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ANALYTICAL STUDY OF SOME SELF-REFERRED OR STATE DEPENDENT FUNCTIONAL EQUATIONS

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Abstract. In this paper, we consider some self reference or state dependent functional equations. Also, we shall discuses a continuous dependence of the solutions of that functional equations on the delay functions.

Розглянуто деякі самопосилаючі або залежні за станом функціональні рівняння, та неперервна залежність розв'язків цього функціонального рівняння від функції затримки.

1. Introduction and preliminaries

Differential equations with state-dependent delays attract interests of specialists since they widely arise in application models such as two-body problem of classical electrodynamics, position control, mechanical models, infection disease transmission, population models, the dynamics of economical systems, etc. This type of differential equations that have been called self-referred and hereditary has been studied in [9] and [23]–[25].

Let $I = [0, b]$, and $C(I)$ consist of all real-valued functions defined and continuous on I with values in $\mathbb{R}$ and endowed with the norm

$$||x|| = \max\{|x(t)| : t \in I\}.$$

Functional equations have been studied in several papers and monographs (see for examples [3]–[6], [11, 12] and [13]). Banaś [3] proved existence of a monotone integrable solution for the functional equation

$$x(t) = f(t, x(\phi(t))), \qquad t \in [0, 1],$$

under certain monotonicity condition by using the technique of measure of noncompactness.

In most of the differential and integral equations with deviating arguments that appear in many literature, the deviation of the argument usually involves only the time itself, However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are devoted to such a kind of differential equations, see references therein.

One of the first papers studying this class of functional equations is the one by Eder [10] who considered the functional differential equation

$$x'(t) = x(x(t)), \qquad t \in A \subset \mathbb{R},$$

while Fečkan [20] studied a functional differential equation of the form

$$x'(t) = f(x(x(t))), \tag{1.1}$$

with $f \in C^1(\mathbb{R})$ by applying the Leray-Schauder theorem. Also, the existence of a Picard iteration method enabled him to approximate a solution of (1.1).

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This kind of problems has been studied by Coman Pravel [8], Oberg [7] and [26]. In [7] existence, uniqueness, and continuous dependence theorems were proved for the differential equation

$$x'(t) = f(t, x(x(t)))$$

with a certain initial condition. Nguyen Minh Tuan et al [28] presented local and global solutions of a system of hereditary and self-referred partial-differential equations.

Here, we shall study existence of a unique solution for some functional equations with state-dependent deviated arguments

$$x(t) = f(t, x(x(\phi(t)))), \qquad t \in I, \tag{1.2}$$

$$x(t) = f(t, x(g(t, x(t)))), \qquad t \in I, \tag{1.3}$$

and

$$x(t) = f(t, x(x(\phi(t))), x(g(t, x(t)))), \qquad t \in I. \tag{1.4}$$

The present paper is organized as follows. In Section 1, we introduce some preliminaries and a brief survey on differential equation with deviating argument. In Section 2, we prove some existence theorems of continuous solution for the functional equations (1.2) and (1.3). In Section 3, we discuss continuous dependence of the state argument on each other.

The existence results will be based on the contraction mapping principle.

In what follows we will need the following property of the superposition operator in the space $C[a, b]$.

Lemma 1.1. [2] *Assume that F is the superposition operator generated by the function $f : [a, b] \times R \to R$. Then F transform the space $C[a, b]$ into itself and is continuous if and only if the function f is continuous on the set $[a, b] \times R$.*

2. Solvability of the functional equation (1.2)

2.1. **Existence Theorem.** Now, equation (1.2) will be investigated under the following assumptions:

(i) $f : I \times I \to \mathbb{R}$ is continuous.

(ii) There exists a positive constant k such that

$$|f(t, x) - f(s, y)| \leq k(|t - s| + |x - y|) \qquad \forall\, t, s \in I \quad \text{and} \quad x, y \in I.$$

(iii) $\phi : I \to I$ satisfies $\phi(0) = 0$ and

$$|\phi(t) - \phi(s)| \leq |t - s| \qquad \forall\, t, s \in I.$$

Define a subset S_L of $C(I)$ by

$$S_L = \{x \in C(I) : |x(t) - x(s)| \leq L|t - s|,\ L > 0,\ t, s \in I\},$$

and define the operator

$$F_1 x(t) = f(t, x(x(\phi(t)))), \qquad t \in I.$$

For $x \in S_L$, we have

$$\begin{aligned} |x(t) - x(0)| &\leq L|t - 0| \\ \Rightarrow |x(t)| &\leq |x(0)| + L|t| \\ &\leq |x(0)| + Lb. \end{aligned}$$

If we choose $L = \frac{b - |x(0)|}{b}$, then $|x(t)| \leq b$ and $L = \frac{b-|x(0)|}{b} \leq 1$. Therefore, $x : I \to I \Rightarrow x(x(t)) = x \circ x(t) : I \to I$,

Theorem 2.1. *Let assumptions (i)–(iii) be satisfied. Furthermore, if $k + kL < 1$, then the functional equation (1.2) has a unique solution $x \in C(I)$.*

Proof. Observe that in view of assumptions (i)–(iii), $F_1 : S_L \to C(I)$ is a continuous operator in $x \in S_L$. Now for $t_1, t_2 \in I$ such that $|t_2 - t_1| < \delta$ and $x \in S_L$, we have

$$\begin{aligned}
|(F_1x)(t_2) - (F_1x)(t_1)| &= |f(t_2, x(x(\phi(t_2)))) - f(t_1, x(x(\phi(t_1))))| \\
&\leq k[|t_2 - t_1| + |x(x(\phi(t_2))) - x(x(\phi(t_1)))|] \\
&\leq k[|t_2 - t_1| + L|x(\phi(t_2)) - x(\phi(t_1))|] \\
&\leq k[|t_2 - t_1| + L^2|\phi(t_2) - \phi(t_1)|] \\
&\leq k[|t_2 - t_1| + L^2|t_2 - t_1|] \\
&\leq (k + kL^2 + kL - kL)|t_2 - t_1| \\
&\leq (1 + kL^2 - kL)|t_2 - t_1| \\
&\leq (1 + kL(L - 1))|t_2 - t_1| \\
&\leq (1 + L - 1)|t_2 - t_1| \\
&\leq L|t_2 - t_1|.
\end{aligned}$$

Then $F_1 : S_L \to S_L$. Now, for $x, y \in S_L$,

$$\begin{aligned}
|(F_1x)(t) - (F_1y)(t)| &= |f(t, x(x(\phi(t)))) - f(t, y(y(\phi(t))))| \\
&= |f(t, x(x(\phi(t)))) - f(t, y(y(\phi(t)))) \\
&\qquad + f(t, x(y(\phi(t)))) - f(t, x(y(\phi(t))))| \\
&\leq |f(t, x(x(\phi(t)))) - f(t, x(y(\phi(t))))| \\
&\qquad + |f(t, y(y(\phi(t)))) - f(t, x(y(\phi(t))))| \\
&\leq kL|x(\phi(t)) - y(\phi(t))| + k|x(y(\phi(t))) - y(y(\phi(t)))|. \\
&\leq kL|x(t) - y(t)| + k|x(t) - y(t)|,
\end{aligned}$$

so that

$$||F_1x - F_1y|| \leq kL||x - y|| + k||x - y|| = (kL + k)||x - y||.$$

Since $kL + k < 1$, F_1 is a contraction mapping and hence there exists a unique solution $x \in S_L \subset C(I)$. □

2.2. **Continuous dependence of solutions of (1.2) on the function ϕ.** Here, we shall study the continuous dependence of solutions of the self referred functional equations (1.2) on the function ϕ.

Definition 2.2. The solutions $x \in C(I)$ of (1.2) is continuously dependent on the function ϕ if $\forall \epsilon > 0, \exists \delta > 0$ such that $||\phi - \tilde{\phi}|| \leq \epsilon$ implies that $||x - \tilde{x}|| \leq \delta$, where $\tilde{x}(t) = f(t, \tilde{x}(\tilde{x}(\tilde{\phi}(t)))), t \in I$.

Theorem 2.3. *Let the assumptions of Theorem 2.1 be satisfied. Then solutions $x \in C(I)$ of the functional equation (1.2) depend continuously on ϕ.*

Proof. Let x and $\tilde{x}$ be two solutions of the state dependent functional equation (1.2). Then

$$\begin{aligned}
|x(t) - \tilde{x}(t)| &= |f(t, x(x(\phi(t)))) - f(t, \tilde{x}(\tilde{x}(\tilde{\phi}(t))))| \\
&= |f(t, x(x(\phi(t)))) - f(t, \tilde{x}(x(\phi(t)))) + f(t, \tilde{x}(x(\phi(t)))) - f(t, \tilde{x}(\tilde{x}(\tilde{\phi}(t))))| \\
&\leq k|x(t) - \tilde{x}(t)| + kL|x(\phi(t)) - \tilde{x}(\tilde{\phi}(t))| \\
&\leq k|x(t) - \tilde{x}(t)| + kL|x(\phi(t)) - \tilde{x}(\phi(t)) - \tilde{x}(\phi(t)) - \tilde{x}(\tilde{\phi}(t))| \\
&\leq k|x(t) - \tilde{x}(t)| + kL(|x(t) - \tilde{x}(t)| + \epsilon_1),
\end{aligned}$$

so that

$$(1-(k+kL))||x-\tilde{x}|| \leq kL\epsilon_1,$$
$$||x-\tilde{x}|| \leq \frac{kL\epsilon_1}{1-(k+kL)} = \epsilon.$$

Since $kL+k<1$, the solutions $x \in C(I)$ of (1.2) are continuously dependent on the function ϕ. □

3. Solvability of the functional equation (1.3)

3.1. **Existence Theorem.** Now, equation (1.3) will be investigated under the following assumptions:

(i^*) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous.

(ii^*) There exists a positive constant k such that

$$|f(t,x)-f(s,y)| \leq k(|t-s|+|x-y|) \qquad \forall\, t,s \in I \text{ and } x,y \in \mathbb{R}.$$

(iii^*) $g : I \times \mathbb{R} \to I$ is continuous and satisfies the Lipschitz condition

$$|g(t,x)-g(s,y)| \leq k_1(|t-s|+|x-y|) \qquad \forall\, t,s \in I \text{ and } x,y \in \mathbb{R}.$$

(iv^*) $1+L^2kk_1 \leq L$.

Define an operator F_2 by

$$F_2x(t) = f(t, x(g(t,x(t)))), \qquad t \in I.$$

Theorem 3.1. *Let assumptions* (i)–(iii) *be satisfied. Furthermore, if* $k(1+Lk_1)<1$, *then the functional equation (1.3) has a unique solution* $x \in C(I)$.

Proof. Observe that in view of assumptions (i^*)-(iv^*), $F_2 : S_L \to C(I)$ is a continuous operator in $x \in S_L$.

Now, for $t_1, t_2 \in I$ such that $|t_2-t_1|<\delta$ and $x \in S_L$, we have

$$\begin{aligned}
|(F_2x)(t_2)-(F_2x)(t_1)| &= |f(t_2, x(g(t_2,x(t_2)))) - f(t_1, x(g(t_1,x(t_1))))| \\
&\leq k[|t_2-t_1| + |x(g(t_2,x(t_2))) - x(g(t_1,x(t_1)))|] \\
&\leq k[|t_2-t_1| + L|g(t_2,x(t_2)) - g(t_1,x(t_1))|] \\
&\leq k[|t_2-t_1| + Lk_1|x(t_2)-x(t_1)| + Lk_1|t_2-t_1|] \\
&\leq k[|t_2-t_1| + L^2k_1|t_2-t_1| + Lk_1|t_2-t_1|] \\
&\leq [k+Lkk_1+L^2kk_1]|t_2-t_1| \\
&\leq [1+L^2kk_1]|t_2-t_1| \\
&\leq L|t_2-t_1|.
\end{aligned}$$

Then $F_1 : S_L \to S_L$. Now, for $x,y \in S_L$,

$$\begin{aligned}
|(F_2x)(t)-(F_2y)(t)| &= |f(t,x(g(t,x(t)))) - f(t,y(g(t,y(t))))| \\
&= |f(t,x(g(t,x(t)))) - f(t,x(g(t,y(t)))) \\
&\qquad + f(t,x(g(t,y(t)))) - f(t,y(g(t,y(t))))| \\
&\leq kL|g(t,x(t)) - g(t,y(t))| + k|x(g(t,y(t)) - y(g(t,y(t))| \\
&\leq kLk_1|x(t)-y(t)| + k|x(t)-y(t)| \\
&\leq [kLk_1+k]|x(t)-y(t)| \\
&\leq k(Lk_1+1)|x(t)-y(t)|.
\end{aligned}$$

Then

$$||F_2x-F_2y|| \leq k(Lk_1+1)||x-y||.$$

Since $k(1+Lk_1)<1$, F_2 is a contraction mapping and hence there exists a unique solution $x\in S_L\subset C(I)$. □

3.2. **Continuous dependence of solutions of (1.3) on the function g.** Here, we shall study the continuous dependence of solutions of the self referred functional equations (1.3) on the function g.

Definition 3.2. A solution $x\in C(I)$ of (1.3) is continuously dependent on the function g if $\forall\epsilon>0,\exists\delta>0$ such that $||g-\tilde{g}||\leq\epsilon$ implies that $||x-\tilde{x}||\leq\delta$, where $\tilde{x}(t)=f(t,\tilde{x}(\tilde{g}(t,\tilde{x}(t))))$, $t\in I$

Theorem 3.3. *Let the assumptions of Theorem 3.1 be satisfied. Then solutions $x\in C(I)$ of the functional equation (1.3) depend continuously on g.*

Proof. Let x and $\tilde{x}$ be two solutions of the state dependent functional equation (1.3). Then

$$\begin{aligned}|x(t)-\tilde{x}(t)| &= |f(t,x(g(t,x(t))))-f(t,\tilde{x}(\tilde{g}(t,\tilde{x}(t))))|\\ &= |f(t,x(g(t,x(t))))-f(t,x(g(t,\tilde{x}(t))))\\ &\qquad +f(t,x(g(t,\tilde{x}(t))))-f(t,\tilde{x}(\tilde{g}(t,\tilde{x}(t))))|\\ &\leq kL|g(t,x(t))-\tilde{g}(t,\tilde{x}(t))|+k|\tilde{x}(\tilde{g}(t,\tilde{x}(t)))-\tilde{x}(g(t,\tilde{x}(t)))\\ &\qquad +\tilde{x}(g(t,\tilde{x}(t)))-x(g(t,\tilde{x}(t)))|\\ &\leq k|x(t)-\tilde{x}(t)|+kLk_1|x(t)-\tilde{x}(t)|+kL\delta.\end{aligned}$$

Hence,

$$(1-k-kLk_1)||x-\tilde{x}||\leq kL\delta,$$
$$||x-\tilde{x}||\leq\frac{kL\delta}{1-k-kLk_1}=\epsilon.$$

Since $k(1+Lk_1)<1$, the solutions $x\in C(I)$ of (1.3) are continuously dependent on the function g. □

4. Solvability of functional equation (1.4) with multi-state dependent deviated arguments

4.1. **Existence Theorem.** Now, equation (1.4) will be investigated under te following assumptions.

($\mathfrak{I}$) $f:I\times I\times\mathbb{R}\to\mathbb{R}$ is continuous.

($\mathfrak{II}$) There exists a positive constant k such that

$$|f(t,x,u)-f(s,y,v)|\leq k(|t-s|+|x-y|+|u-v|)\qquad\forall\,t,s,x,y\in I\text{ and }u,v\in\mathbb{R}.$$

($\mathfrak{III}$) $g:I\times\mathbb{R}\to I$ is continuous and satisfies the Lipschitz condition

$$|g(t,x)-g(s,y)|\leq k_1(|t-s|+|x-y|)\qquad\forall\,t,s\in I\text{ and }x,y\in\mathbb{R}.$$

($\mathfrak{IV}$) $\phi:I\to I$ satisfies $\phi(0)=0$ and

$$|\phi(t)-\phi(s)|\leq|t-s|\qquad\forall\,t,s\in I.$$

($\mathfrak{V}$) $k(1+Lk_1+L^2+L^2k_1)\leq L$.

Define an operator F_3 by

$$F_3x(t)=f(t,x(x(\phi(t))),x(g(t,x(t)))),t\in I.$$

Theorem 4.1. *Let assumptions ($\mathfrak{I}$)–($\mathfrak{V}$) be satisfied. Furthermore, if $k(2+Lk_1+L)<1$, then the functional equation (1.4) has a unique solution $x\in C(I)$.*

Proof. Observe that in view of assumptions $(\mathfrak{I})$-$(\mathfrak{V})$, $F_3 : S_L \to C(I)$ is a continuous operator in $x \in S_L$. Now for $t_1, t_2 \in I$ such that $|t_2 - t_1| < \delta$ and $x \in S_L$, we have

$$\begin{aligned}
|(F_3x)(t_2) - (F_3x)(t_1)| &= |f(t_2, x(x(\phi(t_2))), x(g(t_2, x(t_2))))\\
&\qquad - f(t_1, x(x(\phi(t_1))), x(g(t_1, x(t_1))))|\\
&\le k[|t_2 - t_1| + |x(x(\phi(t_2))) - x(x(\phi(t_1)))|\\
&\qquad + |x(g(t_2, x(t_2))) - x(g(t_1, x(t_1)))|]\\
&\le k[|t_2 - t_1| + L^2|t_2 - t_1| + Lk_1|t_2 - t_1| + L^2k_1|t_2 - t_1|]\\
&\le k[1 + L^2 + Lk_1 + L^2k_1]|t_2 - t_1|\\
&\le L|t_2 - t_1|.
\end{aligned}$$

Then $F_1 : S_L \to S_L$. Now, for $x, y \in S_L$,

$$\begin{aligned}
|(F_3x)(t) - (F_3y)(t)| &= |f(t, x(x(\phi(t))), x(g(t, x(t)))) - f(t, y(y(\phi(t))), y(g(t, y(t))))|\\
&\le k(|x(x(\phi(t))) - y(y(\phi(t)))| + |x(g(t, x(t))) - y(g(t, y(t)))|)\\
&\le k(|x(x(\phi(t))) - x(y(\phi(t)))| + |x(y(\phi(t))) - y(y(\phi(t)))|\\
&\qquad + |x(g(t, x(t))) - x(g(t, y(t)))| + |x(g(t, y(t))) - y(g(t, y(t)))|)\\
&\le k(L|x(\phi(t)) - y(\phi(t))| + |x(y(\phi(t))) - y(y(\phi(t)))|\\
&\qquad + Lk_1|x(t) - y(t)| + |x(g(t, y(t))) - y(g(t, y(t)))|)\\
&\le k(L + 2 + Lk_1)|x(t) - y(t)|\\
&\le k(L + 2 + Lk_1)|x(t) - y(t)|.
\end{aligned}$$

Thus

$$||F_3x - F_3y|| \le k(L + 2 + Lk_1)||x - y||.$$

Since $k(L + 2 + Lk_1) < 1$, F_3 is a contraction mapping and hence there exists a unique solution $x \in S_L \subset C(I)$. □

4.2. **Continuous dependence of solutions of (1.4) on the function g.** Here, we shall study continuous dependence of solutions of the self referred functional equations (1.4) on the function g.

Definition 4.2. A solution $x \in C(I)$ of (1.4) is continuously dependent on the function g if $\forall \epsilon > 0, \exists \delta > 0$ such that $||g - \tilde{g}|| \le \epsilon$ implies that $||x - \tilde{x}|| \le \delta$, where $\tilde{x}(t) = f(t, \tilde{x}(\tilde{x}(\phi(t))), \tilde{x}(\tilde{g}(t, \tilde{x}(t))))$, $t \in I$.

Theorem 4.3. *Let the assumptions of Theorem 4.1 be satisfied. Then solutions $x \in C(I)$ of the functional equation (1.4) depend continuously on g.*

Proof. Let x and $\tilde{x}$ be two solutions of the state dependent functional equation (1.4). We have

$$\begin{aligned}
|x(t) - \tilde{x}(t)| &= |f(t, x(x(\phi(t))), x(g(t, x(t)))) - f(t, \tilde{x}(\tilde{x}(\phi(t))), \tilde{x}(\tilde{g}(t, \tilde{x}(t))))|\\
&= |f(t, x(x(\phi(t))), x(g(t, x(t)))) - f(t, x(\tilde{x}(\phi(t))), x(g(t, \tilde{x}(t))))\\
&\qquad + f(t, x(\tilde{x}(\phi(t))), x(g(t, \tilde{x}(t)))) - f(t, \tilde{x}(\tilde{x}(\phi(t))), \tilde{x}(\tilde{g}(t, \tilde{x}(t))))|\\
&\le kL|g(t, x(t)) - \tilde{g}(t, \tilde{x}(t))| + kL|x(\phi(t)) - \tilde{x}(\phi(t))| + k|x(t) - \tilde{x}(t)|\\
&\qquad + k|\tilde{x}(\tilde{g}(t, \tilde{x}(t))) - \tilde{x}(g(t, \tilde{x}(t))) + \tilde{x}(g(t, \tilde{x}(t))) - x(g(t, \tilde{x}(t)))|\\
&\le kL|g(t, x(t)) - \tilde{g}(t, x(t)) + \tilde{g}(t, x(t)) - \tilde{g}(t, \tilde{x}(t))|\\
&\qquad + kL|x(\phi(t)) - \tilde{x}(\phi(t))| + k|x(t) - \tilde{x}(t)| + k|\tilde{x}(\tilde{g}(t, \tilde{x}(t))) - \tilde{x}(g(t, \tilde{x}(t)))\\
&\qquad + \tilde{x}(g(t, \tilde{x}(t))) - x(g(t, \tilde{x}(t)))|\\
&\le 2kL\delta + kLk_1|x(t) - \tilde{x}(t)| + kL|x(t) - \tilde{x}(t)| + 2k|x(t) - \tilde{x}(t)|.
\end{aligned}$$

Hencce,

$$(1-2k-kL-kLk_1)||x-\tilde{x}|| \leq 2kL\delta$$
$$||x-\tilde{x}|| \leq \frac{2kL\delta}{1-2k-kL-kLk_1} = \epsilon.$$

Since $k(2+L+Lk_1) < 1$, solutions $x \in C(I)$ of (1.4) are continuously dependent on the function g. □

4.3. **Continuous dependence of solutions of (1.4) on the function** ϕ**.** Here, we shall study continuous dependence of solutions of the self referred functional equations (1.4) on the function ϕ.

Definition 4.4. A solution $x \in C(I)$ of (1.4) is continuously dependent on the function ϕ if $\forall \epsilon > 0, \exists \delta > 0$ such that $||\phi - \tilde{\phi}|| \leq \epsilon$ implies that $||x-\tilde{x}|| \leq \delta$, where $\tilde{x}(t) = f(t, \tilde{x}(\tilde{x}(\tilde{\phi}(t))), \tilde{x}(g(t,\tilde{x}(t))))$, $t \in I$.

Theorem 4.5. *Let the assumptions of Theorem 4.1 be satisfied. Then solutions $x \in C(I)$ of the functional equation (1.2) depend continuously on ϕ.*

Proof. Let x and $\tilde{x}$ be two solutions of the state dependent functional equation (1.2). Then

$$\begin{aligned}
|x(t)-\tilde{x}(t)| &= |f(t, x(x(\phi(t))), x(g(t,x(t)))) - f(t, \tilde{x}(\tilde{x}(\tilde{\phi}(t))), \tilde{x}(g(t,\tilde{x}(t))))| \\
&= |f(t, x(x(\phi(t))), x(g(t,x(t)))) - f(t, \tilde{x}(x(\phi(t))), \tilde{x}(g(t,x(t)))) \\
&\quad + f(t, \tilde{x}(x(\phi(t))), \tilde{x}(g(t,x(t)))) - f(t, \tilde{x}(\tilde{x}(\tilde{\phi}(t))), \tilde{x}(g(t,\tilde{x}(t))))| \\
&\leq 2k|x-\tilde{x}| + kL|x(\phi(t)) - \tilde{x}(\tilde{\phi}(t))| + kLk_1|x(t)-\tilde{x}(t)| \\
&\leq 2k|x-\tilde{x}| + kL|x(\phi(t)) - \tilde{x}(\phi(t)) - \tilde{x}(\phi(t)) - \tilde{x}(\tilde{\phi}(t))| + kLk_1|x(t)-\tilde{x}(t)| \\
&\leq 2k|x-\tilde{x}| + kL(|x(t)-\tilde{x}(t)| + \epsilon_1) + kLk_1|x(t)-\tilde{x}(t)|.
\end{aligned}$$

Hence,

$$(1-(2k+kL+kLk_1))||x-\tilde{x}|| \leq kL\epsilon_1$$
$$||x-\tilde{x}|| \leq \frac{kL\epsilon_1}{1-(2k+kL+kLk_1)} = \epsilon.$$

Since $2k+kL+kLk_1 < 1$, solutions $x \in C(I)$ of (1.4) are continuously dependent on the function ϕ. □

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