$ . Then  $g$  in the domain of  $S_0^*$  is continuous at  $v$ , since

$$0 = \sum_{e \sim v} [f_e(v) \partial_\nu g_e(v) - g_e(v) \partial_\nu f_e(v)] = g_{e_m}(v) - g_{e_n}(v). \quad \square$$

The domain  $\mathcal{D}_{min}$  will now be extended, with the aim of finding novel self-adjoint operators  $-D^2$ . Start with a finite partition  $\mathcal{E}(0) = \{E_{n,0}, n = 0, \dots, N\}$  of  $\partial \bar{\mathcal{G}}$  by nonempty clopen sets. Assume that for some  $\delta > 0$  the neighborhoods  $N_\delta(E_{n,0})$  are pairwise disjoint, and have finite volume for  $n = 1, \dots, N$ ; neighborhoods of  $E_{0,0}$  may have infinite volume. Also choose a collection  $\mathcal{H} = \{h_n, n = 1, \dots, N\}$  of functions, with  $h_n$  defined and harmonic in  $N_\delta(E_{n,0})$ ; initially it will suffice to assume that  $h_n \in L^2(N_\delta(E_{n,0}))$ .

Define a domain  $\mathcal{D}_{\mathcal{E}(0), \mathcal{H}}$  of functions which are constant multiples of  $h_n$  in some neighborhood  $N_\epsilon(E_{n,0})$ . More precisely  $\phi \in \mathcal{D}_{\mathcal{E}(0), \mathcal{H}}$  if  $\phi \in \mathcal{D}_{max}$  and for some  $\epsilon$  with  $0 < \epsilon < \delta$ ,

- (i)  $\phi(x) = 0$  for  $x \in N_\epsilon(E_{0,0})$ , and
- (ii) there are constants  $c_n$  (depending on  $\phi$ ) such that  $\phi(x) = c_n h_n(x)$  for  $x \in N_\epsilon(E_{n,0})$ .

Note that  $\mathcal{D}_{\mathcal{E}(0),\mathcal{H}}$  is a dense (algebraic) subspace of  $L^2(\mathcal{G})$ . The finite volume assumption for  $N_\delta(E_n)$ ,  $n = 1, \dots, N$ , insures that  $\phi \in L^2(\mathcal{G})$  if the  $h_n \in \mathcal{H}$  are bounded. Let  $S_{\mathcal{E}(0),\mathcal{H}}$  denote the operator on  $L^2(\mathcal{G})$  acting by  $-D^2$  with domain  $\mathcal{D}_{\mathcal{E}(0),\mathcal{H}}$ . The collection  $\mathcal{H}$  serves as boundary conditions on  $\mathcal{E}(0)$ . (If  $\text{vol}(\mathcal{G})$  is finite,  $E_0$  and condition (i) may be dropped.)

The partition and domain for  $-D^2$  will be extended inductively. Suppose  $\mathcal{E}(j) = \{E_{m,j}\}$  is an already defined finite partition of  $\partial\bar{\mathcal{G}}$  by nonempty clopen sets. Form the partition  $\mathcal{E}(j+1)$  by partitioning the sets  $E_{m,j}$  into two nonempty disjoint clopen sets  $E_{m,j,1}$  and  $E_{m,j,2}$  and putting these into  $\mathcal{E}(j+1)$ . If  $E_{m,j}$  may not be partitioned in this way, add it unchanged to  $\mathcal{E}(j+1)$ .

The associated collection  $\mathcal{H}(j+1)$  of harmonic functions is obtained by assigning the function  $h_{m,j}$  previously assigned to  $E_{m,j}$  to both sets  $E_{m,j,1}$  and  $E_{m,j,2}$ . As above,  $\phi \in \mathcal{D}_{\mathcal{E}(j+1),\mathcal{H}(j+1)}$  if there are constants  $c_{m,j,1}$  and  $c_{m,j,2}$  such that  $\phi(x) = c_{m,j,1}h_{m,j}(x)$  for  $x$  in some  $\epsilon$  neighborhood  $N_\epsilon(E_{n,1})$  and  $\phi(x) = c_{m,j,2}h_{m,j}(x)$  for  $x \in N_\epsilon(E_{n,2})$ . Note that the collection  $\mathcal{H}(j+1)$  of harmonic functions is only assigning a function  $h_{m,j}$  to the subsets of  $E_{m,j}$ , although as sets are split the constants need not be the same. However, if  $c_{m,j,1} = c_{m,j,2}$  for all  $m$ , a subspace of  $\mathcal{D}_{\mathcal{E}(j+1),\mathcal{H}(j+1)}$  is naturally identified with  $\mathcal{D}_{\mathcal{E}(j),\mathcal{H}(j)}$ . The operators  $S_{\mathcal{E}(j),\mathcal{H}(j)}$  act by  $-D^2$  with the increasing domains  $\mathcal{D}_{\mathcal{E}(j),\mathcal{H}(j)}$ . The operator  $S_U$  will act by  $-D^2$  with the domain  $\bigcup_j \mathcal{D}_{\mathcal{E}(j),\mathcal{H}(j)}$ .

**Theorem 4.1.** *The operators  $S_{\mathcal{E}(j),\mathcal{H}(j)}$  and  $S_U$  are symmetric. They have self-adjoint extensions.*

*Proof.* Suppose  $f, g \in \mathcal{D}_{\mathcal{E}(j),\mathcal{H}(j)}$ . Since both are harmonic in a neighborhood of  $\partial\bar{\mathcal{G}}$ , there is a finite graph  $\mathcal{G}_\epsilon$  as in lemma 2.2 such that lemma 2.1 gives

$$\begin{aligned} \int_{\mathcal{G}} (D^2 f)g - \int_{\mathcal{G}} f(D^2 g) &= \int_{\mathcal{G}_\epsilon} (D^2 f)g - \int_{\mathcal{G}_\epsilon} f(D^2 g) \\ &= \sum_{v \in \partial\mathcal{G}_\epsilon} \sum_{e \sim v} [f(v)\partial_\nu g_e(v) - g(v)\partial_\nu f_e(v)]. \end{aligned}$$

$\mathcal{G}_\epsilon$  can be chosen so that near the boundary vertices  $v \in N_\epsilon(E_{m,j})$  there are constants  $c_f, c_g$  such that  $f = c_f h_n$  and  $g = c_g h_n$ . Symmetry follows from

$$f(v)\partial_\nu g_e(v) - g(v)\partial_\nu f_e(v) = c_f c_g h_n(v)\partial_\nu h_{n,e}(v) - c_g c_f h_n(v)\partial_\nu h_{n,e}(v) = 0.$$

The operators  $S_{\mathcal{E}(j),\mathcal{H}(j)}$  and  $S_U$  are densely defined and symmetric on the real Hilbert space  $L^2(\mathcal{G})$ . All such operators have [18, p. 349] self-adjoint extensions on  $L^2_{\mathbb{R}}(\mathcal{G})$  or [7, p. 1231] on the complexification  $L^2_{\mathbb{C}}(\mathcal{G})$ .  $\square$

It is often useful to know that a densely defined symmetric operator  $S$  is nonnegative, since  $S$  will then have a distinguished self-adjoint extension, the Friedrichs extension. To establish nonnegativity of the symmetric operators  $S_{\mathcal{E}(j),\mathcal{H}(j)}$ , restrictions are placed on the harmonic functions. Choose a collection  $\mathcal{K} = \{k_n, n = 1, \dots, N\}$  of harmonic functions as before, with each  $k_n$  satisfying (i) each  $k_n$  has a constant value  $C_n > 0$  on  $E_n$ , (ii)  $k_n \in H_{fin}$ , and (iii)  $\partial_\nu k_n(x) > 0$  for  $x$  in the level sets  $\{k_n(x) = t_n\}$  for  $t_n$  sufficiently close to  $C_n$ . The existence of such functions  $k_n$  is established in lemma 3.4 and Theorem 3.1.

**Theorem 4.2.**  *$S_{\mathcal{E}(j),\mathcal{K}(j)}$  and the corresponding  $S_U$  are nonnegative. There are functions  $f$  in the domain of  $S_{\mathcal{E}(j),\mathcal{K}(j)}$  with quadratic form  $\langle S_{\mathcal{E}(j),\mathcal{K}(j)} f, f \rangle_2$  strictly larger than  $\int_{\mathcal{G}} (f')^2$ .*

*Proof.* Replace the graph  $\mathcal{G}_\epsilon$  of Theorem 4.1 with a graph  $\mathcal{G}_T$  bounded by the level sets  $\{k_n(x) = t_n\}$  and the set  $\{x, d(x, E_0) = t_0\}$ . By lemma 2.1, for  $t_m$  sufficiently small,

$m = 0, \dots, N$  the quadratic form for  $S_{\mathcal{E}(j), \mathcal{K}(j)}$  is

$$\begin{aligned} \int_{\mathcal{G}} (-D^2 f) f &= \sum_{n=1}^N \sum_{v \in k_n^{-1}(t_n)} f \partial_\nu f + \int_{\mathcal{G}_T} (f')^2 \\ &= \sum_{n=1}^N \sum_{v \in k_n^{-1}(t_n)} c_{n,f}^2 k_n(v) \partial_\nu k_n(v) + \int_{\mathcal{G}_T} (f')^2 \geq 0. \end{aligned} \quad (4.1)$$

By lemma 3.5, the sums  $\sum_{v \in k_n^{-1}(t_n)} \partial_\nu k_n(v)$  will have a fixed positive value as  $t_n \downarrow C_n$ . Consequently, the quadratic form for  $S_{\mathcal{E}(j), \mathcal{K}(j)}$  will be strictly larger than  $\int_{\mathcal{G}} (f')^2$  for functions in  $\mathcal{D}_{\mathcal{E}(j), \mathcal{K}(j)}$  with  $c_{n,f}^2 > 0$ .  $\square$

When  $S_{\mathcal{E}(j), \mathcal{K}(j)}$  is nonnegative it has [14, pp. 313-326] a nonnegative self-adjoint Friedrichs extension  $\mathcal{L}_{\mathcal{E}(j), \mathcal{K}(j)}$ . The quadratic form for  $S_{\mathcal{E}(j), \mathcal{K}(j)} + I$  provides an inner product,  $\langle f, g \rangle_F = \langle S_{\mathcal{E}(j), \mathcal{K}(j)} f, g \rangle_2 + \langle f, g \rangle_2$ . Then [14, p. 322] the completion of  $\mathcal{D}_{\mathcal{E}(j), \mathcal{K}(j)}$  with respect to the inner product  $\langle f, g \rangle_F$  yields a closed form  $\tau[f, g]$  extending  $\langle S_{\mathcal{E}(j), \mathcal{K}(j)} f, g \rangle_2$ , with the associated Friedrichs extension  $\mathcal{L}_{\mathcal{E}(j), \mathcal{K}(j)}$  having a domain contained in the form domain of  $\tau$ .

**Corollary 3.** *If  $f$  is in the completion of  $\mathcal{D}_{\mathcal{E}(j), \mathcal{K}(j)}$  with respect to the inner product  $\langle f, g \rangle_F$ , then  $f$  is continuous on  $\bar{\mathcal{G}}$  and absolutely continuous on the edges of  $\mathcal{G}$ , with  $f' \in L^2(\mathcal{G})$ .*

*Suppose for  $n = 1, \dots, N$  the functions  $h_n \in \mathcal{H}$  are constants. Then  $\tau[f, f] = \int_{\mathcal{G}} (f')^2$ .*

*Proof.* The function  $f$  is the limit of a Cauchy sequence  $f_n \in \mathcal{D}_{\mathcal{E}(j), \mathcal{K}(j)}$  with respect to the norm of the inner product  $\langle f, g \rangle_F$ . By (4.1),

$$\|f\|_F^2 \geq \int_{\mathcal{G}} f^2 + (f')^2.$$

Following the proof of Proposition 3.1 with  $\int_{\partial \bar{\mathcal{G}}} f^2 d\mu$  replaced by  $\int_{\mathcal{G}} f^2$ , one concludes that  $f$  is continuous on  $\bar{\mathcal{G}}$  and absolutely continuous on the edges of  $\mathcal{G}$ , with  $f' \in L^2(\mathcal{G})$ . In case  $h_n$  is replaced by a constant, (4.1) simplifies to

$$\langle f, f \rangle_F = \int_{\mathcal{G}} f^2 + (f')^2. \quad \square$$

The quadratic forms thus provide a way to distinguish the Friedrichs extensions for various harmonic functions  $h$ .

**Corollary 4.** *The Friedrichs extensions of  $\mathcal{L}_{\mathcal{E}(j), \mathcal{K}(j)}$  are distinct from the Friedrichs extensions of  $S_{\mathcal{E}, \mathcal{H}}$  obtained when the functions  $h_n \in \mathcal{H}$  are constants.*

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Robert Carlson: [rcarlson@uccs.edu](mailto:rcarlson@uccs.edu)

Department of Mathematics, University of Colorado at Colorado Springs, Colorado Springs, CO 80918 USA

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