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ROTHE TIME-DISCRETIZATION METHOD FOR NOLINEAR PARABOLIC PROBLEMS

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Abstract. In this paper we consider a class of nonlinear parabolic problems whose model is

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left[|\nabla u - \Theta(u)|^{p-2}(\nabla u - \Theta(u))\right] + \beta(u) = f & \text{in} \quad Q_T := (0,T)\times\Omega, \\ u = 0 & \text{on} \quad \Sigma_T := (0,T)\times\partial\Omega, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{$*$}$$

Using time discretization technique and Rothe method we prove existence and uniqueness results for bounded weak solutions.

Використовуючи техніку дискретизації за часом та метод Рота, доведено існування та єдиність слабкого обмеженого розв'язку для нелінійних параболічних задач вигляду $(*)$.

1. Introduction

This paper is devoted to existence and uniqueness results for bounded weak solutions to the following nonlinear parabolic problem:

$$(P)\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(\Phi(\nabla u - \Theta(u))) + \beta(u) = f & \text{in} \quad Q_T := (0,T)\times\Omega, \\ u = 0 & \text{on} \quad \Sigma_T := (0,T)\times\partial\Omega, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^d (d \geq 3)$ is an open bounded domain with Lipschitz boundary $\partial\Omega$, T is a fixed positive number; ∇u is the gradient of u and $\Phi(\xi) := |\xi|^{p-2}\xi$ for all $\xi \in \mathbb{R}^d$ with $1 < p < d$. We make the following assumptions:

- **(A1):** β is a non-decreasing continuous real-valued function on $\mathbb{R}$, surjective, satisfies $\beta(0) = 0$, and $|\beta(x)| \leq M|x|$, where M is a positive constant.
- **(A2):** $f \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.
- **(A3):** Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^d$, $\Theta(0) = 0$, and $|\Theta(x) - \Theta(y)| \leq \lambda|x-y|$ for all $x, y \in \mathbb{R}$, and λ is a positive constant such that

$$\lambda < \frac{(d-p)(\operatorname{meas}(\Omega))^{-1/d}}{2(d-1)p}.$$

The problem (P) arises in various physical contexts like chemical heterogeneous catalysts, non-Newtonian fluids,and the theory of heat conduction in electrically conducting materials (see for example [9, 24, 26]). Here we shall mention two of them which are related to turbulent flows.

Model 1: Filtration of a fluid in a partially saturated porous medium. This flow is governed by the equation

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{1.1}$$

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where p is an unknown pressure, c the water content, k the conductivity of the porous medium, a the matrix heterogeneity and $-e$ the direction of gravity. The Kirchhoff transformation equation

$$u = \int_0^p k(c(\xi))d\xi$$

leads to a differential equation (1.1) of the form

$$\frac{\partial b(u)}{\partial t} = \nabla a[\nabla u + k(b(u))e],$$

where the function b has the same behavior as c.

Model 2: Flow through a porous medium in a turbulent regime This model is governed by the continuity equation

$$\frac{\partial \theta}{\partial t} + \operatorname{div} v = 0,$$

and Darcy's law

$$v = -K(\theta)\operatorname{grad}\phi(\theta),$$

where $\theta(x,t)$ is the volumetric moisture content, $k(\theta)$ is the hydraulic conductivity, and the total potential ϕ is given by

$$\phi(\theta) = \psi(\theta) + z,$$

where $\psi(\theta)$ is the hydrostatic potential and z is the gravitational potential. In turbulent regimes, the flow rate is different from that which can be predicted by the Darcy law, and so several authors proposed a nonlinear relation between v and $K(\theta)\operatorname{grad}\phi$,

$$|v|^{q-2}v = -K(\theta)\operatorname{grad}\phi(\theta), q > 2.$$

If e denotes the unit vector in the vertical direction, we obtain

$$\frac{\partial \theta}{\partial t} - \operatorname{div}\big(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\big) = 0,$$

where

$$\varphi(\theta) = \int_0^\theta K(s)\phi'(s)ds, p = \frac{q}{q-1}.$$

In the last years, the problem (P) or special cases of it have been extensively treated by many authors in elliptic or parabolic case, we invite the reader to see for example the works [2, 3, 4, 5, 10, 13, 16, 18].

We recall that the Euler forward scheme has been used by several authors while studying time discretization of nonlinear parabolic problems, we refer for example to the works [6, 12, 14, 15, 17, 23, 25] for some details.

The advantage of our method is that we can not only obtain existence and uniqueness of weak solutions to the problem (P), but also compute numerical approximations. In the particular case where $\Theta = 0$, the author in [17] showed existence and uniqueness of entropy solutions in Orlicz spaces by using our Rothe time-discretization method.

This work is divided into four sections. In Section 1, we introduce the problem (P) and we state the assumptions. In Section 2, we show some preliminary results and notations, also we state our main result. In Section 3, we discretize the problem (P) by the Euler forward scheme, we show existence and uniqueness of a weak solution for the discretized problems and we give some stabilty results. In the last section, we finish this work by treating convergence and existence results for the problem (P), moreover we confirm the uniqueness of solution.

2. Preliminary results and notations

In this section, we give some notations and definitions and state some results that will be used in this work. For a measurable set Ω in $\mathbb{R}^d$, meas (Ω) denotes its measure, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$, and $\|\cdot\|_{1,p}$ denotes the norm in the Sobolev space $W^{1,p}(\Omega)$. For a Banach space X and $a<b$ we have

$$L^p(a,b;X)=\Big\{u:[a,b]\to X \quad \text{is measurable ;} \quad \int_a^b \|u(t)\|_X^p dt<\infty\Big\},$$

endowed with the norm

$$\|u\|_{L^p(a,b;X)}:=\Big(\int_a^b \|u(t)\|_X^p dt\Big)^{\frac{1}{p}},$$

if $p=\infty$, we have

$$\|u\|_{L^\infty(a,b;X)}:=\sup_{[a,b]}\|u\|_X<\infty. \tag{2.2}$$

Lemma 2.1. *For $\xi,\eta\in\mathbb{R}^d$ and $1<p<\infty$, we have*

$$\frac{1}{p}|\xi|^p-\frac{1}{p}|\eta|^p\le|\xi|^{p-2}\xi(\xi-\eta). \tag{2.3}$$

Proof. We consider the function $f:\mathbb{R}^+\to\mathbb{R}$ defined by $x\mapsto x^p-px+(p-1)$. We have

$$f(x)\ge\min_{y\in\mathbb{R}^+}f(y)=f(1)=0 \text{ for all } x\in\mathbb{R}^+.$$

Therefore, we take $x=\frac{|\eta|}{|\xi|}$ (if $|\xi|=0$, the result is obvious) in the inequality above to get the result of the lemma by using Cauchy-Schwarz inequality. $\square$

Lemma 2.2 ([9]). *Let p,p' be two real numbers such that $p>1,p'>1$ and $\frac{1}{p}+\frac{1}{p'}=1$, Then $\left||\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right|^{p'}\le C\left\{(\xi-\eta)\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\right\}^{\frac{\alpha}{2}}\left\{|\xi|^p+|\eta|^p\right\}^{1-\frac{\alpha}{2}},\forall\xi,\eta\in\mathbb{R}^d$, where $\alpha=2$ if $1<p\le 2$, and $\alpha=p'$ if $p\ge 2$.*

Remark 2.3. Hereinafter, c_i, $i\in\mathbb{N}$, are positive constants independent of N.

Definition 2.4. A measurable function $u:Q_T\to\mathbb{R}$ is a weak solution to nonlinear parabolic problems (P) in Q_T if $u(.,0)=u_0$ in $\Omega, u\in C\left(0,T;L^2(\Omega)\right)\cap L^p\left(0,T;W^{1,p}(\Omega)\right)$, $\frac{\partial u}{\partial t}\in L^2\left(Q_T\right)$ and we have

$$\begin{aligned}\int_0^T\int_\Omega\frac{\partial u}{\partial t}\varphi dxdt+\int_0^T\int_\Omega\Phi(\nabla u-\Theta(u)).\nabla\varphi dxdt+\int_0^T\int_\Omega\beta(u)\varphi dxdt\\ =\int_0^T\int_\Omega f\varphi dxdt,\qquad\forall\varphi\in C^1\left(Q_T\right).\end{aligned} \tag{2.4}$$

Given a constant $k>0$, we define the cut function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$T_k(s):=\begin{cases} s & \text{if} \quad |s|\le k,\\ k \quad \operatorname{sign}(s) & \text{if} \quad |s|>k,\end{cases}$$

where

$$\operatorname{sign}(s):=\begin{cases}1 & \text{if} \quad s>0,\\ 0 & \text{if} \quad s=0,\\ -1 & \text{if} \quad s<0.\end{cases}$$

The following theorem is the main result of this paper.

Theorem 2.5. *Under the assumptions $(A1)$, $(A2)$, and $(A3)$ there exists a unique weak solution for the nonlinear parabolic problem (P).*

3. The semi-discrete problem and stability results

3.1. **The semi-discrete problem.** In this section, we discretize the problem (P) by Euler forward scheme and we study the questions of existence and uniqueness under the assumptions $(A1)$, $(A2)$ and $(A3)$ to the following discretized problems:

$$(P_n)\begin{cases} U_n - \tau \operatorname{div}\left(\Phi\left(\nabla U_n - \Theta\left(U_n\right)\right)\right) + \tau\beta\left(U_n\right) = \tau f_n + U_{n-1} & \text{in} \quad \Omega, \\ U_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $U_0 = u_0$, $N\tau = T, 0 < \tau < 1$, $1 \le n \le N$, $t_n = n\tau$, and

$$f_n(\cdot) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(s,\cdot)ds, \text{in } \ \Omega.$$

A weak solution to the discretized problems (P_n) is a sequence $(U_n)_{0\le n\le N}$ such that $U_0 = u_0$ and U_n is defined by induction as a weak solution to the problem

$$\begin{cases} u - \tau \operatorname{div}(\Phi(\nabla u - \Theta(u))) + \tau\beta\left(u\right) = \tau f_n + U_{n-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

i.e., for $U_n \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ and $\forall\varphi \in W^{1,p}(\Omega), \forall\tau > 0$, we have

$$\int_\Omega U_n\varphi dx + \tau\int_\Omega \Phi(\nabla U_n - \Theta(U_n)).\nabla\varphi dx + \tau\int_\Omega \beta\left(U_n\right)\varphi dx = \int_\Omega \left(\tau f_n + U_{n-1}\right)\varphi dx.$$

Theorem 3.1. *Under the assumptions* $(A1)$, $(A2)$, $(A3)$, *and* $1 < p < d$, *the problem* (P_n) *has a unique weak solution* $(U_n)_{0\le n\le N}$ *and for all* $n = 1, \dots, N$, $U_n \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

For $n = 1$, we denote $U = U_1$, and rewrite the problem (3.5) as

$$\begin{cases} -\tau \operatorname{div}(\Phi(\nabla U - \Theta(U))) + \bar{\beta}(U) = F & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.6}$$

By the assumption $(A2)$, the function $F = \tau f_1 + u_0$ is an element of $L^\infty(\Omega)$ and the function $\bar{\beta}(s) = \tau\beta(s) + s$ is a non decreasing continuous real-valued function on $\mathbb{R}$, surjective, and is such that $\bar{\beta}(0) = 0$. Therefore, according to [7], problem (3.7) has a unique weak solution U_1 in $L^\infty(\Omega)\cap W^{1,p}(\Omega)$. By induction, we deduce in the same manner that the problem (P_n) has a unique weak solution $(U_n)_{0\le n\le N}$ such that $n = 1, \dots, N$, $U_n \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

3.2. **Stability results.** In this section, we prove some a priori estimates for the discrete weak solution $(U_n)_{1\le n\le N}$, which we use later to derive convergence results for the Euler forward scheme.

Theorem 3.2. *Under the assumptions* $(A1)$, $(A2)$, $(A3)$, *and* $1 < p < d$ *there exists a positive constant* $C(u_0, f, F)$ *depending on the data but not on* N *such that for all* $n = 1$, $\dots$, N, *we have*

$$\|U_n\|_\infty \le C\left(u_0, f, F\right),$$

$$\sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 \le C\left(u_0, f, F\right),$$

$$\tau\sum_{i=1}^n \int_\Omega \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx \le C\left(u_0, f, F\right).$$

Proof. **For** (3.8). Let $k > 0$ and $1 \le n \le N$, we have $U_n \in L^\infty(\Omega)$. Then, multiplying (P_n) by $|U_n|^k U_n$ and integrating over Ω, we obtain that

$$\int_\Omega |U_n|^{k+2} dx - \tau\int_\Omega \operatorname{div}\left(\Phi\left(\nabla U_n - \Theta\left(U_n\right)\right)\right)|U_n|^k U_n dx + \tau\int_\Omega \beta\left(U_n\right)|U_n|^k U_n dx$$
$$= \int_\Omega \left(\tau f_n + U_{n-1}\right)|U_n|^k U_n dx.$$

Using Holder's inequality, $(A1)$, $(A2)$, $(A3)$, and that $\Phi(\nabla U_i - \Theta(U_i)).\nabla U_i$ is monotone, we get

$$\|U_n\|_{k+2}^{k+2} \leq \tau c_1 \|U_n\|_{k+1}^{k+1} + \|U_{n-1}\|_{k+2} \|U_n\|_{k+2}^{k+1}. \tag{3.7}$$

Hence,

$$\|U_n\|_{k+2} \leq \tau c_1 \|U_n\|_{k+1}^{k+1} + \|U_{n-1}\|_{k+2}. \tag{3.8}$$

By simple induction, we get that

$$\|U_n\|_{k+2} \leq N c_2 T + \|U_0\|_{k+2}. \tag{3.9}$$

Finally, as $k \to \infty$, we obtain the result (3.8).

For (3.9). Let $1 \leq i \leq N$, replacing φ with U_i, as a test function in (3.6), we get

$$\int_\Omega (U_i - U_{i-1}) U_i dx + \tau \int_\Omega \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx + \tau \int_\Omega \beta(U_i) U_i dx = \int_\Omega \tau f_i U_i dx.$$

With the elementary identity,

$$a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2,$$

we get from (3.15) that

$$\frac{1}{2}\|U_i\|_2^2 - \frac{1}{2}\|U_{i-1}\|_2^2 + \frac{1}{2}\|U_i - U_{i-1}\|_2^2 + \tau \int_\Omega \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx \leq \tau c_3 \|U_i\|_2. \tag{3.10}$$

Now, we take the sum of (3.16) from $i = 1$ to n to get

$$\frac{1}{2}\|U_n\|_2^2 - \frac{1}{2}\|U_0\|_2^2 + \frac{1}{2}\sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 + \tau \sum_{i=1}^n \int_\Omega \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx \leq c_4. \tag{3.11}$$

Hence,

$$\frac{1}{2}\sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 + \tau \sum_{i=1}^n \int_\Omega \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx \leq c_4 + \frac{1}{2}\|U_0\|_2^2. \tag{3.12}$$

Thus

$$\frac{1}{2}\sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 + \tau \sum_{i=1}^n \int_\Omega \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx \leq c_5. \tag{3.13}$$

So

$$\frac{1}{2}\sum_{i=1}^n \|U_i - U_{i-1}\|_2^2 \leq c_5. \tag{3.14}$$

This implies the stability result (3.9).

For (3.10). In view of (3.19) and (3.9), we obtain the stability result (3.6). □

Theorem 3.3. *Let the assumptions* $(A1)$, $(A2)$, $(A3)$, *and* $1 < p < d$ *be satisfied. Then there exists a positive constant* $C(u_0, f, F)$ *depending on the data but not on* N *such that for all* $n = 1, \ldots, N$, *we have*

$$\tau \sum_{i=1}^n \|\beta(U_i)\|_1 \leq C(u_0, f, F),$$

$$\lim_{k \to 0} \sum_{i=1}^n \frac{\tau}{k} \int_{\{|U_i| \leq k\}} \Phi(\nabla U_i - \Theta(U_i)).\nabla U_i dx \leq C(u_0, f, F),$$

$$\sum_{i=1}^n \|U_i - U_{i-1}\|_1 \leq C(u_0, f, F).$$

Proof. **For** (3.21) **and** (3.22)**.** Replacing φ with $T_k(U_i)$ as a test function in (3.6), and dividing this equality by k, taking limits as k approaches 0, we get

$$\|U_i\|_1+\tau\|\beta(U_i)\|_1+\lim_{k\to 0}\frac{\tau}{k}\int_{\{|U_i|\le k\}}\Phi(\nabla U_i-\Theta(U_i)).\nabla U_i dx\le\tau\|f_i\|_1+\|U_{i-1}\|_1. \quad (3.15)$$

Summing (3.24) from $i=1$ to n, we conclude the stability results (3.21) and (3.22).

For (3.23)**.** Replacing φ with $T_\tau(U_i-U_{i-1})$ in (3.6) and dividing this equality by τ we get

$$\int_\Omega (U_i-U_{i-1})\frac{T_\tau(U_i-U_{i-1})}{\tau}dx+\int_{B_\tau^i}\Phi(\nabla U_i-\Theta(U_i)).(\nabla U_i-\nabla U_{i-1})dx$$
$$\le\tau\|\beta(U_i)\|_1+\tau\|f_i\|_1, \quad (3.16)$$

where $B_\tau^i=\{|U_i-U_{i-1}|\le\tau\}$. Applying Lemma 2.1, we obtain

$$\frac{1}{p}|\nabla U_i-\theta(U_i)|^p-\frac{1}{p}|\nabla U_{i-1}-\theta(U_i)|^p\le|\nabla U_i-\theta(U_i)|^{p-2}|\nabla U_i-\theta(U_i)|.(\nabla U_i-\nabla U_{i-1}).$$

Hence,

$$\int_\Omega (U_i-U_{i-1})\frac{T_\tau(U_i-U_{i-1})}{\tau}dx+\int_{B_\tau^i}(\frac{1}{p}|\nabla U_i-\theta(U_i)|^p-\frac{1}{p}|\nabla U_{i-1}-\theta(U_i)|^p)dx$$
$$\le\tau\|\beta(U_i)\|_1+\tau\|f_i\|_1.$$

Summing these inequalities from $i=1$ to n, using the stability result (3.21), we obtain

$$\sum_{i=1}^n\int_\Omega (U_i-U_{i-1})\frac{T_\tau(U_i-U_{i-1})}{\tau}dx\le\frac{1}{p}\int_\Omega|\nabla U_0|^p dx+c_6. \quad (3.17)$$

Then, we let τ tend to 0 in the inequality above to deduce the stability result (3.23). □

4. Convergence and existence results

In this section using the results obtained above, we construct a weak solution of problem (P) and we show that this solution is unique.

4.1. **Proof of Existence.** Let us introduce a piecewise linear extension, called a Rothe function, by

$$\begin{cases} u_N(0):=u_0, \\ u_N(t):=U_{n-1}+(U_n-U_{n-1})\frac{(t-t_{n-1})}{\tau},\ \forall t\in\left]t_{n-1},t_n\right],\ n=1,\ldots,N \quad \text{in } \Omega,\end{cases}$$

and a piecewise constant function

$$\begin{cases} \bar{u}_N(0):=u_0, \\ \bar{u}_N(t):=U_n \quad \forall t\in]t_{n-1},t_n], \quad n=1,\ldots,N \quad \text{in} \quad \Omega,\end{cases} \quad (4.18)$$

where $t_n:=n\tau$. As already shown, for any $N\in\mathbb{N}$, the solution $(U_n)_{1\le n\le N}$ of problems (P_n) is unique. Thus, u_N and $\bar{u}_N$ are uniquely defined and by construction, for any $t\in]t_{n-1},t_n]$, $n=1,\ldots,N$, we have that

i) $\frac{\partial u_N(t)}{\partial t}=\frac{(U_n-U_{n-1})}{\tau}$.
ii) $\bar{u}_N(t)-u_N(t)=(U_n-U_{n-1})\frac{t_n-t}{\tau}$.

From theorem 3.2, for any $N\in\mathbb{N}$, the solution $(U_n)_{1\le n\le N}$ of problems (3.5) is unique. Thus, u_N and $\bar{u}_N$ are uniquely defined. Using the stability results of Theorem 3.3 , we deduce the following a priori estimates concerning the Rothe function u_N and the function $\bar{u}_N$.

Lemma 4.1. *Under the assumptions* $(A1)$, $(A2)$, *and* $(A3)$, *there exists a positive constant* $C(T,u_0,f,F)$ *not depending on* N *such that for all* $N\in\mathbb{N}$, *we have*

$$\|\bar{u}_N-u_N\|^2_{L^2(Q_T)}\leq\frac{1}{N}C(T,u_0,f,F), \tag{4.19}$$

$$\|\bar{u}_N\|_{L^\infty(0,T,L^2(\Omega))}\leq C(T,u_0,f,F), \tag{4.20}$$

$$\|u_N\|_{L^\infty(0,T,L^2(\Omega))}\leq C(T,u_0,f,F), \tag{4.21}$$

$$\|\bar{u}_N\|_{L^p(0,T,W^{1,p}(\Omega))}\leq C(T,u_0,f,F), \tag{4.22}$$

$$\|\beta(\bar{u}_N)\|_{L^1(Q_T)}\leq C(T,u_0,f,F), \tag{4.23}$$

$$\left\|\frac{\partial u_N}{\partial t}\right\|^2_{L^2(Q_T)}\leq C(T,u_0,f,F). \tag{4.24}$$

Proof. **For** (4.29)**.** We have

$$\begin{aligned}\|\bar{u}_N-u_N\|^2_{L^2(Q_T)}&=\int_0^T\int_\Omega|\bar{u}_N-u_N|^2\,dxdt\\&\leq\sum_{i=1}^{i=N}\int_{t^{n-1}}^{t^n}\int_\Omega|U_n-U_{n-1}|^2\left(\frac{t_n-t}{\tau}\right)^2dxdt\leq\frac{1}{N}C(T,u_0,f,F).\end{aligned}$$

We follow the same techniques as above to show estimates (4.30), (4.31), (4.32) and (4.33).

For (4.34)**.** We have for $n=1,\dots,N$ and $t\in(t_{n-1},t_n]$ that

$$\frac{\partial u_N(t)}{\partial t}=\frac{(U_n-U_{n-1})}{\tau}.$$

This implies that

$$\left\|\frac{\partial u_N}{\partial t}\right\|_{L^1(Q_T)}=\int_0^T\int_\Omega\left|\frac{\partial u_N}{\partial t}\right|\mathrm{d}x\ \mathrm{d}t=\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\frac{1}{\tau}\|U_n-U_{n-1}\|_1=\sum_{n=1}^N\|U_n-U_{n-1}\|_1.$$

Using the result (3.23), we obtain estimate (4.34). This finishes the proof of Lemma 4.1.

Now, using the two results (4.30) and (4.31) of Lemma 4.1, the sequences $(u_N)_{N\in\mathbb{N}}$ and $(\bar{u}_N)_{N\in\mathbb{N}}$ are uniformly bounded in $L^\infty\left(0,T,L^2(\Omega)\right)$. Therefore, there exists two elements u and v in $L^\infty\left(0,T,L^2(\Omega)\right)$ such that

$$\begin{aligned}\bar{u}_N&\to^* u\quad\text{in}\quad L^\infty\left(0,T,L^2(\Omega)\right),\\u_N&\to^* v\quad\text{in}\quad L^\infty\left(0,T,L^2(\Omega)\right).\end{aligned}$$

And from the result (4.31) of Lemma 4.1, it follows that

$$u\equiv v.$$

Furthermore, by Lemma 4.1 and the assumption $(A2)$, we get that

$$\begin{aligned}\frac{\partial u_N}{\partial t}\to\frac{\partial u}{\partial t}&\quad\text{in}\quad L^2(Q_T),\\\bar{u}_N\to u&\quad\text{in}\quad L^p\left(0,T,W^{1,p}(\Omega)\right).\end{aligned}$$

From the assumption $(A1)$, we know that

$$\begin{aligned}&\beta(\bar{u}_N)\to\beta(u)\quad a.e.\ \text{in}\ Q_T,\\&|\beta(\bar{u}_N)|\leq M|\bar{u}_N|\in L^1(Q_T).\end{aligned}$$

Then, thanks to the Lebesgue dominated convergence theorem, we deduce that

$$\beta(\bar{u}_N)\to\beta(u)\quad\text{in}\quad L^1(Q_T). \tag{4.25}$$

On the other hand, since $\{\nabla\bar{u}_N-\Theta(\bar{u}_N)\}$ is equiintegrable by the assumption $(A3)$ and the boundedness of $(\bar{u}_N)$ it results that

$$\Phi(\nabla\bar{u}_N-\Theta(\bar{u}_N))\to\Phi(\nabla u-\Theta(u))\quad\text{weakly}\quad\text{in}\quad L^1(Q_T).$$

By reflexivity of $L^{p'}(\Omega)$ and boundedness of $\{\Phi(\nabla \bar{u}_N - \Theta(\bar{u}_N))\}$, we deduce that

$$\Phi(\nabla \bar{u}_N - \Theta(\bar{u}_N)) \to \Phi(\nabla u - \Theta(u)) \quad \text{weakly} \quad \text{in} \quad \left(L^{p'}(Q_T)\right)^d. \tag{4.26}$$

According to Lemma 4.1 and Aubin-Simons compactness result, we get that

$$u_N \to u \quad \text{in} \quad C\left(0, T, L^2(\Omega)\right). \tag{4.27}$$

Now, we show that the limit function u is a weak solution of problem (P). Firstly, we have $u_N(0) = U_0 = u_0$ for all $N \in \mathbb{N}$, then $u(0,.) = u_0$. Secondly, let $\varphi \in C^1(Q_T)$, we rewrite (2.4) in the form

$$\int_0^T \int_\Omega \frac{\partial u_N}{\partial t} \varphi dxdt + \int_0^T \int_\Omega \Phi(\nabla \bar{u}_N - \Theta(\bar{u}_N)).\nabla \varphi dxdt + \int_0^T \int_\Omega \beta(\bar{u}_N)\varphi dxdt = \int_0^T \int_\Omega f_N \varphi dxdt, \tag{4.28}$$

where

$$f_N(t,x) = f_n(x), \qquad \forall t \in]\, t_{n-1}, t_n], \quad n = 1, \ldots, N.$$

Taking limits as $N \to \infty$ in (4.40) and using the above results, we deduce that u is a weak solution of the nonlinear parabolic problem (P).

4.2. **Proof of Uniqueness.** We suppose that there exist two weak solutions u and v of the nonlinear parabolic problem (P), setting φ to $u - v$ as a test function for solution u in (2.4) and replacing φ with $v - u$ as a test function for solution v in (2.4), we obtain that

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}(u-v)dxdt + \int_0^T \int_\Omega \Phi(\nabla u - \Theta(u)).\nabla(u-v)dxdt + \int_0^T \int_\Omega \beta(u)(u-v)dxdt = \int_0^T \int_\Omega f(u-v)dxdt,$$

and

$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}(v-u)dxdt + \int_0^T \int_\Omega \Phi(\nabla v - \Theta(v)).\nabla(v-u)dxdt + \int_0^T \int_\Omega \beta(v)(v-u)dxdt = \int_0^T \int_\Omega f(v-u)dxdt.$$

By summing up the two above equalities, we get

$$\int_0^T \int_\Omega \frac{\partial (u-v)}{\partial t}(u-v)dxdt + \int_0^T \int_\Omega (\Phi(\nabla u - \Theta(u)) - \Phi(\nabla v - \Theta(v))).\nabla(u-v)dxdt + \int_0^T \int_\Omega (\beta(u) - \beta(v))(u-v)dxdt = 0.$$

Using assumption $(A1)$, $(A3)$ and the fact that $\Phi(\nabla u - \Theta(u))$ is monotone, we deduce that

$$u \equiv v. \qquad \square$$

References

[1] R.A., Adams, Sobolev Spaces. Academic Press. 1975.

[2] F. Andreu, N. Igbida, J. M. Mazón, and J.Toledo, L^1 existence and uniqueness results for quasilinear elliptic equations with nonlinear boundary conditions, Ann. Ins. H. Poincaré Anal. Non Linéaire, 20 (2007), 61-89.

[3] E. Azroul, M.B. Benboubker, S. Ouaro, Entropy solutions for nonlinear nonhomogeneous Neumann problems involving the generalized $p(x)$-Laplace operator. J. Appl. Anal. Comput. 3(2), 105-121 (2013).

[4] E. Azroul, B. Farah, A weak solution to quasilinear elliptic problems with perturbed gradient. Rendiconti del Circolo Matematico di Palermo Series (2020).

[5] P.Bènilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, , J.-L. Vasquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa 22, 241–273 (1995).

[6] F. Benzekri, A. El Hachimi, Doubly nonlinear parabolic equations related to the p-Laplacian operator: Semi-discretization. Electron. J. Differential Equations.113(2003), 1-14(2003).

[7] B.K. Bonzi, I. Nyanquini, S. Ouaro, Existence and uniqueness of weak and entropy solutions for homogeneous neumann boundary-value problems involving variable exponents. Electron. J. Differential Equations. 12(2012), 1-19(2012).

[8] H. Brezis, Analyse fonctionnelle. Thèorie et applications. Masson, Paris, 1983.

[9] J.I. Diaz, F. de Thelin, On a nonlinear parabolic problem arising in some models related to turbulent flows. SIAM J. Math. Anal. 25(1994), 10085-1111(1994).

[10] R.-J. DiPerna, P.-L. Lions, On the Cauchy problem for Boltzmann equations: global existence and weak stability. Ann. Math. 130, 321-366 (1989).

[11] J. Droniou, A. Prignet, Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data. Nonlinear Differ. Equ. Appl. 14(12), 181-205 (2007).

[12] A. Eden, , B. Michaux, J.M. Rakotoson, Semi-discretized nonlinear evolution equations as dynamical systems and error analysis. Indiana Univ. Math. J. 39(1990), 737-783(1990).

[13] A. El Hachimi, J. Igbida, Bounded weak solutions to nonlinear elliptic equations. Electronic Journal of Qualitative Theory of Differential Equations (ejqtde). 2009(10), 1-16.

[14] A. El Hachimi, M.R. Sidi Ammi, Thermistor problem: a nonlocal parabolic problem. Electron. J. Differential Equations. 11(2004), 117-128(2004).

[15] A. El Hachimi, H. El Ouardi, Existence and regularity of a global attractor for doubly nonlinear parabolic equations(EJDE).45 (2002), 1-15(2002).

[16] N. Elharrar, J. Igbida, H. Talibi, $p(\cdot)$-Laplacian problem with nonlinear singular terms. Rend. Circ. Mat. Palermo, II. Ser 71, 105–118 (2022).

[17] L. Essafi, S.Hadj Nassar, M. Mohamed Rhoudaf, Rothe time-discretization method for a nonlinear parabolic problem in Orlicz-Sobolev spaces. Journal of Elliptic and Parabolic Equations.5(2009), 69-104(2019).

[18] J. Igbida, Nonlinear elliptic equations with divergence term and without sign condition. Ann. Univ. Craiova Math. Comput. Sci. Ser. 38(2), 7–17 (2011).

[19] S. Jaques, Compact sets in the space $L^p(0;T;B)$. Ann. Math. Pura Appl. 146, 65–96 (1987).

[20] J. Kacur, Method of Rothe in Evolution Equations. Institute of Applied Mathematics, Bratislava (1986).

[21] C. Kouraichi, A. Siai, Equivalence between entropy and renormalized solutions for parabolic equations. Indagat. Math. 26, 679-696 (2015).

[22] J.L. Lions, Quelques Mèthodes de Rèsolution des Problèmes aux Limites nonlinèaires. Dunod, Paris, 1969.

[23] N. Merazga, A. Bouziani, Rothe method for a mixed problem with an integral condition for the two-dimensional diffusion equation, Abstr. Appl. Anal. 2003 (2003), no. 16, 899-922.

[24] K.R. Rajagopal, M. Ruzicka, Mathematical modeling of electrorheological materials. Contin. Mech. Thermodyn. 13, 59-78 (2001).

[25] K. Rektorys, The Method of Discretization in Time and Partial Differential Equations, vol. 4 of Mathematics and Its Applications (East European Series), D. Reidel, Dordrecht, The Netherlands. View at MathSciNet (1982).

[26] Ruzicka, M., : Electrorheological fluids. modeling and mathematical theory, lecture notes in mathematics. Springer, Berlin (2000).

[27] K. Shin, S. Kang, Doubly Nonlinear Parabolic Equations Involving p-Laplacian Operators Via Time-Discretization Method. Bull. Korean Math. Soc. 49 (2012), No. 6, pp. 1179-1192.

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