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INDETERMINACY CONDITIONS FOR THE MATRIX NEVANLINNA-PICK PROBLEM AND RATIONAL MATRIX FUNCTIONS OF THE FIRST AND SECOND KIND

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Abstract. We obtain new indeterminacy conditions for the matrix Nevanlinna-Pick interpolation problem. These conditions are formulated in terms of the convergence of two matrix series. The elements of these series are rational matrix functions of the first and second kind.

Отримано нові умови невизначеності для матричної інтерполяційної задачі Неванлінни-Піка. Ці умови формулюється в термінах збіжності двох матричних рядів. Елементами цих рядів є раціональні матриць-функції першого і другого роду.

1. Introduction

By $\mathbb{C}^m$ denote the linear space of columns of complex numbers $x = \operatorname{col}\big(x_1\, x_2\, \dots\, x_m\big)$ of size m with the inner product $(x, y) = \sum_{j=1}^m \bar{x}_i y_j$. By $\mathbb{C}^{m\times n}$ denote the set of complex matrices with m rows and n columns. By $\mathbb{C}_H^{m\times m}$ denote the set of all Hermitian matrices. An Hermitian matrix A is called nonnegative if $(x, Ax) \ge 0$ $\forall x \in \mathbb{C}^m$. By $\mathbb{C}_\ge^{m\times m}$ denote the set of nonnegative matrices. A nonnegative matrix A is called positive if $(x, Ax) > 0$ for any nonzero vector $x \in \mathbb{C}^m$. By $\mathbb{C}_>^{m\times m}$ denote the set of positive matrices. By $I_m \in \mathbb{C}^{m\times m}$ denote the identity matrix and by $O_{m\times n} \in \mathbb{C}^{m\times n}$ denote the zero matrix. We will often omit the subscripts of the identity matrix and the zero matrix if these subscripts are clear from the context. For Hermitian matrices A, B we write $A > B$ ($A \ge B$) if $A - B \in \mathbb{C}_>^{m\times m}$ ($A - B \in \mathbb{C}_\ge^{m\times m}$). If a matrix A is invertible, then by A^{-*} denote the matrix $\big(A^{-1}\big)^*$. If $f(z)$ is a matrix-valued function, then by $f^*(z)$ denote the matrix-valued function $(f(z))^*$. Let $f(z)$ be an invertible matrix function. By $f^{-1}(z)$ and $f^{-*}(z)$ denote matrix-valued functions $(f(z))^{-1}$ and $\big(\big(f(z)\big)^{-1}\big)^*$ respectively. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

A holomorphic matrix-valued function $w : \mathbb{C}_+ \to \mathbb{C}^{m\times m}$ is said to be Nevanlinna if

$$\frac{w(z) - w^*(z)}{2\,i} \ge O, \qquad \forall z \in \mathbb{C}_+.$$

By $\mathcal{R}_m$ denote the class of Nevanlinna matrix-valued functions of fixed order $m \ge 1$.

In the Nevanlinna–Pick matrix interpolation problem, it is required to describe all Nevanlinna matrix-valued functions $w \in \mathcal{R}_m$ such that

$$w(z_j) = w_j, \qquad j \in \mathbb{N}, \tag{1.1}$$

given a sequence of interpolation nodes

$$z_1, z_2, \dots, z_n, \dots \subset \mathbb{C}_+, \qquad z_j \ne z_k, \quad j \ne k,$$

and a sequence of interpolated values

$$w_1, w_2, \dots, w_n, \dots \subset \mathbb{C}^{m\times m}.$$

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By $\mathcal{F}_\infty$ we denote the set of all solutions of problem (1.1).

The Nevanlinna–Pick matrix interpolation problem and its numerous analogs were studied in the works of many authors [3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 18]. The monograph [2] contains an analysis of the state-of-the art of the theory of interpolation problems for Nevanlinna functions and their analogs.

We will also consider truncated Nevanlinna–Pick matrix interpolation problems,

$$w(z_j) = w_j, \qquad 1 \le j \le n. \tag{1.2}$$

By $\mathcal{F}_n$ we denote the solution set of problem (1.2). Note that

$$\mathcal{F}_\infty = \cap_{n=1}^\infty \mathcal{F}_n, \qquad \mathcal{F}_{n+1} \subset \mathcal{F}_n.$$

With the truncated Nevanlinna–Pick matrix interpolation problem, we associate the following block matrices:

$$T_n = \begin{pmatrix} z_1^{-1}I & & \\ & \ddots & \\ & & z_n^{-1}I \end{pmatrix}, \quad K_n = T_n^{-1} \begin{pmatrix} \frac{w_1 - w_1^*}{z_1 - \bar{z}_1} & \dots & \frac{w_1 - w_n^*}{z_1 - \bar{z}_n} \\ \vdots & \ddots & \vdots \\ \frac{w_n - w_1^*}{z_n - \bar{z}_1} & \dots & \frac{w_n - w_n^*}{z_n - \bar{z}_n} \end{pmatrix} T_n^{-1*}, \tag{1.3}$$

$$u_n = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad v_n = \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix}, \quad R_{T_n}(z) = (I - zT_n)^{-1}. \tag{1.4}$$

It is easy to see that *the fundamental identity*

$$T_n K_n - K_n T_n^* = v_n u_n^* - u_n v_n^*. \tag{1.5}$$

holds. The fundamental identities play an important role in the theory of interpolation problems (see, for example [16, 18, 25]).

We first recall some facts about the matrix Nevanlinna-Pick interpolation problem (see [16, 17, 18]). Assume that the truncated problems (1.2) are *completely indeterminate*,

$$K_n > O_{nm \times nm}, \qquad n \in \mathbb{N}. \tag{1.6}$$

In this case, the sets $\mathcal{F}_n$ and $\mathcal{F}_\infty$ are nonempty (see [16, 18]).

Consider the sets

$$\mathcal{Z}_n = \{z_1, \dots, z_n\}, \qquad \mathcal{Z}_\infty = \bigcup_{n=1}^\infty \mathcal{Z}_n, \qquad \overline{\mathcal{Z}}_n = \{\bar{z}_1, \dots, \bar{z}_n\}, \qquad \overline{\mathcal{Z}}_\infty = \bigcup_{n=1}^\infty \overline{\mathcal{Z}}_n.$$

Let us fix a point $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$ and consider the set of matrices

$$\mathcal{K}_\infty(z_0) = \{w(z_0) : w \in F_\infty\}.$$

Then there are matrices $c_\infty(z_0) \in \mathbb{C}^{m\times m}$, $r_\infty(z_0) \in \mathbb{C}_\ge^{m\times m}$, and $\rho_\infty(z_0) \in \mathbb{C}_\ge^{m\times m}$ such that (see [16, 17, 18])

$$\mathcal{K}_\infty(z_0) = \{c_\infty(z_0) + r_\infty(z_0) V \rho_\infty(z_0) : V^*V \le I\}. \tag{1.7}$$

From the geometric point of view, the set $\mathcal{K}_\infty(z_0)$ can be regarded as the matrix disk centered at the point $c_\infty(z_0)$ with the left radius $r_\infty(z_0)$ and the right radius $\rho_\infty(z_0)$.

The set of matrices $\mathcal{K}_\infty(z_0)$ is called *the Weyl limit disk* at the point z_0. Let

$$m_+ = \operatorname{rank} r_\infty(z_0), \qquad m_- = \operatorname{rank} \rho_\infty(z_0).$$

The matrix Nevanlinna-Pick interpolation problem (1.1) is called *completely indeterminate* if $m_+ = m_- = m$. We will only consider the completely indeterminate case. The application of Orlov's theorem [20] yields

$$\operatorname{rank} r_\infty(z) = \operatorname{rank} \rho_\infty(z) = m \qquad \forall z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty.$$

Therefore, in order that the matrix Nevanlinna-Pick interpolation problem (1.1) be completely indeterminate, it is necessary that

$$r_\infty(z) > O, \qquad \rho_\infty(z) > O \tag{1.8}$$

for any points $z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$ and it is sufficient that the inequalities (1.8) were satisfied at least at one point $z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$.

For $n > 1$, consider the block partition of the matrices

$$K_n = \begin{pmatrix} K_{n-1} & B_n \\ B_n^* & C_n \end{pmatrix}.$$

By definition, put

$$\widehat{K}_1 = K_1, \qquad \widehat{K}_n = C_n - B_n^* K_{n-1}^{-1} B_n, \quad n > 1. \tag{1.9}$$

The following expressions are obvious ($n > 1$):

$$K_n = \begin{pmatrix} I & O \\ B_n^* K_{n-1}^{-1} & I \end{pmatrix} \begin{pmatrix} K_{n-1} & O \\ O & \widehat{K}_n \end{pmatrix} \begin{pmatrix} I & K_{n-1}^{-1} B_n \\ O & I \end{pmatrix}. \tag{1.10}$$

Whence,

$$\widehat{K}_n > O_{m\times m}, \qquad n \geq 1.$$

With a Nevanlinna–Pick matrix interpolation problem (1.1) we associate two sequences of rational matrix-valued functions of the first and second kind as

$$P_1(z) = \widehat{K}_1^{-1/2} R_{T_1}(z) v_1, \quad P_j(z) = \widehat{K}_j^{-1/2}(-B_j^* K_{j-1}^{-1},\ I) R_{T_j}(z) v_j, \quad j > 1, \tag{1.11}$$

$$Q_1(z) = -\widehat{K}_1^{-1/2} R_{T_1}(z) u_1, \quad Q_j(z) = -\widehat{K}_j^{-1/2}(-B_j^* K_{j-1}^{-1},\ I) R_{T_j}(z) u_j, \quad j > 1, \tag{1.12}$$

respectively.

A finite and infinite *Blaschke products* are defined as ($z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$)

$$\mathcal{B}_n(z) = \prod_{j=1}^{n} \frac{\bar{z}_j}{z_j} \cdot \frac{z - z_j}{z - \bar{z}_j}, \qquad \mathcal{B}(z) = \prod_{j=1}^{\infty} \frac{\bar{z}_j}{z_j} \cdot \frac{z - z_j}{z - \bar{z}_j}, \tag{1.13}$$

respectively.

We consider infinite matrix column vectors of the type

$$V = \operatorname{col}(V_1, V_2, V_3, \dots), \qquad V_j \in \mathbb{C}^{m\times m}.$$

We denote by $\ell^2(\mathbb{C}^{m\times m})$ the set of all infinite matrix columns V for which the matrix series $\sum_{j=1}^{\infty} V_j^* V_j$ converges.

Using the rational matrix-valued functions (1.11) and (1.12), we construct the infinite matrix columns

$$\pi(z) = \operatorname{col}(P_1(z), P_2(z), P_3(z), \dots), \tag{1.14}$$

$$\xi(z) = \operatorname{col}(Q_1(z), Q_2(z), Q_3(z), \dots). \tag{1.15}$$

The following theorem is the main result of this paper.

Theorem 1.1. *Suppose the Nevanlinna–Pick matrix interpolation problem* (1.1) *is given, the conditions* (1.6) *are satisfied, and the infinite matrix columns* $\pi(z), \xi(z)$ *are defined by* (1.14), (1.15) *respectively. Then the following conditions are equivalent:*

1) *the Nevanlinna–Pick interpolation problem* (1.1) *is completely indeterminate;*
2) *for some point* $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$, *the infinite matrix column* $\pi(z_0)$ *belongs to* $\ell^2(\mathbb{C}^{m\times m})$ *and the infinite Blaschke product* $\mathcal{B}(z_0)$ *converges;*
3) *for some point* $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$, *the infinite matrix column* $\xi(z_0)$ *belongs to* $\ell^2(\mathbb{C}^{m\times m})$ *and the infinite Blaschke product* $\mathcal{B}(z_0)$ *converges;*
4) *for some point* $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$, *both infinite matrix columns* $\pi(z_0)$ *and* $\pi(\bar{z}_0)$ *belong to* $\ell^2(\mathbb{C}^{m\times m})$.

For the classical Hamburger moment problem ($m = 1$) similar statements 1)–3) listed in Theorem 1.1 have been proved in [26, Theorem 3] (see also [1] and [19]). For matrix Hamburger moment problem ($m > 1$) similar results have been obtained in [11]. In this paper we prove the similar results to the case of the Nevanlinna–Pick matrix interpolation problem. We use methods of the theory of J-contractive analytic matrix-valued functions due to V. P. Potapov (see[16, 17, 18, 21, 22, 23, 24]).

2. Preliminaries

In this section, we summarize a number of basic well known definitions and facts about the Nevanlinna–Pick matrix interpolation problem.

If $w \in \mathcal{R}_m$, then (see [2])

$$w(z) = \mu z + \nu + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \sigma(dt), \tag{2.16}$$

where $\mu \in \mathbb{C}_{\geq}^{m\times m}$, $\nu \in \mathbb{C}_H^{m\times m}$, and σ is a matrix Hermitian non-negative measure on $\mathbb{R}$ such that the integral $\int_{-\infty}^{\infty}(1 + t^2)^{-1}\sigma(dt)$ is convergent.

A matrix-valued function $w \in \mathcal{R}_m$ is a solution of the truncated matrix Nevanlinna–Pick problem (1.2) if and only if it satisfies *the Fundamental Matrix Inequality (FMI) of V.P. Potapov*,

$$\begin{pmatrix} K_n & R_{T_n}(z)(v_n w(z) - u_n) \\ \left(R_{T_n}(z)(v_n w(z) - u_n)\right)^* & (w(z) - w^*(z))/(z - \bar{z}) \end{pmatrix} \geq O, \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n. \tag{2.17}$$

FMI follows from (1.2) and (2.16) (see [16, 18, 25]).

We multiply FMI (2.17) on right by the matrix

$$\begin{pmatrix} I & -K_n^{-1} R_n(z)(v_n w(z) - u_n) \\ O & I \end{pmatrix},$$

and on the left by the adjoint matrix. We obtain that FMI (2.17) is equivalent to the following matrix inequality:

$$\frac{w(z) - w^*(z)}{z - \bar{z}} - (v_n w(z) - u_n)^* R_n^*(z) K_n^{-1} R_n(z)(v_n w(z) - u_n) \geq O, \ z \in \mathbb{C}_+ \setminus \mathcal{Z}_n. \tag{2.18}$$

Consider *the resolvent matrix* for the truncated problem (1.2) (see [16, 18]),

$$\begin{aligned} U_n(z) = \begin{pmatrix} \alpha_n(z) & \beta_n(z) \\ \gamma_n(z) & \delta_n(z) \end{pmatrix} &= \begin{pmatrix} I + z v_n^* R_{T_n^*}(z) K_n^{-1} u_n & -z v_n^* R_{T_n^*}(z) K_n^{-1} v_n \\ z u_n^* R_{T_n^*}(z) K_n^{-1} u_n & I - z u_n^* R_{T_n^*}(z) K_n^{-1} v_n \end{pmatrix} \\ &= I_{2m} + z \begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) K_n^{-1} \begin{pmatrix} u_n & -v_n \end{pmatrix}. \end{aligned} \tag{2.19}$$

By definition, put

$$\mathcal{J} = \begin{pmatrix} O & -iI \\ iI & O \end{pmatrix} \in \mathbb{C}^{2m\times 2m}.$$

It is obvious that $\mathcal{J}^2 = I, \ \ \mathcal{J}^* = \mathcal{J}$. We have

$$\mathcal{V}\mathcal{J}\mathcal{V}^* = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

Here the matrix

$$\mathcal{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} iI & I \\ -iI & I \end{pmatrix}$$

is unitary. Therefore,

$$\det \mathcal{J} = (-1)^m. \tag{2.20}$$

Lemma 2.1. *If $U_n(z)$ is the resolvent matrix (2.19), then*

$$\mathcal{J} - U_n(z)\mathcal{J}U_n^*(\lambda) = i(z-\overline{\lambda})\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z)K_n^{-1}R_{T_n^*}^*(\lambda)(v_n\ u_n). \tag{2.21}$$

Proof. Using (2.19) and (1.5), we get

$$\begin{aligned}
&\mathcal{J} - U_n(z)\mathcal{J}U_n^*(\lambda) = \mathcal{J} \\
&- \left\{I_{2m} + z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z)K_n^{-1}\,(u_n\ -v_n)\right\}\mathcal{J}\left\{I_{2m} + \lambda\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(\lambda)K_n^{-1}(u_n\ -v_n)\right\}^* \\
&= -z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z)K_n^{-1}(u_n\ -v_n)\mathcal{J} - \bar{\lambda}\mathcal{J}\begin{pmatrix} u_n^* \\ -v_n^* \end{pmatrix} K_n^{-1}R_{T_n^*}^*(\lambda)(v_n\,u_n) \\
&- z\bar{\lambda}\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z)K_n^{-1}\underbrace{(u_n\ -v_n)\mathcal{J}\begin{pmatrix} u_n^* \\ -v_n^* \end{pmatrix}}_{i(K_nT_n^*-T_nK_n)} K_n^{-1}R_{T_n^*}^*(\lambda)(v_n\,u_n) = i\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) \\
&\times\Big(zK_n^{-1}(I-\bar{\lambda}T_n) - \bar{\lambda}(I-zT_n^*)K_n^{-1} - z\bar{\lambda}T_n^*K_n^{-1} + z\bar{\lambda}K_n^{-1}T_n\Big)R_{T_n^*}^*(\lambda)(v_n\,u_n) \\
&= i(z-\bar{\lambda})\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z)K_n^{-1}R_{T_n^*}^*(\lambda)(v_n\,u_n).
\end{aligned}$$

This shows that formula (2.21) is true. ☐

Substituting $\bar{z}$ for λ in (2.21), we get $\mathcal{J} - U_n(z)\mathcal{J}U_n^*(\bar{z}) = O$. This immediately implies the invertibility of $U_n(z)$ and the equality

$$U_n^{-1}(z) = \mathcal{J}U_n^*(\overline{z})\mathcal{J}. \tag{2.22}$$

Substituting $\bar{z}$ for z and λ into (2.21) and multiply (2.21) on the left and on the right by $\mathcal{J}$. Using (2.22) and the obvious equality $R_{T_n^*}(\bar{z}) = R_{T_n}^*(z)$, $R_{T_n^*}^*(\bar{z}) = R_{T_n}(z)$, we get

$$\begin{aligned}
\mathcal{J} - U_n^{-*}(z)\mathcal{J}U_n^{-1}(z) &= i(\overline{z}-z)\mathcal{J}\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)(v_n\ u_n)\mathcal{J} \\
&= -i(\overline{z}-z)\begin{pmatrix} -u_n^*R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)u_n & u_n^*R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)v_n \\ v_n^*R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)u_n & -v_n^*R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)v_n \end{pmatrix}.
\end{aligned} \tag{2.23}$$

Hence,

$$\frac{U_n^{-*}(z)\mathcal{J}U_n^{-1}(z)}{i(\overline{z}-z)} = \frac{\mathcal{J}}{i(\overline{z}-z)} - \mathcal{J}\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)(v_n\ u_n)\mathcal{J}.$$

Multiply this equality on the left by the matrix $(I\ w^*(z))$ and on the right by the adjoint matrix, we obtain

$$\begin{aligned}
&(I\ w^*(z))\frac{U_n^{-*}(z)\mathcal{J}U_n^{-1}(z)}{i(\overline{z}-z)}\begin{pmatrix} I \\ w(z) \end{pmatrix} \\
&\qquad = \frac{w(z)-w^*(z)}{z-\overline{z}} - (v_nw(z)-u_n)^*R_{T_n}^*(z)K_n^{-1}R_{T_n}(z)(v_nw(z)-u_n).
\end{aligned} \tag{2.24}$$

Combining (2.18) and (2.24), we obtain

$$(I\ w^*(z))\,\frac{U_n^{-*}(z)\mathcal{J}U_n^{-1}(z)}{i(\overline{z}-z)}\begin{pmatrix} I \\ w(z) \end{pmatrix} \geq O, \qquad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n. \tag{2.25}$$

By definition, the matrix-valued function

$$W_n(z) = U_n^{-*}(z)\mathcal{J}U_n^{-1}(z) \tag{2.26}$$

is called *the Weyl matrix* for the truncated problem (1.2) (see [16, 18]). It follows from (2.26) and (2.22) that

$$W_n^{-1}(z) = JW_n(\bar{z})J,$$

i.e., the Weyl matrix is invertible. Consider the $m \times m$ block partition of the Weyl matrix

$$W_n(z) = \begin{pmatrix} -a_{11}(z) & a_{12}(z) \\ a_{12}^*(z) & -a_{22}(z) \end{pmatrix}.$$

Using (2.23), we get

$$W_n(z) = \begin{pmatrix} -a_{11}(z) & a_{12}(z) \\ a_{12}^*(z) & -a_{22}(z) \end{pmatrix} =$$
$$\begin{pmatrix} -i(\bar{z}-z)u_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) u_n & i(\bar{z}-z)u_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) v_n - iI \\ i(\bar{z}-z)v_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) u_n + iI & -i(\bar{z}-z)v_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) v_n \end{pmatrix}. \quad (2.27)$$

It now follows (see (1.3)-(1.4)) that $a_{11}(z) > O,\ a_{22}(z) > O,\ a_{11}(\bar{z}) < O$ and $a_{22}(\bar{z}) < O$ for any $z \in \mathbb{C}_+ \setminus \mathcal{Z}_n$. Therefore,

$$W_n(z) = \begin{pmatrix} I & -a_{12}(z)a_{22}^{-1}(z) \\ O & I \end{pmatrix} \begin{pmatrix} -a_{11}(z) + a_{12}(z)a_{22}^{-1}(z)a_{12}^*(z) & O \\ O & -a_{22}(z) \end{pmatrix}$$
$$\times \begin{pmatrix} I & O \\ -a_{22}^{-1}(z)a_{12}^*(z) & I \end{pmatrix}, \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n. \quad (2.28)$$

We have

$$-a_{22}^{-1}(\bar{z}) = -a_{11}(z) + a_{12}(z)a_{22}^{-1}(z)a_{12}^*(z), \qquad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n. \quad (2.29)$$

Indeed,

$$W_n^{-1}(z) = JW_n^*(\bar{z})J = \begin{pmatrix} -a_{22}(\bar{z}) & -a_{12}^*(\bar{z}) \\ -a_{12}(\bar{z}) & -a_{11}(\bar{z}) \end{pmatrix}, \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

Hence,

$$\begin{pmatrix} -a_{22}(\bar{z}) & -a_{12}^*(\bar{z}) \\ -a_{12}(\bar{z}) & -a_{11}(\bar{z}) \end{pmatrix} \begin{pmatrix} -a_{11}(z) & a_{12}(z) \\ a_{12}^*(z) & -a_{22}(z) \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}, \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

Therefore,

$$a_{22}(\bar{z})a_{11}(z) - a_{12}^*(\bar{z})a_{12}^*(z) = I, \quad -a_{22}(\bar{z})a_{12}(z) + a_{12}^*(\bar{z})a_{22}(z) = O, \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

It now follows that

$$a_{11}(z) - a_{22}^{-1}(\bar{z})a_{12}^*(\bar{z})a_{12}^*(z) = a_{22}^{-1}(\bar{z}), \quad -a_{22}(\bar{z})a_{12}(z)a_{22}^{-1}(z) + a_{12}^*(\bar{z}) = O, \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

Combining these formulas, we obtain

$$a_{22}^{-1}(\bar{z}) = a_{11}(z) - a_{12}(z)a_{22}^{-1}(z)a_{12}^*(z), \quad z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

Formula (2.29) is proved.

Using (2.28) and (2.29), we get ($z \in \mathbb{C}_+ \setminus \mathcal{Z}_n$) that

$$W_n(z) = \begin{pmatrix} I & -a_{12}(z)a_{22}^{-1}(z) \\ O & I \end{pmatrix} \begin{pmatrix} -a_{22}(\bar{z}) & O \\ O & -a_{22}(z) \end{pmatrix} \begin{pmatrix} I & O \\ -a_{22}^{-1}(z)a_{12}^*(z) & I \end{pmatrix}.$$

By definition, put

$$c_n(z) = a_{22}^{-1}(z)a_{12}^*(z) = \left(i(\bar{z}-z)v_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) v_n\right)^{-1}$$
$$\times \left(i(\bar{z}-z)u_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) v_n - iI\right)^*, \quad (2.30)$$

$$r_n(z) = a_{22}(z) = \left(i(\bar{z}-z)v_n^* R_{T_n}^*(z) K_n^{-1} R_{T_n}(z) v_n\right)^{-1/2} > O, \quad (2.31)$$

$$\rho_n(z) = -a_{22}(\bar{z}) = \left(i(\bar{z}-z)v_n^* R_{T_n}^*(\bar{z}) K_n^{-1} R_{T_n}(\bar{z}) v_n\right)^{-1/2} > O. \quad (2.32)$$

It now follows that

$$W_n(z) = \begin{pmatrix} I & -c_n^*(z) \\ O & I \end{pmatrix} \begin{pmatrix} \rho_n^2(z) & O \\ O & -r_n^{-2}(z) \end{pmatrix} \begin{pmatrix} I & O \\ -c_n(z) & I \end{pmatrix}, \ z \in \mathbb{C}_+ \setminus \mathcal{Z}_n. \quad (2.33)$$

Combining (2.25) and (2.33), we obtain that $w \in \mathcal{F}_n$ if and only if it satisfies the inequality

$$(I\ w^*(z))\begin{pmatrix} I & -c_n^*(z) \\ O & I \end{pmatrix}\begin{pmatrix} \rho_n^2(z) & O \\ O & -r_n^{-2}(z) \end{pmatrix}\begin{pmatrix} I & O \\ -c_n(z) & I \end{pmatrix}\begin{pmatrix} I \\ w(z) \end{pmatrix} \geq O,\ z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

It now follows that

$$\left\{\rho_n^{-1}(z)(w^*(z) - c_n^*(z)))r_n^{-1}(z)\right\} \cdot \left\{r_n^{-1}(z)(w(z) - c_n(z))\rho_n^{-1}(z)\right\} \leq I,\ z \in \mathbb{C}_+ \setminus \mathcal{Z}_n.$$

Therefore,

$$r_n^{-1}(z)(w(z) - c_n(z))\rho_n^{-1}(z) = V, \quad V^*V \leq I,$$

i.e.,

$$w(z) = c_n(z) + r_n(z)V\rho_n(z), \quad V^*V \leq I. \tag{2.34}$$

Conversely, if $V \in \mathbb{C}^{m\times m}$ and $V^*V \leq I$, then there is $w \in \mathcal{F}_n$ such that equality (2.34) holds. Consider the set of matrices

$$\mathcal{K}_n(z) = \{w(z) : w \in F_n\}.$$

We have proved that

$$\mathcal{K}_n(z) = \{c_n(z) + r_n(z)V\rho_n(z) : V^*V \leq I\}. \tag{2.35}$$

From the geometric point of view, the set $\mathcal{K}_n(z)$ can be regarded as a matrix disk centered at the point $c_n(z)$ with the left radius $r_n(z)$ and the right radius $\rho_n(z)$. The set of matrices $\mathcal{K}_n(z)$ is called *the Weyl disk* at the point z (see [16, 17, 18]). Let $n_2 > n_1 > 0$ be given integers. It follows from the obvious inclusion $\mathcal{F}_{n_2} \subset \mathcal{F}_{n_1}$ that $\mathcal{K}_{n_2} \subset \mathcal{K}_{n_1}$.

Let $\mathcal{K}_\infty(z)$ and $\mathcal{K}_n(z)$ be recorded as (1.7) and (2.35) respectively. We have (see [2]) that $\mathcal{K}_\infty(z) = \bigcap_{n=1}^\infty \mathcal{K}_n(z)$ and

$$c_\infty(z) = \lim_{n\to\infty} c_n(z), \quad r_\infty(z) = \lim_{n\to\infty} r_n(z), \quad \rho_\infty(z) = \lim_{n\to\infty} \rho_n(z). \tag{2.36}$$

For $n > 1$, consider the block partition of the matrices $T_n, R_{T_n}, u_n, v_n, K_n$

$$T_n = \begin{pmatrix} T_{n-1} & O_{(n-1)m\times m} \\ O_{m\times(n-1)m} & z_n^{-1}I_m \end{pmatrix},\ R_{T_n}(z) = \begin{pmatrix} R_{T_{n-1}}(z) & O_{(n-1)m\times m} \\ O_{m\times(n-1)m} & (1 - z\,z_n^{-1})^{-1}I_m \end{pmatrix},$$

$$u_n = \begin{pmatrix} u_{n-1} \\ w_n \end{pmatrix}, \quad v_n = \begin{pmatrix} v_{n-1} \\ I_m \end{pmatrix}, \quad K_n = \begin{pmatrix} K_{n-1} & B_n \\ B_n^* & C_n \end{pmatrix}. \tag{2.37}$$

We have (see (1.10))

$$\begin{aligned} K_n^{-1} &= \begin{pmatrix} I & -K_{n-1}^{-1}B_n \\ O & I \end{pmatrix}\begin{pmatrix} K_{n-1}^{-1} & O \\ O & \widehat{K}_n^{-1} \end{pmatrix}\begin{pmatrix} I & O \\ -B_n^*K_{n-1}^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} I & -K_{n-1}^{-1}B_n \\ O & I \end{pmatrix}\left\{\begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ O & \widehat{K}_n^{-1} \end{pmatrix}\right\}\begin{pmatrix} I & O \\ -B_n^*K_{n-1}^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} I & -K_{n-1}^{-1}B_n \\ O & I \end{pmatrix}\begin{pmatrix} O & O \\ O & \widehat{K}_n^{-1} \end{pmatrix}\begin{pmatrix} I & O \\ -B_n^*K_{n-1}^{-1} & I \end{pmatrix} \end{aligned}$$

Hence,

$$K_n^{-1} = \begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} -K_{n-1}^{-1}B_n \\ I \end{pmatrix}\widehat{K}_n^{-1}(-B_n^*K_{n-1}^{-1}\ \ I). \tag{2.38}$$

Further, let $(n > 1)$

$$\widehat{v}_1 = v_1,\ \widehat{u}_1 = u_1,\ \widehat{v}_n = -B_n^*K_{n-1}^{-1}v_{n-1} + I,\ \widehat{u}_n = -B_n^*K_{n-1}^{-1}u_{n-1} + w_n. \tag{2.39}$$

Let us substitute the block representations (1.10) and (2.37) into the main identity (1.5). Using (2.39), we obtain the induced identities ($n > 1$)

$$K_{n-1}T_{n-1}^*K_{n-1}^{-1}B_n - B_n\bar{z}_n^{-1}I_m = v_{n-1}\widehat{u}_n^* - u_{n-1}\widehat{v}_n^*, \tag{2.40}$$

$$(z_n^{-1}I_m)\widehat{K}_n - \widehat{K}_n(\bar{z}_n^{-1}I_m) = \widehat{v}_n\widehat{u}_n^* - \widehat{u}_n\widehat{v}_n^*. \tag{2.41}$$

The matrix-valued functions

$$\begin{aligned} b_j(z) &= I + z\begin{pmatrix} v_j^* \\ u_j^* \end{pmatrix}\begin{pmatrix} -K_{j-1}^{-1}B_j \\ I \end{pmatrix}\frac{1}{1-\bar{z}_j^{-1}z}\widehat{K}_j^{-1}(-B_j^*K_{j-1}^{-1}\ I)(u_j\ -v_j) \\ &= I + z\begin{pmatrix} \widehat{v}_j^* \\ \widehat{u}_j^* \end{pmatrix}\frac{1}{1-\bar{z}_j^{-1}z}\widehat{K}_j^{-1}(\widehat{u}_j\ -\widehat{v}_j), \quad j \geq 1 \end{aligned} \tag{2.42}$$

are called *Blaschke–Potapov factors.*

Theorem 2.2. *The resolvent matrix $U_n(z)$ for the truncated problem (1.2) defined by (2.19) can be expressed as a product of Blaschke–Potapov factors defined by (2.42),*

$$U_n(z) = b_1(z)\cdot b_2(z)\cdot \ldots \cdot b_n(z). \tag{2.43}$$

Proof. The proof is conducted by induction on n. For $n = 1$, there is nothing to prove. By the inductive assumption, formula (2.43) is true for $U_{n-1}(z)$. This implies that

$$\begin{aligned} U_{n-1}(z)b_n(z) &= \left\{I_{2m} + z\begin{pmatrix} v_{n-1}^* \\ u_{n-1}^* \end{pmatrix}R_{T_{n-1}^*}(z)K_{n-1}^{-1}(u_{n-1}\ -v_{n-1})\right\} \\ &\times\left\{I_{2m} + z\begin{pmatrix} \widehat{v}_n^* \\ \widehat{u}_n^* \end{pmatrix}\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(\widehat{u}_n\ -\widehat{v}_n)\right\} \\ &= I + z\begin{pmatrix} v_{n-1}^* \\ u_{n-1}^* \end{pmatrix}R_{T_{n-1}^*}(z)K_{n-1}^{-1}(u_{n-1}\ -v_{n-1}) + z\begin{pmatrix} \widehat{v}_n^* \\ \widehat{u}_n^* \end{pmatrix}\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(\widehat{u}_n\ -\widehat{v}_n) \\ &+ z^2\begin{pmatrix} v_{n-1}^* \\ u_{n-1}^* \end{pmatrix}R_{T_{n-1}^*}(z)K_{n-1}^{-1}\underbrace{(u_{n-1}\ -v_{n-1})\begin{pmatrix} \widehat{v}_n^* \\ \widehat{u}_n^* \end{pmatrix}}_{-K_{n-1}T_{n-1}^*K_{n-1}^{-1}B_n+B_n\bar{z}_n^{-1}}\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(\widehat{u}_n\ -\widehat{v}_n) \\ &= I + z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix}R_{T_n^*}(z)\begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix}(u_n\ -v_n) \\ &+ z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix}\begin{pmatrix} -K_{n-1}^{-1}B_n \\ I \end{pmatrix}\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(-B_n^*K_{n-1}^{-1}\ I)(u_n\ -v_n) \\ &- z^2\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix}R_{T_n^*}(z)T_n^*\begin{pmatrix} K_{n-1}^{-1} \\ O \end{pmatrix}B_n\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(-B_n^*K_{n-1}^{-1}\ I)(u_n\ -v_n) \\ &+ z^2\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix}R_{T_n^*}(z)\begin{pmatrix} K_{n-1}^{-1} \\ O \end{pmatrix}B_n\bar{z}_n^{-1}\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(-B_n^*K_{n-1}^{-1}\ I)(u_n\ -v_n) \\ &= I + z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix}R_{T_n^*}(z)\left\{\begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix}\right. \\ &+ \left[\begin{pmatrix} -K_{n-1}^{-1}B_n \\ I \end{pmatrix} - zT_n^*\begin{pmatrix} -K_{n-1}^{-1}B_n \\ I \end{pmatrix} - zT_n^*\begin{pmatrix} K_{n-1}^{-1} \\ O \end{pmatrix}B_n + z\begin{pmatrix} K_{n-1}^{-1} \\ O \end{pmatrix}B_n\bar{z}_n^{-1}\right] \\ &\times \left.\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(-B_n^*K_{n-1}^{-1}\ I)\right\}(u_n\ -v_n) \\ &= I + z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix}R_{T_n^*}(z)\left\{\begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix}\right. \\ &+ \left[\begin{pmatrix} -K_{n-1}^{-1}B_n(1-\bar{z}_n^{-1}z) \\ I \end{pmatrix} + \begin{pmatrix} O \\ -Iz\bar{z}_n^{-1} \end{pmatrix}\right]\left.\frac{1}{1-\bar{z}_n^{-1}z}\widehat{K}_n^{-1}(-B_n^*K_{n-1}^{-1}\ I)\right\}(u_n\ -v_n) \end{aligned}$$

$$= I + z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z)\left\{\begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} -K_{n-1}^{-1}B_n \\ I \end{pmatrix}\widehat{K}_n^{-1}(-B_n^* K_{n-1}^{-1} \ I)\right\}(u_n \ - v_n)$$
$$= I + z\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) K_n^{-1}(u_n \ - v_n) = U_n(z).$$

The third equality in this chain follows from (2.40) and the seventh from (2.38).

Consequently formula (2.43) is true for any $n \in \mathbb{N}$ (see also [8, 10, 16, 21]). □

Now if we recall (2.42), (2.41), and Lemma 2.1, we get

$$\mathcal{J} - b_j(z)\mathcal{J}b_j^*(\lambda) = i(z-\overline{\lambda})\begin{pmatrix} \widehat{v}_j^* \\ \widehat{u}_j^* \end{pmatrix}\frac{1}{1-\bar{z}_j^{-1}z}\widehat{K}_j^{-1}\frac{1}{1-z_j^{-1}\overline{\lambda}}(\widehat{v}_j \ \widehat{u}_j) \quad \forall j \in \mathbb{N}. \tag{2.44}$$

Substituting $\bar{z}$ for λ in (2.44), we get

$$\mathcal{J} - b_j(z)\mathcal{J}b_j^*(\bar{z}) = O. \tag{2.45}$$

This immediately implies the invertibility of $b_j(z)$ and the equality

$$b_j^{-1}(z) = \mathcal{J}b_j^*(\bar{z})\mathcal{J}.$$

Substituting $x \in \mathbb{R}$ for z into (2.45) we obtain

$$\mathcal{J} - b_j(x)\mathcal{J}b_j^*(x) = O.$$

Hence,

$$b_j^{-1}(x)\mathcal{J}b_j^{-*}(x) - \mathcal{J} = O.$$

Theorem 2.3. *Let the Blaschke–Potapov factors $b_j(z)$ be defined in (2.42) and let the resolvent matrix $U_n(z)$ be defined in (2.19). Then:*

1) *the matrices $\widehat{K}_j$ can be expressed as*
$$\widehat{K}_j = z_j\frac{\widehat{u}_j\widehat{v}_j^* - \widehat{v}_j\widehat{u}_j^*}{z_j - \bar{z}_j}\bar{z}_j > O$$
and, for all $j \geq 1$, the matrices $\widehat{u}_j$ and $\widehat{v}_j$ are nondegenerate;
2) *the Blaschke–Potapov factors (2.42) have the representations*
$$b_j(z) = I + \frac{z(\bar{z}_j - z_j)}{z_j(\bar{z}_j - z)}\mathcal{P}_j,$$
in which the matrices $\mathcal{P}_j$ are expressed by the formulas
$$\mathcal{P}_j = \begin{pmatrix} \widehat{v}_j^* \\ \widehat{u}_j^* \end{pmatrix}\Big(\frac{\widehat{u}_j\widehat{v}_j^* - \widehat{v}_j\widehat{u}_j^*}{i}\Big)^{-1}(\widehat{v}_j \ \widehat{u}_j)\mathcal{J}$$
and satisfy the conditions
$$\mathcal{P}_j^2 = -\mathcal{P}_j, \quad \mathcal{P}_j\mathcal{J} \geq O;$$
3) *the J-forms (2.44) of the Blaschke–Potapov factors can be calculated by the formulas*
$$\mathcal{J} - b_j(z)\mathcal{J}b_j^*(z) = \left(\left|\frac{z-z_j}{z-\bar{z}_j}\right|^2 - 1\right)\mathcal{P}_j\mathcal{J};$$
4) *the determinants of the Blaschke–Potapov factors can be calculated by the formulas*
$$\det b_j(z) = \left(\frac{\bar{z}_j}{z_j}\cdot\frac{z-z_j}{z-\bar{z}_j}\right)^m;$$
5)
$$\det U_n^{-1}(z) = \prod_{j=1}^{n}\left(\frac{z_j}{\bar{z}_j}\cdot\frac{z-\bar{z}_j}{z-z_j}\right)^m; \tag{2.46}$$

6) *the traces of the matrices $\mathcal{P}_j$ are*

$$\operatorname{tr}\mathcal{P}_j = -m.$$

Proof. The proof of Theorem 2.3 was given in [13, 16, 18]. ☐

3. Rational matrix-valued functions of the first and the second kind

Theorem 3.1. *Suppose the sequence $(P_j(z))_{j=1}^{\infty}$ of rational matrix-valued functions of the first kind (1.11) is associated with a Nevanlinna–Pick matrix interpolation problem (1.1), the matrix-valued function $w(z)$ is a solution of the interpolation problem (1.1), and*

$$w(z) = \mu z + \nu + \int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\sigma(dt).$$

Then the rational matrix-valued functions $P_j(z)$ are orthonormal in the sense that

$$\int_{-\infty}^{\infty} P_j(t)\sigma(dt)P_\ell^*(t) + \lim_{t\to\infty} t^2 P_j(t)\mu P_\ell^*(t) = \delta_{j\ell}I, \quad \delta_{j\ell} = \left\{\begin{array}{ll} 1, & j=\ell \\ 0, & j\neq\ell \end{array}\right. . \tag{3.47}$$

Proof. 1) Using (1.1) and (2.16), we get $(p,q\in\mathbb{N})$

$$\begin{aligned}
\frac{w_p - w_q^*}{z_p - \bar{z}_q} &= \frac{1}{z_p-\bar{z}_q}\left(\mu z_p + \nu + \int_{-\infty}^{\infty}\left(\frac{1}{t-z_p} - \frac{t}{1+t^2}\right)\sigma(dt)\right.\\
&\qquad \left. -\mu\bar{z}_q - \nu - \int_{-\infty}^{\infty}\left(\frac{1}{t-\bar{z}_q} - \frac{t}{1+t^2}\right)\sigma(dt)\right)\\
&= \mu + \frac{1}{z_p-\bar{z}_q}\int_{-\infty}^{\infty}\left(\frac{1}{t-z_p} - \frac{1}{t-\bar{z}_q}\right)\sigma(dt)\\
&= \mu + \int_{-\infty}^{\infty}\frac{1}{t-z_p}\cdot\frac{1}{t-\bar{z}_q}\sigma(dt).
\end{aligned}$$

Therefore, we have $(j\geq 1)$

$$\begin{aligned}
&\int_{-\infty}^{\infty} R_{T_j}(t)v_j\sigma(dt)v_j^*R_{T_j}^*(t) + \lim_{t\to\infty} t^2 R_{T_j}(t)v_j\mu v_j^* R_{T_j}^*(t)\\
&= \int_{-\infty}^{\infty} R_{T_j}(t)v_j\sigma(dt)v_j^*R_{T_j}^*(t) + T_j^{-1}v_j\mu v_j^* T_j^{-1^*}\\
&= T_j^{-1}\left(\int_{-\infty}^{\infty} T_jR_{T_j}(t)v_j\sigma(dt)v_j^*R_{T_j}^*(t)T_j^* + v_j\mu v_j^*\right)T_j^{-1^*}\\
&= T_j^{-1}\left(\int_{-\infty}^{\infty}(tI-T_j^{-1})^{-1}v_j\sigma(dt)v_j^*(tI-T_j^{-1^*})^{-1} + v_j\mu v_j^*\right)T_j^{-1^*}\\
&= T_j^{-1}\left(\int_{-\infty}^{\infty}\frac{1}{t-z_p}\cdot\frac{1}{t-\bar{z}_q}\sigma(dt) + \mu\right)_{p,q=1}^{j} T_j^{-1^*}\\
&= T_j^{-1}\left(\frac{w_p-w_q^*}{z_p-\bar{z}_q}\right)_{p,q=1}^{j} T_j^{-1^*} = K_j.
\end{aligned}$$

It now follows that $(j=\ell>1)$

$$\begin{aligned}
&\int_{-\infty}^{\infty} P_j(t)\sigma(dt)P_j^*(t) + \lim_{t\to\infty} t^2P_j(t)\mu P_j^*(t) = \widehat{K}_j^{-1/2}(-B_j^*K_{j-1}^{-1},\ I)\\
&\times\left(\int_{-\infty}^{\infty} R_{T_j}(t)v_j\sigma(dt)v_j^*R_{T_j}^*(t) + \lim_{t\to\infty} t^2R_{T_j}(t)v_j\mu v_j^*R_{T_j}^*(t)\right)\begin{pmatrix}-K_{j-1}^{-1}B_j\\ I\end{pmatrix}\widehat{K}_j^{-1/2}\\
&= \widehat{K}_j^{-1/2}(-B_j^*K_{j-1}^{-1},\ I)\begin{pmatrix}K_{j-1} & B_j\\ B_j^* & C_j\end{pmatrix}\begin{pmatrix}-K_{j-1}^{-1}B_j\\ I\end{pmatrix}\widehat{K}_j^{-1/2} = \widehat{K}_j^{-1/2}\widehat{K}_j\widehat{K}_j^{-1/2} = I.
\end{aligned}$$

Now let us prove (3.47) under the condition that $j \neq \ell$. Let, for example, $j > \ell > 1$. We obtain

$$\int_{-\infty}^{\infty} R_{T_j}(t) v_j \sigma(dt) v_\ell^* R_{T_\ell}^*(t) + \lim_{t\to\infty} t^2 R_{T_j}(t) v_j \mu v_\ell^* R_{T_\ell}^*(t) = T_j^{-1} \left(\frac{w_p - w_q^*}{z_p - \bar{z}_q} \right)_{p,q=1}^{j,\ell} T_\ell^{-1^*} = K_j \cdot \begin{pmatrix} I_{\ell m} \\ O_{\ell m \times (j-\ell)m} \end{pmatrix}.$$

Thus we have

$$\begin{aligned} &\int_{-\infty}^{\infty} P_j(t)\sigma(dt) P_\ell^*(t) + \lim_{t\to\infty} t^2 P_j(t) \mu P_\ell^*(t) = \widehat{K}_j^{-1/2} (-B_j^* K_{j-1}^{-1},\ I) \\ &\times \left(\int_{-\infty}^{\infty} R_{T_j}(t) v_j \sigma(dt) v_\ell^* R_{T_\ell}^*(t) + \lim_{t\to\infty} t^2 R_{T_j}(t) v_j \mu v_\ell^* R_{T_\ell}^*(t) \right) \begin{pmatrix} -K_{\ell-1}^{-1} B_\ell \\ I \end{pmatrix} \widehat{K}_\ell^{-1/2} \\ &= \widehat{K}_j^{-1/2} (-B_j^* K_{j-1}^{-1},\ I) \begin{pmatrix} K_{j-1} & B_j \\ B_j^* & C_j \end{pmatrix} \begin{pmatrix} I_{\ell m} \\ O_{\ell m \times (j-\ell)m} \end{pmatrix} \begin{pmatrix} -K_{\ell-1}^{-1} B_\ell \\ I \end{pmatrix} \widehat{K}_\ell^{-1/2} \\ &= \widehat{K}_j^{-1/2} (O,\ \widehat{K}_j) \begin{pmatrix} I_{\ell m} \\ O_{\ell m \times (j-\ell)m} \end{pmatrix} \begin{pmatrix} -K_{\ell-1}^{-1} B_\ell \\ I \end{pmatrix} \widehat{K}_\ell^{-1/2} = O. \end{aligned}$$

In the same way, equality (3.47) can be proved for other cases (see also [10]). □

Lemma 3.2. *Suppose the sequences* $(P_j(z))_{j=1}^\infty$ *and* $(Q_j(z))_{j=1}^\infty$ *of rational matrix-valued functions of the first and the second kind* (1.11) *and* (1.12)*, respectively, are associated with a Nevanlinna–Pick matrix interpolation problem* (1.1)*. Then*

$$\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) K_n^{-1} R_{T_n^*}^*(\lambda)(v_n\ u_n) = \sum_{j=1}^n \begin{pmatrix} P_j^*(\bar{z}) P_j(\bar{\lambda}) & -P_j^*(\bar{z}) Q_j(\bar{\lambda}) \\ -Q_j^*(\bar{z}) P_j(\bar{\lambda}) & Q_j^*(\bar{z}) Q_j(\bar{\lambda}) \end{pmatrix}, \tag{3.48}$$

where the block matrices $K_n, u_n, v_n, R_{T_n^*}(z)$ *are defined by* (1.3) *and* (1.4)*.*

Proof. We will use formulas (1.9) and (2.38). The proof is by induction on n. The basis step $n = 1$ says

$$\begin{pmatrix} v_1^* \\ u_1^* \end{pmatrix} R_{T_1^*}(z) K_1^{-1} R_{T_1^*}^*(\lambda)(v_1\ u_1) = \begin{pmatrix} P_1^*(\bar{z}) P_1(\bar{\lambda}) & -P_1^*(\bar{z}) Q_1(\bar{\lambda}) \\ -Q_1^*(\bar{z}) P_1(\bar{\lambda}) & Q_1^*(\bar{z}) Q_1(\bar{\lambda}) \end{pmatrix},$$

which is true. Here is the induction step.

$$\begin{aligned} &\begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) K_n^{-1} R_{T_n^*}^*(\lambda)(v_n\ u_n) = \\ &= \begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) \left\{ \begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} -K_{n-1}^{-1} B_n \\ I \end{pmatrix} \widehat{K}_n^{-1} (-B_n^* K_{n-1}^{-1}\ \ I) \right\} R_{T_n^*}^*(\lambda)(v_n\ u_n) \\ &= \begin{pmatrix} v_{n-1}^* \\ u_{n-1}^* \end{pmatrix} R_{T_{n-1}^*}(z) K_{n-1}^{-1} R_{T_{n-1}^*}^*(\lambda)(v_{n-1}\ u_{n-1}) \\ &+ \begin{pmatrix} v_n^* \\ u_n^* \end{pmatrix} R_{T_n^*}(z) \begin{pmatrix} -K_{n-1}^{-1} B_n \\ I \end{pmatrix} \widehat{K}_n^{-1} (-B_n^* K_{n-1}^{-1}\ \ I) R_{T_n^*}^*(\lambda)(v_n\ u_n) \\ &= \sum_{j=1}^{n-1} \begin{pmatrix} P_j^*(\bar{z}) P_j(\bar{\lambda}) & -P_j^*(\bar{z}) Q_j(\bar{\lambda}) \\ -Q_j^*(\bar{z}) P_j(\bar{\lambda}) & Q_j^*(\bar{z}) Q_j(\bar{\lambda}) \end{pmatrix} + \begin{pmatrix} P_n^*(\bar{z}) P_n(\bar{\lambda}) & -P_n^*(\bar{z}) Q_n(\bar{\lambda}) \\ -Q_n^*(\bar{z}) P_n(\bar{\lambda}) & Q_n^*(\bar{z}) Q_n(\bar{\lambda}) \end{pmatrix} \\ &= \sum_{j=1}^{n} \begin{pmatrix} P_j^*(\bar{z}) P_j(\bar{\lambda}) & -P_j^*(\bar{z}) Q_j(\bar{\lambda}) \\ -Q_j^*(\bar{z}) P_j(\bar{\lambda}) & Q_j^*(\bar{z}) Q_j(\bar{\lambda}) \end{pmatrix}. \end{aligned}$$

This implies that equality (3.48) is true for any $n \in \mathbb{N}$. □

Theorem 3.3. *Under the conditions of Lemma 3.2, we have:*

1) *inequality* (2.18) *can be written as*

$$\sum_{j=1}^{n}(P_j(z)w(z)+Q_j(z))^*(P_j(z)w(z)+Q_j(z)) \le \frac{w(z)-w^*(z)}{z-\bar z},\ z\in\mathbb{C}_+\setminus\mathcal{Z}_n; \quad (3.49)$$

2) *the resolvent matrix* (2.19) *can be written as*

$$U_n(z)=I_{2m}+z\sum_{j=1}^{n}\begin{pmatrix} -P_j^*(\bar z)Q_j(0) & -P_j^*(\bar z)P_j(0) \\ Q_j^*(\bar z)Q_j(0) & Q_j^*(\bar z)P_j(0)\end{pmatrix}; \quad (3.50)$$

3) *the Blaschke–Potapov factors* (2.42) *can be written as*

$$b_j(z)=I+\frac{z}{1-\bar z_j^{-1}z}\begin{pmatrix} -P_j^*(0)Q_j(0) & -P_j^*(0)P_j(0) \\ Q_j^*(0)Q_j(0) & Q_j^*(0)P_j(0)\end{pmatrix};$$

4) *the J-form* (2.21) *of the resolvent matrix can be calculated by the formula*

$$\mathcal{J}-U_n(z)\mathcal{J}U_n^*(\lambda)=i(z-\overline{\lambda})\sum_{j=1}^{n}\begin{pmatrix} P_j^*(\bar z)P_j(\bar\lambda) & -P_j^*(\bar z)Q_j(\bar\lambda) \\ -Q_j^*(\bar z)P_j(\bar\lambda) & Q_j^*(\bar z)Q_j(\bar\lambda)\end{pmatrix};$$

5) *the J-form* (2.23) *can be calculated by the formula*

$$\mathcal{J}-U_n^{-*}(z)\mathcal{J}U_n^{-1}(z)=i(\bar z-z)\sum_{j=1}^{n}\mathcal{J}\begin{pmatrix} P_j^*(z)P_j(\lambda) & -P_j^*(z)Q_j(\lambda) \\ -Q_j^*(z)P_j(\lambda) & Q_j^*(z)Q_j(\lambda)\end{pmatrix}\mathcal{J};$$

6) *the J-forms* (2.44) *of the Blaschke–Potapov factors can be calculated by the formulas*

$$\mathcal{J}-b_j(z)\mathcal{J}b_j^*(\lambda)=\frac{i(z-\overline{\lambda})}{(1-\bar z_j^{-1}z)(1-z_j^{-1}\bar\lambda)}\begin{pmatrix} P_j^*(0)P_j(0) & -P_j^*(0)Q_j(0) \\ -Q_j^*(0)P_j(0) & Q_j^*(0)Q_j(0)\end{pmatrix};$$

7) *the center* (2.30), *the left radius* (2.31), *and the right radius* (2.32) *of the Weyl disk* (2.35) *can be calculated by the formulas*

$$c_n(z)=\Big(i(\bar z-z)\sum_{j=1}^{n}P_j^*(z)P_j(z)\Big)^{-1}\Big(-i(\bar z-z)\sum_{j=1}^{n}Q_j^*(z)P_j(z)-iI\Big)^*,$$

$$r_n(z)=\Big(i(\bar z-z)\sum_{j=1}^{n}P_j^*(z)P_j(z)\Big)^{-1/2}>O,$$

$$\rho_n(z)=\Big(i(\bar z-z)\sum_{j=1}^{n}P_j^*(\bar z)P_j(\bar z)\Big)^{-1/2}>O,$$

respectively;

8) *the center, the left radius, and the right radius of the limit Weyl disk* (1.7) *can be calculated by the formulas* (*see* (2.36))

$$c_\infty(z)=\lim_{n\to\infty}\Big\{\Big(i(\bar z-z)\sum_{j=1}^{n}P_j^*(z)P_j(z)\Big)^{-1}\Big(-i(\bar z-z)\sum_{j=1}^{n}Q_j^*(z)P_j(z)-iI\Big)^*\Big\},$$

$$r_\infty(z)=\lim_{n\to\infty}\Big\{\Big(i(\bar z-z)\sum_{j=1}^{n}P_j^*(z)P_j(z)\Big)^{-1/2}\Big\}, \quad (3.51)$$

$$\rho_\infty(z)=\lim_{n\to\infty}\Big\{\Big(i(\bar z-z)\sum_{j=1}^{n}P_j^*(\bar z)P_j(\bar z)\Big)^{-1/2}\Big\}, \quad (3.52)$$

respectively.

Proof. 1. We claim that

$$\big(v_n w(z) - u_n\big)^* R_n^*(z) K_n^{-1} R_n(z)\big(v_n w(z) - u_n\big) = \sum_{j=1}^{n} \big(P_j(z)w(z) + Q_j(z)\big)^* \big(P_j(z)w(z) + Q_j(z)\big).$$

The proof is by induction on n. The basis step $n=1$ says

$$\begin{aligned}&(v_1 w(z) - u_1)^* R_1^*(z) K_1^{-1} R_1(z)(v_1 w(z) - u_1)\\ &\quad = (\widehat{K}_1^{-1/2} R_1(z) v_1 w(z) - \widehat{K}_1^{-1/2} R_1(z) u_1)^* (\widehat{K}_1^{-1/2} R_1(z) v_1 w(z) - \widehat{K}_1^{-1/2} R_1(z) u_1)\\ &\quad = (P_1(z)w(z) + Q_1(z))^*(P_1(z)w(z) + Q_1(z)),\end{aligned}$$

which is true. Here is the induction step.

$$\begin{aligned}&(v_n w(z) - u_n)^* R_n^*(z) K_n^{-1} R_n(z)(v_n w(z) - u_n)\\ &= (v_n w(z) - u_n)^* R_n^*(z) \left\{ \begin{pmatrix} K_{n-1}^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} -K_{n-1}^{-1} B_n \\ I \end{pmatrix} \widehat{K}_n^{-1} (-B_n^* K_{n-1}^{-1} \quad I) \right\}\\ &\times R_n(z)(v_n w(z) - u_n) = (v_{n-1} w(z) - u_{n-1})^* R_{n-1}^*(z) K_{n-1}^{-1} R_{n-1}(z)(v_{n-1} w(z) - u_{n-1})\\ &+ (v_n w(z) - u_n)^* R_n^*(z) \begin{pmatrix} -K_{n-1}^{-1} B_n \\ I \end{pmatrix} \widehat{K}_n^{-1} (-B_n^* K_n^{-1} \quad I) R_n(z)(v_n w(z) - u_n)\\ &= \sum_{j=1}^{n-1} (P_j(z)w(z) + Q_j(z))^* (P_j(z)w(z) + Q_j(z)) + (P_n(z)w(z) + Q_n(z))^*\\ &+ (P_n(z)w(z) + Q_n(z)) = \sum_{j=1}^{n} (P_j(z)w(z) + Q_j(z))^* (P_j(z)w(z) + Q_j(z)).\end{aligned}$$

This and (2.18) imply that the matrix inequality (3.49) is true for any $n \in \mathbb{N}$. Statements 2) – 7) follow from (3.48). Statement 8) follows from (2.36) and statement 7). □

Lemma 3.4. *Let $(A_j)_{j=1}^{\infty} \subset \mathbb{C}^{m\times m}$ and $(B_j)_{j=1}^{\infty} \subset \mathbb{C}^{m\times m}$ be the sequences of matrices such that the series*

$$\sum_{j=1}^{\infty} A_j^* A_j, \qquad \sum_{j=0}^{\infty} B_j^* B_j \tag{3.53}$$

converge. Then the series

$$\sum_{j=1}^{\infty} \big(A_j - B_j\big)^* \big(A_j - B_j\big), \qquad \sum_{j=1}^{\infty} B_j^* A_j \tag{3.54}$$

converge.

Proof. By $A_j(kl)$ and $A_j^* A_j(kl)$ denote the components of the matrix A_j and $A_j^* A_j$, i.e.,

$$A_j = \Big(A_j(kl)\Big)_{k,l=1}^{m}, \qquad A_j^* A_j = \Big(A_j^* A_j(kl)\Big)_{k,l=1}^{m} = \Big(\sum_{p=1}^{m} \overline{A_j(pk)} \cdot A_j(pl)\Big)_{k,l=1}^{m}.$$

It follows from (3.53) that the sequences of the diagonal components

$$\Big(A_j^* A_j(kk)\Big)_{j=1}^{\infty} = \Big(\sum_{p=1}^{m} \overline{A_j(pk)} \cdot A_j(pk)\Big)_{j=1}^{\infty} = \Big(\sum_{p=1}^{m} |A_j(pk)|^2\Big)_{j=1}^{\infty}, \quad 1 \le k \le m$$

converge. It now follows that any m^2 sequences $(A_j(pk))_{j=1}^{\infty}$, $1 \le p, k \le m$ belong to $\ell^2(\mathbb{C})$. By the same argument, any m^2 sequences $(B_j(pk))_{j=1}^{\infty}$, $1 \le p, k \le m$, belong to $\ell^2(\mathbb{C})$. This implies that series (3.54) converge. □

Lemma 3.5. *Let two sequences $(P_j)_{j=0}^{\infty}$ and $(Q_j)_{j=0}^{\infty}$ of rational matrix functions of the first and the second kind be given by (1.11) and (1.12), respectively. Then:*

1) *for any matrix-valued function $w(z) \in \mathcal{F}_\infty$ and any $z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$, the series*
$$\sum_{j=1}^{\infty} \big(P_j(z)w(z) - Q_j(z)\big)^* (P_j(z)w(z) - Q_j(z)) \tag{3.55}$$
converges;

2) *if for some $z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$ one of the series*
$$\sum_{j=1}^{\infty} P_j^*(z)P_j(z), \qquad \sum_{j=1}^{\infty} Q_j^*(z)Q_j(z) \tag{3.56}$$
converges, then both series (3.56) converge.

Proof. 1) This statement follows from inequality (3.49).

2) Without loss of generality, we can assume that the series $\sum_{j=1}^{\infty} P_j^*(z)P_j(z)$ converges for some $z \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$. Consequently both series ($w \in \mathcal{F}_\infty$)
$$\sum_{j=1}^{\infty} (P_j(z)w(z))^* P_j(z)w(z), \quad \sum_{j=1}^{\infty} \big(P_j(z)w(z) - Q_j(z)\big)^* (P_j(z)w(z) - Q_j(z))$$
converge. From Lemma 3.4 it follows that the series
$$\sum_{j=1}^{\infty} Q_j^*(z)Q_j(z) = \sum_{j=1}^{\infty} \big(P_j(z)w(z) - P_j(z)w(z) + Q_j(z)\big)^* (P_j(z)w(z) - P_j(z)w(z) + Q_j(z))$$
converges. □

4. Proof of Theorem 1.1

Proof. Now we are able to present the proof of Theorem 1.1.

1) ⇒ 2) Suppose that the Nevanlinna–Pick matrix interpolation problem (1.1) is completely indeterminate and some point $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$. Then (see (1.8))
$$r_\infty(z_0) > O, \quad \rho_\infty(z_0) > O. \tag{4.57}$$
Using (2.26) and (2.33), we get
$$\det\Big(U_n^{-*}(z_0)\mathcal{J}U_n^{-1}(z_0)\Big) = \det\Big(\rho_n^2(z_0)\Big) \det\Big(-r_n^{-2}(z_0)\Big).$$
Now if we recall (2.46) and (2.20), we obtain
$$(-1)^m \Big|\prod_{j=1}^{n} \Big(\frac{z_j}{\bar z_j} \cdot \frac{z - \bar z_j}{z - z_j}\Big)^m\Big|^2 = (-1)^m \Big(\det \rho_n(z_0)\Big)^2 \Big(\det r_n(z_0)\Big)^{-2}.$$
Using (1.13), we get
$$\det\Big(r_n(z_0)\Big) = |\mathcal{B}_n(z_0))|^m \det\Big(\rho_n(z_0)\Big).$$
It now follows that
$$\det\Big(r_\infty(z_0)\Big) = |\mathcal{B}(z_0))|^m \det\Big(\rho_\infty(z_0)\Big), \tag{4.58}$$
i.e., the infinite Blaschke product $\mathcal{B}(z_0)$ converges. It follows from (3.51) and (4.57) that
$$r_\infty(z_0) = \lim_{n\to\infty} \Big\{ \Big(i(\bar z_0 - z_0) \sum_{j=1}^{n} P_j^*(z_0)P_j(z_0)\Big)^{-1/2} \Big\} > O.$$

Consequently the series $\sum_{j=1}^{\infty} P_j^*(z_0)P_j(z_0)$ converges, i.e., the infinite matrix column $\pi(z_0)$ (see (1.14)) belongs to $\ell^2(\mathbb{C}^{m\times m})$.

2) $\Rightarrow$ 3) Suppose for some point $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$ the infinite matrix column $\pi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$ and the infinite Blaschke product $\mathcal{B}(z_0)$ converges. Then it follows from Lemma 3.5 that $\xi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$.

3) $\Rightarrow$ 4) Suppose for some point $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$ the infinite matrix column $\xi(z_0)$ (see (1.15)) belongs to $\ell^2(\mathbb{C}^{m\times m})$ and the infinite Blaschke product $\mathcal{B}(z_0)$ converges. Then it follows from Lemma 3.5 that $\pi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$. This implies that

$$r_\infty(z_0) = \lim_{n\to\infty}\left\{\left(i(\bar{z}_0 - z_0)\sum_{j=1}^{n} P_j^*(z_0)P_j(z_0)\right)^{-1/2}\right\} > O.$$

By (4.58), we have that $\rho_\infty(z_0) > O$. Now if we recall (3.52), we get

$$\rho_\infty(z_0) = \lim_{n\to\infty}\left\{\left(i(\bar{z}_0 - z_0)\sum_{j=1}^{n} P_j^*(\bar{z}_0)P_j(\bar{z}_0)\right)^{-1/2}\right\} > O.$$

Consequently the infinite matrix column $\pi(\bar{z}_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$.

4) $\Rightarrow$ 1) Suppose for some point $z_0 \in \mathbb{C}_+ \setminus \mathcal{Z}_\infty$ both infinite matrix columns $\pi(z_0)$ and $\pi(\bar{z}_0)$ belong to $\ell^2(\mathbb{C}^{m\times m})$. Then two series $\sum_{j=1}^{\infty} P_j^*(z_0)P_j(z_0)$ and $\sum_{j=1}^{\infty} P_j^*(\bar{z}_0)P_j(\bar{z}_0)$ are convergent. It now follows that

$$r_\infty(z_0) = \lim_{n\to\infty}\left\{\left(i(\bar{z}_0 - z_0)\sum_{j=1}^{n} P_j^*(z_0)P_j(z_0)\right)^{-1/2}\right\} > O,$$

$$\rho_\infty(z_0) = \lim_{n\to\infty}\left\{\left(i(\bar{z}_0 - z_0)\sum_{j=1}^{n} P_j^*(\bar{z}_0)P_j(\bar{z}_0)\right)^{-1/2}\right\} > O.$$

This completes the proof of Theorem 1.1. □

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