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EXISTENCE OF SOLUTIONS FOR NONLINEAR INTEGRO-DYNAMIC EQUATIONS WITH MIXED PERTURBATIONS OF THE SECOND TYPE VIA KRASNOSELSKII'S FIXED POINT THEOREM

ABDERRAHIM GUERFI AND ABDELOUAHEB ARDJOUNI

Abstract. We prove the existence of solutions of a nonlinear integro-dynamic equation with mixed perturbations of the second type on time scales. The main tool employed here is Krasnoselskii's fixed point theorem. An example is given to illustrate the main results.

Доведено існування розв'язків нелінійного інтегродинамічного рівняння зі змішаними збуреннями за часовою шкалою другого типу. Основним використаним інструментом є теорема Красносельського про нерухому точку. Наведено приклад для ілюстрації основних результатів.

1. Introduction

A time scale is an arbitrary nonempty closed subset of real numbers. It combines the traditional areas of continuous and discrete analysis into one theory. This concept is a fairly new idea and was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [7]. Next, Bohner and Peterson [4] and [5] published two textbooks in this area, more and more researchers were getting involved in this fast-growing field of mathematics. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1, 2, 3, 4, 5, 6, 7, 8], [10, 11, 12] and papers therein).

Dhage [6] discussed the following first order hybrid differential equation with mixed perturbations of the second type

$$\begin{cases} \frac{d}{dt}\left[\frac{u(t)-f(t,u(t))}{g(t,u(t))}\right] = h\left(t,u\left(t\right)\right), \ t\in\left[t_0,t_0+a\right], \\ u\left(t_0\right)=u_0\in\mathbb{R}, \end{cases}$$

where $t_0, a \in \mathbb{R}$ with $a>0$, $g:[t_0,t_0+a]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $f,h:[t_0,t_0+a]\times\mathbb{R}\to\mathbb{R}$ are continuous functions. The author developed a theory of hybrid differential equations with mixed perturbations of the second type and gave some original and interesting results.

Let $\mathbb{T}$ be a time scale and $J=[t_0,t_0+a]_{\mathbb{T}}=[t_0,t_0+a]\cap\mathbb{T}$ be a bounded interval in $\mathbb{T}$ for some $t_0,a\in\mathbb{R}$ with $a>0$. Let $C_{rd}(J\times\mathbb{R},\mathbb{R})$ denote the class of rd-continuous functions $f:J\times\mathbb{R}\to\mathbb{R}$. In [11], Zhao et al. discussed the following dynamic equation with mixed perturbations of the second type on time scales:

$$\begin{cases} \left[\frac{u(t)-f(t,u(t))}{g(t,u(t))}\right]^{\Delta} = h\left(t,u\left(t\right)\right), \ t\in J, \\ u\left(t_0\right)=u_0, \end{cases}$$

where $g\in C_{rd}(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $f,h\in C_{rd}(J\times\mathbb{R},\mathbb{R})$. The authors established an existence theorem for the dynamic equation under mixed Lipschitz and Carathéodory conditions using a fixed point theorem in Banach algebra due to Dhage.

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In this paper, we discuss the existence of solutions for the following nonlinear integro-dynamic equation with mixed perturbations of the second type

$$\begin{cases} \left(\frac{u(t)-f(t,u(t))}{p(t)+\int_0^t g(s,u(s))\Delta s}\right)^{\Delta} = h\left(t,u\left(t\right)\right), \ t\in[0,T]_{\mathbb{T}}, \\ u\left(0\right) = f\left(0,u\left(0\right)\right)+p\left(0\right)\theta, \end{cases} \tag{1.1}$$

where $\theta \in \mathbb{R}$, $g, f, h \in C_{rd}([0,T]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ and $p \in C_{rd}\left([0,T]_{\mathbb{T}}, \mathbb{R}\backslash \{0\}\right)$. To show the existence of solutions, we transform (1.1) into an equivalent integral equation and then use Krasnoselskii's fixed point theorem. The mixed perturbation in (1.1) is the second type because it is under derivative.

This paper is organized as follows. In Section 2, we introduce some notations and definitions, and state some preliminary material needed in later sections. Also, we present Krasnoselskii's fixed point theorem. In Sections 3, we give and prove our main results on the existence. Finally, we provide an example to illustrate our obtained results.

2. Preliminaries

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator σ and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. These operators allow the elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T}\backslash \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}^k = \mathbb{T}\backslash \{m\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1 ([4])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}\left(\mathbb{T}\right) = C_{rd}\left(\mathbb{T}, \mathbb{R}\right).$$

Definition 2.2 ([4])**.** For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)\left[\sigma(t) - s\right]\right| < \varepsilon \left|\sigma(t) - s\right| \text{ for all } s \in U.$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of f on $\mathbb{T}^k$.

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

We are now ready to state some properties of the delta-derivative of f. Note that $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.3 ([4])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar. We have*

(i) $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.
(ii) $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$.
(iii) *The product rules*

$$\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t), \\ (fg)^{\Delta}(t) &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t). \end{aligned}$$

(iv) *If* $g(t)g^{\sigma}(t) \neq 0$, *then*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

Definition 2.4 ([4])**.** If $F^{\Delta}(t) = f(t)$ and $t, t_0 \in \mathbb{T}$, we define the delta-integral by

$$\int_{t_0}^{t} f(s)\,\Delta s = F(t) - F(t_0).$$

If $\mathbb{T} = \mathbb{R}$, then $\int_{t_0}^{t} f(s)\,\Delta s$ corresponds to the Cauchy integral $\int_{t_0}^{t} f(s)\,ds$, and if $\mathbb{T} = \mathbb{Z}$, then $\int_{t_0}^{t} f(s)\,\Delta s = \sum_{s=t_0}^{t-1} f(s)$.

Now, we give the following definition which will be essential in our analysis.

Definition 2.5. A map $Q : [0,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is an L^1_{Δ}-Carathéodory function if it satisfies the following conditions.

(c1) For each $x \in \mathbb{R}$, the mapping $t \to Q(t,x)$ is Δ-measurable.

(c2) For almost all $t \in [0,\infty] \cap \mathbb{T}$, the mapping $x \to Q(t,x)$ is continuous on $\mathbb{R}$.

(c2) For each $k > 0$, there exists $\alpha_k \in L^1_{\Delta}([0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that, for almost all $t \in [0,\infty)_{\mathbb{T}}$ and for all x with $|x| < k$, we have $|Q(t,x)| \leq \alpha_k(t)$.

The proof of the main results in the next section is based upon an application of the following Krasnoselskii's fixed point theorem.

Theorem 2.6 (Krasnoselskii's fixed point theorem [9])**.** *Let* $\mathbb{M}$ *be a non-empty closed bounded convex subset of a Banach space* $(\mathbb{B}, \|.\|)$*. Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *map* $\mathbb{M}$ *into* $\mathbb{B}$ *such that*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$ *for all* $x, y \in \mathbb{M}$,

(ii) $\mathcal{A}$ *is continuous and compact,*

(iii) $\mathcal{B}$ *is a contraction with constant* $r < 1$.

Then there is $z \in \mathbb{M}$, *with* $\mathcal{A}z + \mathcal{B}z = z$.

3. Main Results

In this section, we discuss the existence results for (1.1). Let $[0,T]_{\mathbb{T}} = [0,T] \cap \mathbb{T}$ be a bounded interval in $\mathbb{T}$ with $T > 0$. And let $C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ be the Banach space of all rd-continuous functions defined on the compact interval $[0,T]_{\mathbb{T}}$, endowed with the norm

$$\|u\| = \sup_{t \in [0,T]_{\mathbb{T}}} |u(t)|.$$

By $L^1_{\Delta}([0,T]_{\mathbb{T}}, \mathbb{R})$, we denote the space of Lebesgue Δ-integrable functions on $[0,T]_{\mathbb{T}}$ equipped with the norm $\|.\|_{L^1_{\Delta}}$ defined by

$$\|u\|_{L^1_{\Delta}} = \int_0^T |u(s)|\,\Delta s.$$

We consider the following set of assumptions:

(A_0) $g, f, h \in C_{rd}([0,T]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ and $p \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R}\backslash\{0\})$.

(A_1) There exists a constant $l > 0$ such that

$$|f(t,u) - f(t,v)| \leq l\,|u - v|,$$

for all $t \in [0,T]_{\mathbb{T}}$ and $u, v \in \mathbb{R}$.

(A_2) There exist functions $H, G \in L^1_{\Delta}([0,T]_{\mathbb{T}}, \mathbb{R}^+)$ such that

$$|h(t,u)| \leq H(t),\ |g(t,u)| \leq G(t),\ t \in [0,T]_{\mathbb{T}},\ u \in \mathbb{R}.$$

(A_3) There exists a constant $K_p > 0$ such that

$$|p(t_2) - p(t_1)| \leq K_p\,|t_2 - t_1| \text{ for all } t_1, t_2 \in [0,T]_{\mathbb{T}}.$$

Let us start by defining what we mean by a solution of the problem (1.1).

Definition 3.1. A function $u \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ is said to be a solution of (1.1) if u satisfies the equation

$$\left(\frac{u(t) - f(t,u(t))}{p(t) + \int_0^t g(s,u(s))\,\Delta s} \right)^{\Delta} = h(t,u(t)), \; t \in [0,T]_{\mathbb{T}}, \tag{3.2}$$

with the initial condition

$$u(0) = f(0,u(0)) + p(0)\,\theta. \tag{3.3}$$

For the existence of solutions for the problem (1.1), we need the following lemma.

Lemma 3.2. *Suppose that (A_0) holds. Then, $u \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ solves (1.1) if and only if it is a solution of the integral equation*

$$u(t) = f(t,u(t)) + \left(p(t) + \int_0^t g(s,u(s))\,\Delta s \right) \left(\int_0^t h(s,u(s))\,\Delta s + \theta \right). \tag{3.4}$$

Proof. Let u be a solution of the problem (1.1). Applying the Δ-integral to (1.1) from 0 to t, we obtain

$$\left(\frac{u(t) - f(t,u(t))}{p(t) + \int_0^t g(s,u(s))\,\Delta s} \right) - \left(\frac{u(0) - f(0,u(0))}{p(0)} \right) = \int_0^t h(s,u(s))\,\Delta s,$$

that is,

$$u(t) = f(t,u(t)) + \left(p(t) + \int_0^t g(s,u(s))\,\Delta s \right) \times \left(\frac{u(0) - f(0,u(0))}{p(0)} + \int_0^t h(s,u(s))\,\Delta s \right).$$

Substituting the initial condition in the above equality implies that

$$u(t) = f(t,u(t)) + \left(p(t) + \int_0^t g(s,u(s))\,\Delta s \right) \times \left(\frac{p(0)\,\theta + f(0,u(0)) - f(0,u(0))}{p(0)} + \int_0^t h(s,u(s))\,\Delta s \right).$$

So,

$$u(t) = f(t,u(t)) + \left(p(t) + \int_0^t g(s,u(s))\,\Delta s \right) \left(\int_0^t h(s,u(s))\,\Delta s + \theta \right).$$

Therefore, (3.4) holds.

Conversely, suppose that u satisfies the equation (3.4). We rewrite (3.4) in the form

$$\frac{u(t) - f(t,u(t))}{p(t) + \int_0^t g(s,u(s))\,\Delta s} = \int_0^t h(s,u(s))\,\Delta s + \theta.$$

By direct differentiation and substituting $t = 0$ in (3.4) we obtain (1.1). The proof is completed. □

Now we will give the following existence theorem for (1.1).

Theorem 3.3. *Assume that the hypotheses (A_0)–(A_3) hold. Furthermore, if*

$$l < 1, \tag{3.5}$$

then the problem (1.1) has a solution defined on $[0,T]_{\mathbb{T}}$.

Proof. Set $\mathbb{B} = C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ and define a subset $\mathbb{M}$ of $\mathbb{B}$ by

$$\mathbb{M} = \{u \in \mathbb{B} : \|u\| \leq N\},$$

where

$$N = \frac{F_0 + \left(K_p T + |p(0)| + \|G\|_{L^1_\Delta}\right)\left(\|H\|_{L^1_\Delta} + |\theta|\right)}{1 - l},$$

with $F_0 = \sup_{t \in [0,T]_{\mathbb{T}}} |f(t, 0)|$. Clearly, $\mathbb{M}$ is a closed, convex and bounded subset of the Banach space $\mathbb{B}$.

Define two operators $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{B}$ by

$$(\mathcal{A}u)(t) = \left(p(t) + \int_0^t g(s, u(s))\,\Delta s\right)\left(\int_0^t h(s, u(s))\,\Delta s + \theta\right),\ t \in [0,T]_{\mathbb{T}}, \tag{3.6}$$

and

$$(\mathcal{B}u)(t) = f(t, u(t)),\ t \in [0,T]_{\mathbb{T}}. \tag{3.7}$$

Now, by Lemma 3.2, the problem (1.1) is equivalent to the operator equation

$$(\mathcal{A}u)(t) + (\mathcal{B}u)(t) = u(t),\ t \in [0,T]_{\mathbb{T}}.$$

We shall use Krasnoselskii's fixed point theorem to prove that there exists at least one fixed point of the operator $\mathcal{A} + \mathcal{B}$ in $\mathbb{M}$. The proof will be given in several steps.

Step 1. We prove that $\mathcal{B}$ is a contraction with constant $l < 1$. Let $u, v \in \mathbb{M}$. Then by (A_1), we get

$$|(\mathcal{B}u)(t) - (\mathcal{B}v)(t)| = |f(t, u(t)) - f(t, v(t))| \leq l\,|u(t) - v(t)| \leq l\,\|u - v\|.$$

for all $t \in [0,T]_{\mathbb{T}}$. Taking supremum over t, we have

$$\|\mathcal{B}u - \mathcal{B}v\| \leq l\,\|u - v\|$$

for all $u, v \in \mathbb{M}$. Then, by (3.5), $\mathcal{B}$ is a contraction operator on $\mathbb{M}$ with the constant $l < 1$.

Step 2. We prove that $\mathcal{A}$ is a compact and continuous operator on $\mathbb{M}$ into $\mathbb{B}$. Firstly, we prove that $\mathcal{A}$ is a compact operator on $\mathbb{M}$. It is enough to show that $\mathcal{A}(\mathbb{M})$ is a uniformly bounded and equicontinuous subset in $\mathbb{B}$. On the one hand, let $u \in \mathbb{M}$ be arbitrary. Then by (A_2), we obtain

$$\begin{aligned}
|(\mathcal{A}u)(t)| &\leq \left(|p(t)| + \int_0^t |g(s, u(s))|\,\Delta s\right)\left(\int_0^t |h(s, u(s))|\,\Delta s + |\theta|\right) \\
&\leq \left(K_p t + |p(0)| + \int_0^t |G(s)|\,\Delta s\right)\left(\int_0^t |H(s)|\,\Delta s + |\theta|\right) \\
&\leq \left(K_p T + |p(0)| + \|G\|_{L^1_\Delta}\right)\left(\|H\|_{L^1_\Delta} + |\theta|\right),
\end{aligned}$$

for all $t \in [0,T]_{\mathbb{T}}$. Taking supremum over t, we get

$$\|\mathcal{A}u\| \leq \left(K_p T + |p(0)| + \|G\|_{L^1_\Delta}\right)\left(\|H\|_{L^1_\Delta} + |\theta|\right),$$

for all $u \in \mathbb{M}$. This shows that $\mathcal{A}(\mathbb{M})$ is uniformly bounded.

On the other hand, let $t_1, t_2 \in [0,T]_{\mathbb{T}}$ be arbitrary with $t_1 < t_2$. Then for any $u \in \mathbb{M}$, we get

$$\begin{aligned}
&|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| \\
&= \left| \left(p(t_2) + \int_0^{t_2} g(s,u(s))\,\Delta s \right) \left(\int_0^{t_2} h(s,u(s))\,\Delta s + \theta \right) \right. \\
&\qquad \left. - \left(p(t_1) + \int_0^{t_1} g(s,u(s))\,\Delta s \right) \left(\int_0^{t_1} h(s,u(s))\,\Delta s + \theta \right) \right| \\
&\leq \left(|p(t_2)| + \int_0^{t_2} |g(s,u(s))|\,\Delta s \right) \left| \int_0^{t_2} h(s,u(s))\,\Delta s - \int_0^{t_1} h(s,u(s))\,\Delta s \right| \\
&\qquad + \left(|p(t_2) - p(t_1)| + \left| \int_0^{t_2} g(s,u(s))\,\Delta s - \int_0^{t_1} g(s,u(s))\,\Delta s \right| \right) \\
&\qquad \times \left(\int_0^{t_1} |h(s,u(s))|\,\Delta s + |\theta| \right) \\
&\leq \left(|p(t_2)| + \int_0^{t_2} |G(s)|\,\Delta s \right) \left| \int_{t_1}^{t_2} |h(s,u(s))|\,\Delta s \right| \\
&\qquad + \left(K_p |t_2 - t_1| + \left| \int_{t_1}^{t_2} |g(s,u(s))|\,\Delta s \right| \right) \left(\int_0^{t_1} |H(s)|\,\Delta s + |\theta| \right) \\
&\leq \left(|p(t_2)| + \|G\|_{L^1_\Delta} \right) \left| \int_{t_1}^{t_2} H(s)\,ds \right| \\
&\qquad + \left(K_p |t_2 - t_1| + \left| \int_{t_1}^{t_2} G(s)\,ds \right| \right) \left(\|H\|_{L^1_\Delta} + |\theta| \right) \\
&= \left(|p(t_2)| + \|G\|_{L^1_\Delta} \right) |\phi(t_2) - \phi(t_1)| \\
&\qquad + \left(\|H\|_{L^1_\Delta} + |\theta| \right) \left(K_p |t_2 - t_1| + |\varphi(t_2) - \varphi(t_1)| \right),
\end{aligned}$$

where $\varphi(t) = \int_0^t G(s)\,ds$ and $\phi(t) = \int_0^t H(s)\,ds$. Since the functions ϕ and φ are continuous on compact $[0,T]_{\mathbb{T}}$, they are uniformly continuous. Hence, for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|t_2 - t_1| < \delta \quad \Longrightarrow \quad |(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| < \varepsilon, \tag{3.8}$$

for all $t_1, t_2 \in [0,T]_{\mathbb{T}}$ and $u \in \mathbb{M}$. This shows that $\mathcal{A}(\mathbb{M})$ is an equicontinuous subset of $\mathbb{B}$. Then, $\mathcal{A}(\mathbb{M})$ is a uniformly bounded and equicontinuous subset of $\mathbb{B}$, so it is relatively compact by Arzela-Ascoli theorem. Thus, $\mathcal{A}$ is a compact operator on $\mathbb{M}$.

Next, we prove that $\mathcal{A}$ is continuous on $\mathbb{M}$. Let $\{u_n\}$ be a sequence in $\mathbb{M}$ converging to $u \in \mathbb{M}$. Then by the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\lim_{n\to\infty} (\mathcal{A}u_n)(t) &= \lim_{n\to\infty} \left[\left(p(t) + \int_0^t g(s,u_n(s))\,\Delta s \right) \left(\int_0^t h(s,u_n(s))\,\Delta s + \theta \right) \right] \\
&= \left(p(t) + \int_0^t \left[\lim_{n\to\infty} g(s,u_n(s)) \right] \Delta s \right) \left(\int_0^t \left[\lim_{n\to\infty} h(s,u_n(s)) \right] \Delta s + \theta \right) \\
&= \left(p(t) + \int_0^t g(s,u(s))\,\Delta s \right) \left(\int_0^t h(t,u(t))\,\Delta s + \theta \right) \\
&= (\mathcal{A}u)(t),
\end{aligned}$$

for all $t \in [0,T]_{\mathbb{T}}$. This shows that $\{\mathcal{A}u_n\}$ converges pointwise to $\mathcal{A}u$ on $[0,T]_{\mathbb{T}}$. Moreover, the sequence $\{\mathcal{A}u_n\}$ is equicontinuous by a similar proof of (3.8). Therefore, $\{\mathcal{A}u_n\}$ converges uniformly to $\mathcal{A}u$ and hence $\mathcal{A}$ is a continuous operator on $\mathbb{M}$.

Step 3. $\mathcal{A}u+\mathcal{B}v\in\mathbb{M}$ for all $u,v\in\mathbb{M}$. For any $u,v\in\mathbb{M}$ and $t\in[0,T]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\left|(\mathcal{A}u)(t)+(\mathcal{B}v)(t)\right| \\
&\leq\left(|p(t)|+\int_0^t|g(s,u(s))|\,\Delta s\right)\left(\int_0^t|h(s,u(s))|\,\Delta s+|\theta|\right)+|f(t,v(t))| \\
&\leq\left(K_pt+|p(0)|+\int_0^t|G(s)|\,ds\right)\left(\int_0^t|H(s)|\,ds+|\theta|\right) \\
&+|f(t,v(t))-f(t,0)|+|f(t,0)| \\
&\leq\left(K_pT+|p(0)|+\|G\|_{L^1_\Delta}\right)\left(\|H\|_{L^1_\Delta}+|\theta|\right)+l\,\|v\|+F_0 \\
&\leq N.
\end{aligned}$$

This shows that $\mathcal{A}u+\mathcal{B}v\in\mathbb{M}$ for all $u,v\in\mathbb{M}$.

Thus, all the conditions of Theorem 2.6 are satisfied and hence the operator equation $\mathcal{A}z+\mathcal{B}z=z$ has a solution in $\mathbb{M}$. Therefore, the problem (1.1) has a solution defined on $[0,T]_{\mathbb{T}}$. □

Example 3.4. Let us consider the following integro-dynamic equation

$$\begin{cases}\left(\frac{u(t)-\frac{1}{8}\sin u(t)}{\pi+\sin t+\int_0^t\sin u(s)\Delta s}\right)^{\Delta}=\cos u(t)\,,\ t\in[0,1]_{\mathbb{T}}\,,\\ u(0)=\frac{1}{8}\sin u(0)+\pi,\end{cases}\tag{3.9}$$

where $T=1,\ \theta=1,\ f(t,u(t))=\frac{1}{8}\sin u(t)\,,\ p(t)=\pi+\sin t,\ g(t,u(t))=\sin u(t)\,,$ $h(t,u(t))=\cos u(t)$. Let $l=\frac{1}{8}$, $K_p=1$, $G(t)=1$, $H(t)=1$. Then hypotheses (A_0)–(A_3) hold. Since

$$l=\frac{1}{8}<1,$$

(3.5) holds. Therefore, by Theorem 3.3, the problem (3.9) has a solution.

4. Conclusion

In this paper, we have studied the existence of solutions for the nonlinear integro-dynamic equation with mixed perturbations of the second type. We have presented an existence theorem for the problem (1.1) under some sufficient conditions due to Krasnoselskii's fixed point theorem. The main results have been well illustrated with the help of an example.

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Abderrahim Guerfi: abderrahimg21@gmail.com
Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, University of Annaba, P.O. Box 12, Annaba, 23000, Algeria

Abdelouaheb Ardjouni: abd_ardjouni@yahoo.fr
Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria

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