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FACTORIZATIONS OF GENERALIZED SCHUR FUNCTIONS AND PRODUCTS OF PASSIVE SYSTEMS

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Dedicated to Professor Vladimir Derkach on the occasion of his seventieth birthday.

Abstract. Factorizations of Pontryagin space operator-valued generalized Schur functions are studied. Main tools are products of contractive operator colligations, or cascade connections of passive discrete-time systems. The well-known notion of regular factorizations of ordinary Schur functions is extended to the generalized Schur class functions by using canonical reproducing kernel Pontryagin space models. Factorizations stronger than the regular factorization are also introduced to obtain characterizations in the case where the products of observable co-isometric (controllable isometric) systems preserve the observability (controllability). These factorizations are related to backwards shift invariant regular subspaces of de Branges–Rovnyak spaces, and they can alternatively be viewed as regular factorizations of generalized Schur functions with certain extreme properties. Moreover, their properties are linked with how the optimality is preserved under the product of optimal passive systems.

Досліджено факторизацію операторнозначних узагальнених функцій Шура нп просторі Понтрягіна. Основними інструментами є добутки стискаючих операторних з'єднань, або каскадних зв'язків пасивних систем з дискретним часом. Добре відоме поняття регулярних факторизацій звичайних функцій Шура поширюється на узагальнені функції класу Шура за допомогою канонічних відтворюючих ядер для моделей простору Понтрягіна. Також вводяться факторизації більш сильні, ніж звичайна факторизація, для отримання характеристик, у випадку, коли добутки спостережуваних коізометричних (керованих ізометричних) систем зберігають спостережуваність (керованість). Ці факторизації пов'язані з регулярними підпросторами просторів де Бранжа-Ровняка, які є інваріантними відносно зворотнього зсуву, і їх можна також розглядати як регулярні факторизації узагальнених функцій Шура з певними екстремальними властивостями. Крім того, їх властивості пов'язані з тим, як зберігається оптимальність відносно добутку оптимальних пасивних систем.

1. Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be separable Pontryagin spaces with the same finite negative index. A function θ analytic at the origin with values in the set of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$, which is denoted by $\mathcal{L}(\mathcal{U},\mathcal{Y})$, belongs to the *generalized Schur class* $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ if the $\mathcal{L}(\mathcal{Y})$-valued Schur kernel

$$K_\theta(w,z)=\frac{1-\theta(z)\theta^*(w)}{1-z\bar{w}},\qquad w,z\in\rho(\theta), \tag{1.1}$$

where $\rho(\theta)$ is the maximal domain of holomorphy of θ, has κ negative squares ($\kappa=0,1,2,\ldots$). That is, no Hermitian matrix of the form

$$\left(\langle K_\theta(w_j,w_i)f_j,f_i\rangle_{\mathcal{Y}}\right)_{i,j=1}^n,\quad w_1,\ldots,w_n\in\rho(\theta),\quad f_1,\ldots,f_n\in\mathcal{Y},$$

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where $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ is the indefinite inner product of the space $\mathcal{Y}$, has more than κ negative eigenvalues, and there exists at least one such matrix with exactly κ negative eigenvalues. The class $\mathbf{S}_0(\mathcal{U},\mathcal{Y})$ is denoted by $\mathbf{S}(\mathcal{U},\mathcal{Y})$, and the values of $\theta\in\mathbf{S}(\mathcal{U},\mathcal{Y})$ are contractive. If $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, then $\mathbf{S}(\mathcal{U},\mathcal{Y})$ is the *ordinary Schur class*, i.e., the unit ball of $H^\infty(\mathcal{L}(\mathcal{U},\mathcal{Y}))$.

In this paper, the factorizations of the generalized Schur function $\theta\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ of the form

$$\theta=\theta_2\theta_1,\qquad \theta_1\in\mathbf{S}_{\kappa_1}(\mathcal{U},\mathcal{Y}_1),\qquad \theta_2\in\mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y}), \tag{1.2}$$

where $\mathcal{Y}_1$ is a Pontryagin space with the same negative index as $\mathcal{U}$ and $\mathcal{Y}$, are studied by using products of contractive *operator colligation* realizations of the generalized Schur functions. An *operator colligation*, or as it will be called by using the notations arising from the system theory, linear discrete-time *system* $\Sigma=(T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ consists of the *state space* $\mathcal{X}$ which is a Pontryagin space with the negative index κ, Pontryagin spaces $\mathcal{U}$ and $\mathcal{Y}$ with the same negative index, which is not related to κ, and the bounded linear system operator T_Σ of the form

$$T_\Sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}:\begin{pmatrix}\mathcal{X}\\ \mathcal{U}\end{pmatrix}\to\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\end{pmatrix}, \tag{1.3}$$

where $\binom{\mathcal{X}}{\mathcal{U}}$ is the direct orthogonal sum $\mathcal{X}\oplus\mathcal{U}$ with respect to the indefinite inner product. The *transfer function* of the system (1.3) is an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined by

$$\theta_\Sigma(z)=D+zC(I-zA)^{-1}B,\quad z^{-1}\in\rho(A),$$

where $\rho(A)$ is the resolvent set of A, so at least θ_Σ is defined and holomorphic in a neighbourhood of the origin. The operator $A\in\mathcal{L}(\mathcal{X})$ is called the main operator of Σ. The notation $\Sigma=(A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ is often used instead of $\Sigma=(T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$. For systems $\Sigma_1=(A_1,B_1,C_1,D_1;\mathcal{X}_1,\mathcal{U},\mathcal{Y}_1;\kappa_1)$ and $\Sigma_2=(A_2,B_2,C_2,D_2;\mathcal{X}_2,\mathcal{Y}_1,\mathcal{Y};\kappa_2)$ with transfer functions θ_{Σ_1} and θ_{Σ_2}, a *product* or *cascade connection*, see (3.32), produces a system $\Sigma_2\circ\Sigma_1=\Sigma$ with the system operator

$$T_\Sigma=\begin{pmatrix}A_1 & 0 & B_1\\ B_2C_1 & A_2 & B_2D_1\\ D_2C_1 & C_2 & D_2D_1\end{pmatrix}=\begin{pmatrix}I_{\mathcal{X}_1} & 0 & 0\\ 0 & A_2 & B_2\\ 0 & C_2 & D_2\end{pmatrix}\begin{pmatrix}A_1 & 0 & B_1\\ 0 & I_{\mathcal{X}_2} & 0\\ C_1 & 0 & D_1\end{pmatrix}:\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\\ \mathcal{U}\end{pmatrix}\to\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\\ \mathcal{Y}\end{pmatrix}$$

and the transfer function $\theta_\Sigma=\theta_{\Sigma_2}\theta_{\Sigma_1}$.

A system $\Sigma=(T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ is called *passive (isometric, co-isometric, conservative)* if the system operator T_Σ of Σ is contractive (isometric, co-isometric, unitary), with respect to the underlying indefinite inner products. In the literature, conservative systems are often called unitary. The system $\Sigma=(T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ is a *realization* of $\theta\in\mathbf{S}_{\kappa'}(\mathcal{U},\mathcal{Y})$, if the transfer function of Σ coincides with θ in some neighbourhood of the origin. It is well-known that every $\theta\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ has the so-called *canonical* isometric, co-isometric and unitary realizations, see [2, Chapter 2], with certain minimality properties, which are, in the language of system theory, respectively, controllability, observability and simplicity. By taking restrictions of canonical realizations, one obtains minimal passive realizations of θ. Consider now the functions θ, θ_1 and θ_2 as in (1.2), and the realizations $\Sigma_1=(T_{\Sigma_1};\mathcal{X}_1,\mathcal{U},\mathcal{Y}_1;\kappa_1)$ and $\Sigma_2=(T_{\Sigma_2};\mathcal{X}_2,\mathcal{Y}_1,\mathcal{Y};\kappa_2)$ of θ_1 and θ_2. Then the product $\Sigma=\Sigma_2\circ\Sigma_1$ is a realization of θ, and Σ is passive (isometric, co-isometric, conservative) if Σ_1 and Σ_2 are. Other system theoretical qualitative properties are not necessarily preserved under the cascade connection of passive systems, and the aim of this paper is to obtain necessary and sufficient conditions when such properties are preserved.

In the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and θ, θ_1 and θ_2 are ordinary Schur functions, it is known that the product $\Sigma=\Sigma_2\circ\Sigma_1$ of simple conservative realizations of θ_1 and θ_2 is simple conservative if and only if the corresponding factorization is *regular* in a sense of Brodskiĭ [17] and Sz.-Nagy and Foias [39], see Definition 4.3. Regular factorizations are widely studied, and accounts are given, for instance, in [13] and [14]. It in well-known, that finding a regular factorization is equivalent to finding an invariant subspace of a completely non-unitary Hilbert space contraction. In [9, Theorem 8.1], Arov et al. generalized the notion of a regular factorization to the setting where $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U}, \mathcal{Y}_1)$ and $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1, \mathcal{Y})$, but where $\mathcal{U}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ are still Hilbert spaces. Their definition, as well as the original one for ordinary Schur functions in [17] and [39], depends strongly on the Hilbert space specific functional model of Sz.-Nagy and Foias from [39], and therefore cannot be instantly applied to Pontryagin space operator-valued functions. However, another well-known model, which goes back to the work of de Branges and Rovnyak [19, 20], can be also applied in Pontryagin space setting, see [2, Chapter 2]. By using the de Branges–Rovnyak reproducing kernel model, the notion of regular factorization is extended to the Pontryagin space operator-valued generalized Schur functions in Definition 3.2 and Theorem 3.3.

Instead of products of simple conservative systems, one can consider the product of another canonical realization or other realizations with certain minimality or optimality properties. In finite dimensional system theory, transfer functions are rational matrix functions, and their factorizations via cascade connections of systems are widely studied, see for instance [15]. However, for non-rational (generalized) Schur functions, these subjects are not widely studied. An account was given by Khanh in [25], where he introduced $(+)$-*regular* and $(-)$-*regular* factorizations of ordinary Schur functions, based on the functional model of Sz.-Nagy and Foias. For Khanh's definitions, see Definition 4.3. These factorizations are, in system theoretical sense, stronger than the Brodskiĭ's regular factorizations, and Khanh proved that the product of observable (controllable, minimal) conservative system is observable (controllable, minimal) conservative if and only if the corresponding factorizations $\theta = \theta_2\theta_1$ of the transfer functions is $(+)$-regular ($(-)$-regular, $(+)$- and $(-)$-regular) [25, Corollary 1 and Theorem 4]. Nevertheless, such realizations exist only for certain (generalized) Schur functions. Namely, for those with zero right (left, right and left) defect functions, see [8, Proposition 4] for ordinary Schur functions and [31, Theorem 4.8] for generalized Schur functions. When this happens, $(+)$-regular ($(-)$-regular, $(+)$-regular and $(-)$-regular) factorizations turn out to be equivalent to regular factorization, as it follows from Proposition 4.2. However, by using de Branges–Rovnyal model instead of the model of Sz.-Nagy and Foias, Khanh's definition of $(\pm)$-regularity can be extended to cover the class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Finding a $(+)$-regular factorization of θ is equivalent to finding a backward shift invariant subspace of the generalized de Branges–Rovnyak space induced by the kernel (1.1). A typical example of $(+)$-regular ($(-)$-regular) factorization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, is the right (left) Kreĭn–Langer factorization $\theta = \theta_r B_r^{-1}$ ($\theta = B_l^{-1}\theta_l$), where B_r^{-1} and B_l^{-1} are inverse Blaschke products and θ_r and θ_l are ordinary Schur functions. It will be shown in Theorem 3.3 that the product of the observable co-isometric (controllable isometric) systems is observable co-isometric (controllable isometric) if and only if the corresponding factorization is $(+)$-regular ($(-)$-regular). Moreover, a product $\Sigma_2 \circ \Sigma_1$ of κ-*admissible* (see the definition from the page 70) observable (controllable, simple, minimal) passive systems is κ-admissible observable (controllable, simple, minimal), if the corresponding factorization is $(+)$-regular ($(-)$-regular, regular, $(+)$- and $(-)$-regular). Such results are new also in the standard Hilbert space settings. In the case of rational functions, the results obtained here do not fall under the known results of minimal factorization of rational matrix functions either, as Example 3.5 shows.

The rest of the paper is organized as follows. In Section 2, the background and known fundamental results of passive systems and their connections to the generalized Schur functions needed in this paper are recalled, mostly without proofs. However, Lemma 2.7 and Proposition 2.8 are new, and their proofs are provided.

In Section 3, the definitions of regular and ($\pm$)-regular factorizations are given. The results covering the products of canonical realizations, passive realization and ($\pm$)-regular factorizations are derived. In the last part of this section, products of optimal or *-optimal systems are considered. In the standard Hilbert space case, such products were studied by Hang in [24] and Hang and Khanh in [26]. Their results do not cover Theorem 3.8 even in the standard case.

Section 4 concerns the connection between regular invariant subspaces and factorization of Pontryagin space operator-valued generalized Schur function. It is also shown in Proposition 4.2, that for generalized Schur functions with certain extremality properties, regular factorizations are equivalent to (+)-regular or ($-$)-regular factorizations.

2. Contractive operator colligation realizations

Consider a system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa) = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, with transfer function θ. Define the *adjoint*, or *dual* system of Σ to be $\Sigma^* = (T_\Sigma^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}; \kappa) = (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}; \kappa)$, where the adjoints are, as well as all the adjoints in this paper, calculated with respect to the indefinite inner product. For general theory of indefinite inner product spaces and their operators, we refer to [11, 16, 23]. An easy calculation shows that the transfer function of Σ^* is $\theta^{\#}(z) = \theta^*(\bar{z})$, where the notation $\theta^*(z)$ is used instead of $(\theta(z))^*$. It is known that $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ if and only if $\theta^{\#}_\kappa \in \mathbf{S}_\kappa(\mathcal{Y}, \mathcal{U})$; see [2, Theorem 2.5.2].

The following subspaces

$$\mathcal{X}^c := \overline{\operatorname{span}}\,\{\operatorname{ran} A^n B : n = 0, 1, \ldots\}, \tag{2.4}$$

$$\mathcal{X}^o := \overline{\operatorname{span}}\,\{\operatorname{ran} A^{*n} C^* : n = 0, 1, \ldots\}, \tag{2.5}$$

$$\mathcal{X}^s := \overline{\operatorname{span}}\,\{\operatorname{ran} A^n B, \operatorname{ran} A^{*m} C^* : n, m = 0, 1, \ldots\}, \tag{2.6}$$

of the state space $\mathcal{X}$ of $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ are called the controllable, observable and simple subspaces, respectively. All the notions related to the topology and continuity are considered to be with respect to unique strong topology induced by any fundamental symmetry of Pontryagin space in question. The system is said to be *controllable* (*observable*, *simple*) if $\mathcal{X}^c = \mathcal{X}$ ($\mathcal{X}^o = \mathcal{X}$, $\mathcal{X}^s = \mathcal{X}$) and *minimal* if it is both controllable and observable. When Ω is some sufficiently small symmetric neighbourhood of the origin, that is, $\bar{z} \in \Omega$ whenever $z \in \Omega$ and $(I - zA)^{-1}$ exists for all $z \in \Omega$, then also

$$\mathcal{X}^c = \overline{\operatorname{span}}\,\{\operatorname{ran}\,(I - zA)^{-1} B : z \in \Omega\}, \tag{2.7}$$

$$\mathcal{X}^o = \overline{\operatorname{span}}\,\{\operatorname{ran}\,(I - zA^*)^{-1} C^* : z \in \Omega\}, \tag{2.8}$$

$$\mathcal{X}^s = \overline{\operatorname{span}}\,\{\operatorname{ran}\,(I - zA)^{-1} B, \operatorname{ran}\,(I - wA^*)^{-1} C^* : z, w \in \Omega\}. \tag{2.9}$$

It is clear from (2.4)–(2.6) that Σ is controllable (observable, simple, minimal) if and only if the adjoint Σ^* is observable (controllable, simple, minimal). Moreover, since contractions between Pontryagin spaces with the same negative index are bi-contractions (cf. eg. [23, Corollary 2.5]), Σ is passive (isometric, co-isometric, conservative) whenever Σ^* is passive (co-isometric, isometric, conservative). It is known from [31, Proposition 2.4] that the transfer function of the passive system $\Sigma = (T_\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ belongs to the generalized Schur class $\mathbf{S}_{\kappa'}(\mathcal{U}, \mathcal{Y})$ with the index κ' not larger than the negative index κ of the state space $\mathcal{X}$. Passive systems and generalized Schur functions are deeply interconnected. For Pontryagin space operator-valued case, isometric, co-isometric and conservative systems were studied, for instance, in [1, 2, 18, 23], and for Hilbert space operator-valued case, in [21, 22]. Passive systems for Pontryagin space operator-valued case were studied in [30, 31] and for Hilbert space operator-valued case, in [9, 10, 29, 32].

In general, if the negative index of the state space of passive system Σ coincides with the index of its transfer function, Σ is called *κ-admissible*, and its behaviour resembles Hilbert space passive systems in many aspects.

For $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, it is usually desirable to have a realization of θ such that the system operator is unitary, or close to, and the state space is as small as possible in a certain sense. The following combinations are always possible. For proofs of part (i)–(iii), see [2, Chapter 2], and for part (iv), [31, Lemma 2.8].

Lemma 2.1. *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Then there exist a κ-admissible passive realization $\Sigma = (T_\Sigma;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa)$ of θ, and if desired, it can be chosen such that it has one of the following properties:*

(i) *simple conservative;*
(ii) *controllable isometric;*
(iii) *observable co-isometric;*
(iv) *minimal passive;*

A realization with one of the properties (i)–(iii) of Lemma 2.1 is essentially unique, since if Σ' is another κ-admissible realization of Σ such that Σ and Σ' both have the same property of (i)–(iii) of Lemma 2.1, they are unitarily similar, which means that they differ only by a unitary change of state variable; see [2, Theorem 2.1.3]. The precise definition is that two realizations

$$\Sigma_1 = (A_1,B_1,C_1,D_1;\mathcal{X}_1,\mathcal{U},\mathcal{Y};\kappa_1), \qquad \Sigma_2 = (A_2,B_2,C_2,D_2;\mathcal{X}_2,\mathcal{U},\mathcal{Y};\kappa_2)$$

of the same function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ are called *unitarily similar* if $D_1 = D_2$ and there exists a unitary operator $U : \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$A_1 = U^{-1}A_2U, \quad B_1 = U^{-1}B_2, \quad C_1 = C_2U. \tag{2.10}$$

Minimal passive realizations of θ are not, in general, unique in the sense described above. Instead, they are *weakly similar*, see [31, Proposition 2.2], [33, Theorem 2.17] and [37, p. 702]

One way to produce the realizations in Lemma 2.1 is to use reproducing kernel theory, reproducing kernel Pontryagin spaces and apply de Branges–Rovnyak complementary space theory. For ordinary Schur functions, this idea goes back to [20, 19]. It is well-known; see [2, 5, 23, 28, 35, 36], that if $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, then the kernel (1.1) generates the reproducing kernel Pontryagin space $\mathcal{H}(\theta)$ with the negative index κ. The spaces $\mathcal{H}(\theta)$ are called *generalized de Branges–Rovnyak spaces*, and the elements in $\mathcal{H}(\theta)$ are functions defined on $\rho(\theta)$ with values in $\mathcal{Y}$. Under the assumption that the negative index of the Pontryagin spaces $\mathcal{U}$ and $\mathcal{Y}$ coincides, for a $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function θ holomorphic in a neighbourhood Ω of the origin, the kernel (1.1) has κ negative squares if and only if the related $\mathcal{L}(\mathcal{Y}\oplus\mathcal{U})$-valued kernel

$$D_\theta(w,z) = \begin{pmatrix} K_\theta(w,z) & \dfrac{\theta(z)-\theta(\overline{w})}{z-\overline{w}} \\ \dfrac{\theta^\#(z)-\theta^\#(\overline{w})}{z-\overline{w}} & K_{\theta^\#}(w,z) \end{pmatrix}, \qquad w,z \in \Omega, \tag{2.11}$$

has κ negative squares; see [2, Theorem 2.5.2]. The Pontryagin space generated by the kernel (2.11) is denoted by $\mathcal{D}(\theta)$. The spaces $\mathcal{H}(\theta)$ and $\mathcal{D}(\theta)$ can be chosen as state spaces of an observable co-isometric realization and a simple conservative realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, respectively. For the proof, see [2, Theorems 2.2.1 and 2.3.1].

Lemma 2.2. *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, and let $\mathcal{H}(\theta)$ and $\mathcal{D}(\theta)$ be the Pontryagin spaces induced by the reproducing kernels (1.1) and (2.11). Then:*

(i) *The system* $\Sigma_1 = (A_1, B_1, C_1, D, \mathcal{H}(\theta), \mathcal{U}, \mathcal{Y}, \kappa)$, *where*

$$\begin{cases} A_1 : h(z) \mapsto \dfrac{h(z) - h(0)}{z}, & B_1 : u \mapsto \dfrac{\theta(z) - \theta(0)}{z} u, \\ C_1 : h(z) \mapsto h(0), & D : u \mapsto \theta(0) u, \end{cases} \tag{2.12}$$

is an observable co-isometric realization of θ. *Moreover, for every* $h \in \mathcal{H}(\theta)$, *we have* $C_1(I - zA_1)^{-1} h = h(z)$.

(ii) *The system* $\Sigma_2 = (A_2, B_2, C_2, D, \mathcal{D}(\theta), \mathcal{U}, \mathcal{Y}, \kappa)$, *where*

$$\begin{cases} A_2 : \begin{pmatrix} h(z) \\ k(z) \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{h(z) - h(0)}{z} \\ zk(z) - \theta^{\#}(z) h(0) \end{pmatrix}, & B_2 : u \mapsto \begin{pmatrix} \dfrac{\theta(z) - \theta(0)}{z} u \\ \left(I_{\mathcal{U}} - \theta^{\#}(z) \theta^{\# *}(0)\right) u \end{pmatrix}, \\ C_2 : \begin{pmatrix} h \\ k \end{pmatrix} \mapsto h(0), & D : u \mapsto \theta(0) u, \end{cases} \tag{2.13}$$

is a simple conservative realization of θ. *Moreover, for* $\begin{pmatrix} h \\ k \end{pmatrix} \in \mathcal{D}(\theta)$,

$$C_2(I - zA_2)^{-1} \begin{pmatrix} h \\ k \end{pmatrix} = h(z) \quad \textit{and} \quad B_2^*(I - zA_2^*)^{-1} \begin{pmatrix} h \\ k \end{pmatrix} = k(z).$$

The systems in Lemma 2.2 are called the *canonical co-isometric realization* and the *canonical unitary* (or *conservative*) *realization* of θ, respectively, and the operator A_1 in (2.12) is a *backward shift*. These realizations will be used in Sections 3 and 4.

A *dilation* of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ is any system of the form $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa)$, where

$$\widehat{\mathcal{X}} = \mathcal{D} \oplus \mathcal{X} \oplus \mathcal{D}_*, \quad \widehat{A}\mathcal{D} \subset \mathcal{D}, \quad \widehat{A}^* \mathcal{D}_* \subset \mathcal{D}_*, \quad \widehat{C}\mathcal{D} = \{0\}, \quad \widehat{B}^* \mathcal{D}_* = \{0\}.$$

The spaces $\mathcal{D}$ and $\mathcal{D}_*$ are required to be Hilbert spaces. The system operator $T_{\widehat{\Sigma}}$ of $\widehat{\Sigma}$ is of the form

$$T_{\widehat{\Sigma}} = \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} & \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & C & C_1 \end{pmatrix} & D \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{D} \\ \mathcal{X} \\ \mathcal{D}_* \end{pmatrix} \\ \mathcal{Y} \end{pmatrix},$$

$$\widehat{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix}, \qquad \widehat{C} = \begin{pmatrix} 0 & C & C_1 \end{pmatrix}.$$

The system Σ is called a *restriction* of $\widehat{\Sigma}$. Since $\mathcal{X}$ clearly is a *regular subspace* of $\widehat{\mathcal{X}}$, i.e., it is a Pontryagin space with the inherited inner product, there exists the unique orthogonal projection $P_{\mathcal{X}}$ from $\widehat{\mathcal{X}}$ to $\mathcal{X}$. Let $\widehat{A}\restriction_{\mathcal{X}}$ be the restriction of $\widehat{A}$ to the subspace $\mathcal{X}$. Then, the system Σ has a representation $\Sigma = (P_{\mathcal{X}}\widehat{A}\restriction_{\mathcal{X}}, P_{\mathcal{X}}\widehat{B}, \widehat{C}\restriction_{\mathcal{X}}, D; P_{\mathcal{X}}\widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \kappa)$. Dilations and restrictions are denoted by $\widehat{\Sigma} = \mathrm{dil}_{\mathcal{X} \to \widehat{\mathcal{X}}} \Sigma$ and $\Sigma = \mathrm{res}_{\widehat{\mathcal{X}} \to \mathcal{X}} \widehat{\Sigma}$, mostly without subscripts when the corresponding state spaces are clear. A calculation shows that the transfer functions of the original system and all of its dilation coincide in a neighbourhood of the origin. By taking suitable restrictions, one can often obtain new realizations with desired properties, as the lemma below, taken from [31, Lemma 2.8], shows.

Lemma 2.3. *Let* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ *be a passive system such that the spaces* $(\mathcal{X}^o)^\perp$, $(\mathcal{X}^c)^\perp$ *and* $(\mathcal{X}^s)^\perp$ *are Hilbert subspaces of* $\mathcal{X}$. *Then the system operator* T *of* Σ

has the following representations;

$$T = \left(\begin{pmatrix} A_1 & A_2 \\ 0 & A_o \end{pmatrix} \quad \begin{pmatrix} B_1 \\ B_o \end{pmatrix} \atop \begin{pmatrix} 0 & C_o \end{pmatrix} \quad D \right) : \left(\begin{pmatrix} (\mathcal{X}^o)^\perp \\ \mathcal{X}^o \end{pmatrix} \atop \mathcal{U} \right) \to \left(\begin{pmatrix} (\mathcal{X}^o)^\perp \\ \mathcal{X}^o \end{pmatrix} \atop \mathcal{Y} \right), \tag{2.14}$$

$$T = \left(\begin{pmatrix} A_3 & 0 \\ A_4 & A_c \end{pmatrix} \quad \begin{pmatrix} 0 \\ B_c \end{pmatrix} \atop \begin{pmatrix} C_1 & C_c \end{pmatrix} \quad D \right) : \left(\begin{pmatrix} (\mathcal{X}^c)^\perp \\ \mathcal{X}^c \end{pmatrix} \atop \mathcal{U} \right) \to \left(\begin{pmatrix} (\mathcal{X}^c)^\perp \\ \mathcal{X}^c \end{pmatrix} \atop \mathcal{Y} \right), \tag{2.15}$$

$$T = \left(\begin{pmatrix} A_5 & 0 \\ 0 & A_s \end{pmatrix} \quad \begin{pmatrix} 0 \\ B_s \end{pmatrix} \atop \begin{pmatrix} 0 & C_s \end{pmatrix} \quad D \right) : \left(\begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}^s \end{pmatrix} \atop \mathcal{U} \right) \to \left(\begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}^s \end{pmatrix} \atop \mathcal{Y} \right), \tag{2.16}$$

$$T = \left(\begin{pmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A' & A'_{23} \\ 0 & 0 & A'_{33} \end{pmatrix} \quad \begin{pmatrix} B'_1 \\ B' \\ 0 \end{pmatrix} \atop \begin{pmatrix} 0 & C' & C'_1 \end{pmatrix} \quad D \right) : \left(\begin{pmatrix} (\mathcal{X}^o)^\perp \\ \overline{P_{\mathcal{X}^o}\mathcal{X}^c} \\ \mathcal{X}^o \cap (\mathcal{X}^c)^\perp \end{pmatrix} \atop \mathcal{U} \right) \to \left(\begin{pmatrix} (\mathcal{X}^o)^\perp \\ \overline{P_{\mathcal{X}^o}\mathcal{X}^c} \\ \mathcal{X}^o \cap (\mathcal{X}^c)^\perp \end{pmatrix} \atop \mathcal{Y} \right). \tag{2.17}$$

The restrictions

$$\Sigma_o = (A_o, B_o, C_o, D; \mathcal{X}^o, \mathcal{U}, \mathcal{Y}; \kappa), \tag{2.18}$$
$$\Sigma_c = (A_c, B_c, C_c, D; \mathcal{X}^c, \mathcal{U}, \mathcal{Y}; \kappa), \tag{2.19}$$
$$\Sigma_s = (A_s, B_s, C_s, D; \mathcal{X}^s, \mathcal{U}, \mathcal{Y}; \kappa), \tag{2.20}$$
$$\Sigma' = (A', B', C', D; \overline{P_{\mathcal{X}^o}\mathcal{X}^c}, \mathcal{U}, \mathcal{Y}; \kappa) \tag{2.21}$$

of Σ are passive, and Σ_o is observable, Σ_c is controllable, Σ_s is simple and Σ' is minimal. If Σ is observable or controllable, then so are Σ_o, Σ_c and Σ_s. For any $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and any z in a sufficiently small symmetric neighbourhood of the origin, the following holds:

$$\begin{aligned} A^n B &= A_c^n B_c = A_s^n B_s, \\ (I - zA)^{-1} B &= (I - zA_s)^{-1} B_s = (I - zA_c)^{-1} B_c, \\ A^{*n} C^* &= A_o^{*n} C_o^* = A_s^{*n} C_s^*, \\ (I - zA^*)^{-1} C^* &= (I - zA_s^*)^{-1} C_s^* = (I - zA_c^*)^{-1} C_c^*. \end{aligned}$$

Moreover, if Σ is co-isometric (isometric), then so are Σ_o and Σ_s (Σ_c and Σ_s).

Remark 2.4. The restrictions (2.18)–(2.21) are called the observable, the controllable, the simple and the first minimal restrictions of Σ, respectively. It is not explicitly stated in [31, Lemma 2.8] that Σ_o, Σ_c and Σ_s are controllable or observable whenever Σ is. However, in those cases it easily follows from (2.14)–(2.17) that the systems either coincide with Σ or they are first minimal restrictions, which are shown to be minimal, and the claim follows. Moreover, if the transfer function θ of Σ belongs to $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, i.e., when Σ is a κ-admissible realization of θ, the conditions of Lemma 2.3 are always satisfied.

Contrary to Lemma 2.3, dilations of a given κ-admissible realization Σ of θ may not have such strong properties. There always exists κ-admissible conservative dilation of Σ [31, Proposition 2.2], and if Σ is observable co-isometric or controllable isometric, a simple conservative dilation can be obtained [2, Section 2.4]. However, it may happen that a minimal passive κ-admissible realization has no simple conservative dilation, see [9, Example 6.8].

Denote $E_{\mathcal{X}}(x) = \langle x, x \rangle_{\mathcal{X}}$ for the vector x in an indefinite inner product space $\mathcal{X}$. For $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, a

κ-admissible passive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is called *optimal* if, for any κ-admissible passive realization $\Sigma_0 = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ of θ,

$$E_{\mathcal{X}}\left(\sum_{n=0}^{N} A^n B u_n\right) \leq E_{\mathcal{X}_0}\left(\sum_{n=0}^{N} {A_0}^n B_0 u_n\right) \tag{2.22}$$

for any $N \in \mathbb{N}_0$ and $\{u_n\}_{n=0}^N \subset \mathcal{U}$. Conversely, an observable passive κ-admissible realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is called *-*optimal* if, for any observable κ-admissible passive realization $\Sigma_0 = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y}; \kappa)$ of θ,

$$E_{\mathcal{X}}\left(\sum_{n=0}^{N} A^n B u_n\right) \geq E_{\mathcal{X}_0}\left(\sum_{n=0}^{N} {A_0}^n B_0 u_n\right)$$

for any $N \in \mathbb{N}_0$ and $\{u_n\}_{n=0}^N \subset \mathcal{U}$. The requirement that the considered realizations are κ-admissible is essential; see [31, Example 3.1], as well as the requirement of the observability in the definition of *-optimality. Indeed, otherwise any isometric κ-admissible realization would be *-optimal, as follows from Lemma 2.5 below. Lemma 2.5 illustrates also another point; to prove the optimality of a system, it is sufficient to check the inequality (2.22) for all minimal κ-admissible realizations; see (2.23) below. For a proofof Lemma 2.5, see [31, Lemma 3.3].

Lemma 2.5. *Let*

$$\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}, \kappa), \qquad \widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, D; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}, \kappa),$$
$$\Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa),$$

be realizations of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ such that Σ is passive, $\widehat{\Sigma}$ is a passive dilation of Σ and Σ' is the first minimal restriction of $\widehat{\Sigma}$. Then

$$E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right) \leq E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}. \tag{2.23}$$

Moreover, for any isometric realization $\Sigma_1 = (A_1, B_1, C_1, D; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, \kappa)$ of θ,

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right) \leq E_{\mathcal{X}_1}\left(\sum_{k=0}^{n} A_1^k B_1 u_k\right), \qquad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}, \tag{2.24}$$

and for any co-isometric realization $\Sigma_2 = (A_2, B_2, C_2, D; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}, \kappa)$ of θ,

$$E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^{*k} C^* y_k\right) \leq E_{\mathcal{X}_2}\left(\sum_{k=0}^{n} A_2^{*k} C_2^* y_k\right), \qquad n \in \mathbb{N}_0, \quad y_k \in \mathcal{Y}, \tag{2.25}$$

In the Hilbert space case, the existence of optimal minimal realizations of an ordinary Schur function θ was first proved by Arov in [6]. He constructed such a realization by using (right) defect function, or what is the same thing, the maximal analytic minorant of $I - \theta^*(\zeta)\theta(\zeta)$. In [7], Arov et al. proved that the first minimal restriction, see Remark 2.4, of the simple conservative realization of θ is optimal minimal passive. The same is true for the generalized Schur class $\mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$; see [31, Theorem 3.5], or [32, Theorem 4.3] for the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. However, an optimal realization needs not to be minimal. Indeed, any co-isometric observable realization is optimal. In Hilbert space case, this is known, and seems to be first stated by Ando in [4, Corollary 5.6], without using the notion of the optimality. Ando's approach is to use the canonical observable co-isometric realization, where the state space is de Branges–Rovnyak space, see Lemma 2.2. His result holds also in general, and a direct proof, which does not use properties of the canonical realization, will be given. It then directly follows, that an optimal minimal realization can also be obtained by taking controllable restriction of the observable co-isometric system, as it will be proved in Proposition 2.8 below. For ordinary Hilbert space operator-valued Schur functions, such realizations are considered in [7, Section 5].

In Hilbert spaces, it is easy to deduce that a contractive densely defined linear relation can be extended to everywhere defined contractive linear operator. The same is true for Pontryagin spaces with the same negative index. For a proof, see for instance [2, Section 1.4].

Lemma 2.6. *Let S be a linear relation in $\mathcal{U} \times \mathcal{Y}$, where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, such that the domain of S is dense in $\mathcal{U}$ and*

$$\langle Su, Su \rangle_{\mathcal{Y}} \leq \langle u, u \rangle_{\mathcal{U}}, \tag{2.26}$$

for every $u \in \operatorname{dom}(S)$. Then S has a unique continuous extension to a contractive operator $\bar{S} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. If the equality holds in (2.26), the operator $\bar{S}$ is an isometry which is unitary if and only if the range of S is dense in $\mathcal{Y}$.

By using the extension results of Lemma 2.6, another extensively used lemma can be obtained.

Lemma 2.7. *Let*

$$\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa), \qquad \Sigma' = (A', B', C', D; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \kappa), \tag{2.27}$$

be passive κ-admissible realizations of θ.

(i) *If Σ' is isometric controllable, then there exists a contractive linear operator $Z : \mathcal{X}' \to \mathcal{X}$ such that the identites*

$$ZA'^{n}B' = A^{n}B, \qquad Z(I - zA')^{-1}B' = (I - zA)^{-1}B, \tag{2.28}$$

$$Z^{*}A^{*n}C^{*} = A'^{*k}C'^{*}, \qquad Z^{*}(I - zA^{*})^{-1}C^{*} = (I - zA'^{*})^{-1}C'^{*} \tag{2.29}$$

hold for every $n \in \mathbb{N}_0$ and every z in a sufficiently small symmetric neighbourhood of the origin. In addition, if Σ is also controllable, then Z has a dense range.

(ii) *If Σ' is simple conservative, then there exists a contractive linear operator $Z : \mathcal{X}' \to \mathcal{X}$ which has a dense range such that the identity (2.28) and*

$$ZA^{*'n}C^{*'} = A^{*n}C^{*}, \qquad Z(I - zA'^{*})^{-1}C^{*'} = (I - zA^{*'n})^{-1}C^{*'}, \tag{2.30}$$

hold. In addition, if Σ is also simple, then Z has a dense range.

Proof. Both cases will be proved simultaneously, the expressions in brackets refer to the case where the realizations in (2.27) are simple passive and Σ' is conservative. For vectors of the form

$$x' = \sum_{k=0}^{n} A'^{k}B'u_k \left[+ \sum_{k=0}^{n} A^{*'k}C^{*'}y_k \right], \quad n \in \mathbb{N}_0, \quad \{u_k\}_{k=0}^{n} \subset \mathcal{U}, \quad [\{y_k\}_{k=0}^{n} \subset \mathcal{Y}],$$

define a linear relation,

$$Zx' = \sum_{k=0}^{n} A^{k}Bu_k \left[+ \sum_{k=0}^{n} A^{*k}C^{*}y_k \right].$$

Since Σ' controllable [simple], Z is densely defined, and if Σ is controllable [simple], it has a dense range. We have $CA^{n}B = C'A'^{n}B'$, since the transfer functions of the systems coincide. Moreover, since Σ' is isometric [conservative], it follows from (2.24) [and (2.25)] of Lemma 2.5 that

$$
\begin{aligned}
E(Zx')_{\mathcal{X}} &= E\left(\sum_{k=0}^{n} A^k B u_k\right)_{\mathcal{X}} \left[+2\Re\left\langle \sum_{k=0}^{n} A^k B u_k, \sum_{j=0}^{n} A^{*j} C^* y_j \right\rangle_{\mathcal{X}} + E\left(\sum_{k=0}^{n} A^{*k} C^* y_k\right)_{\mathcal{X}}\right] \\
&\le E\left(\sum_{k=0}^{n} A'^k B' u_k\right)_{\mathcal{X}'} \left[+2\Re\left(\sum_{k=0}^{n}\sum_{j=0}^{n} \langle CA^j A^k B u_k, y_j\rangle_{\mathcal{Y}}\right) + E\left(\sum_{k=0}^{n} A^{*'k} C^{*'} y_k\right)_{\mathcal{X}'}\right] \\
&= E\left(\sum_{k=0}^{n} A'^k B' u_k\right)_{\mathcal{X}'} \left[+2\Re\left(\sum_{k=0}^{n}\sum_{j=0}^{n} \left\langle C' A'^j A'^k B' u_k, y_j\right\rangle_{\mathcal{Y}}\right) + E\left(\sum_{k=0}^{n} A^{*'k} C^{*'} y_k\right)_{\mathcal{X}'}\right] \\
&= E(x')_{\mathcal{X}},
\end{aligned}
$$

which implies that Z is contractive. By Lemma 2.6, it has an extension, still denoted by Z, which is a contractive everywhere defined linear operator, and if Σ is controllable [simple], it has a dense range. By definition, $ZA'^n B' = A^n B$ [and $ZA'^{*n} C'^* = A^{*n} C^*$]. Then by continuity and the Neumann series, (2.28) [and (2.30)] hold, and the part (ii) is proved. By using the identity $CA^n B = C' A'^n B'$ and (2.28), one obtains

$$
B'^* A'^{*n} A'^{*k} C'^* = B^* A^{*n} A^{*k} C^* = B'^* A'^{*n} Z^* A^{*k} C^* \quad \text{for} \quad n, k \in \mathbb{N}_0.
$$

Since Σ' is controllable, this implies $Z^* A^{*k} C^* = A'^{*k} C'^*$ and (2.29) follows, so the proof is complete. ☐

Here is the result on optimal systems, which was promised before Lemma 2.6.

Proposition 2.8. *An observable co-isometric realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is optimal, and its controllable restriction is optimal minimal.*

Proof. Let $\Sigma' = (A', B', C', D; \mathcal{X}'; \mathcal{U}, \mathcal{Y}; \kappa)$ and $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa)$ be an observable co-isometric and a minimal passive realization of θ, respectively. To prove that Σ' is optimal, it follows from Lemma 2.5 that it is enough to ensure that

$$
E_{\mathcal{X}'}\left(\sum_{k=0}^{n} A'^k B' u_k\right) \le E_{\mathcal{X}}\left(\sum_{k=0}^{n} A^k B u_k\right), \quad n \in \mathbb{N}_0, \quad u_k \in \mathcal{U}. \tag{2.31}
$$

The duals Σ'^* and Σ^* are controllable isometric and minimal passive realization of $\theta^{\#}$, respectively. By Lemma 2.7, there exists a contraction $Z : \mathcal{X}' \to \mathcal{X}$ such that $Z^* A^n B = A'^n B'$, and therefore inequality (2.31) holds. By Lemma 2.3, the controllable restriction $\Sigma_c = (A'_c, B'_c, C'_c, D; \mathcal{X}'^c, \mathcal{U}, \mathcal{Y}; \kappa)$ of Σ' is minimal, and it holds $A_c'^n B'_c = A'^n B'$. Therefore Σ'_c is also optimal. ☐

3. Products, regular and (±)-regular factorizations

The *product* or *cascade connection* of two systems

$$
\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1), \qquad \Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)
$$

is a system $\Sigma_2 \circ \Sigma_1 = (T_{\Sigma_2 \circ \Sigma_1}; \mathcal{X}_1 \oplus \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \kappa_1 + \kappa_2)$ such that

$$
T_{\Sigma_2 \circ \Sigma_1} = \begin{pmatrix} \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} & \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} \\ \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} & D_2 D_1 \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}. \tag{3.32}
$$

Written in the form (1.3), one has $\mathcal{X} = \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix}$ and

$$
A = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix}, \quad C = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix}, \quad D = D_2 D_1. \tag{3.33}
$$

In particular, $A_2 = A\restriction_{\mathcal{X}_2}$, $A\mathcal{X}_2 \subset \mathcal{X}_2$ and

$$\begin{pmatrix} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ D_2C_1 & C_2 & D_2D_1 \end{pmatrix} = \begin{pmatrix} I_{\mathcal{X}_1} & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & I_{\mathcal{X}_2} & 0 \\ C_1 & 0 & D_1 \end{pmatrix} : \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{Y} \end{pmatrix}. \tag{3.34}$$

The product $\Sigma_2 \circ \Sigma_1$ is defined when the incoming space of Σ_2 is the outgoing space of Σ_1. It is easy to check that $\theta_{\Sigma_2 \circ \Sigma_1} = \theta_{\Sigma_2}\theta_{\Sigma_1}$ whenever both functions are defined. For the dual system one has $(\Sigma_2 \circ \Sigma_1)^* = \Sigma_1^* \circ \Sigma_2^*$. From the identity (3.34) it follows that the product $\Sigma_2 \circ \Sigma_1$ is conservative (isometric, co-isometric, passive) whenever Σ_1 and Σ_2 are. Also, if the product is isometric (co-isometric, conservative) and one factor of the product is conservative, then the other factor must be isometric (co-isometric, conservative). If $\Sigma_2 \circ \Sigma_1$ is controllable (observable, simple, minimal), then so are Σ_1 and Σ_2; see [2, Theorem 1.2.1]. The converse statement is not true. The following lemma gives necessary and sufficient conditions when the product is observable, controllable or simple. If the incoming and outgoing spaces are assumed to be Hilbert spaces, the simple case is first treated in [9, Lemma 7.4], and the other cases in [29, Lemma 3.3]. The proofs given therein can be applied word by word. In particular, the definiteness of the incoming and outgoing spaces used in [29] do not play any role.

Lemma 3.1. *Let*

$$\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1), \qquad \Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$$

and $\Sigma = \Sigma_2 \circ \Sigma_1$. *Let* Ω *be a neighbourhood of the origin such that the transfer function* $\theta_\Sigma = \theta_{\Sigma_2}\theta_{\Sigma_1}$ *of* Σ *is analytic in* Ω. *Consider the equations*

$$\theta_{\Sigma_2}(z)C_1(I - zA_1)^{-1}x_1 = -C_2(I - zA_2)^{-1}x_2 \quad \text{for all } z \in \Omega; \tag{3.35}$$

$$\theta_{\Sigma_1}^{\#}(z)B_2^*(I - zA_2^*)^{-1}x_2 = -B_1^*(I - zA_1^*)^{-1}x_1 \quad \text{for all } z \in \Omega, \tag{3.36}$$

where $x_1 \in \mathcal{X}_1$ *and* $x_2 \in \mathcal{X}_2$. *Then* Σ *is observable if and only if* (3.35) *has only a trivial solution, and* Σ *is controllable if and only if* (3.36) *has only a trivial solution. Moreover,* Σ *is simple if and only if the pair of equations consisting of* (3.35) *and* (3.36) *has only a trivial solution.*

Consider $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U}, \mathcal{Y}_1)$ and $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1, \mathcal{Y})$, where $\mathcal{U}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Then $\theta_2\theta_1 \in \mathbf{S}_{\kappa'}(\mathcal{U}, \mathcal{Y})$ where $\kappa' \leq \kappa_1 + \kappa_2$ [2, Theorem 4.1.1]. Conditions when equality holds are derived in [2, Theorem 4.1.1]. When $\mathcal{U}$, $\mathcal{Y}_1$, and $\mathcal{Y}$ are Hilbert spaces, this happens when there are no pole cancellations, see [21, Proposition 7.11] and [2, Theorem 4.2.1]. In a setting where $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and $\theta_2 \in \mathbf{S}_{\kappa_1}(\mathcal{Y}_1, \mathcal{Y})$ are fixed, an extended version of Leech's factorization theorem, see [3, Theorem 8], can be used to determine if there exists $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U}, \mathcal{Y}_1)$ such that $\theta = \theta_2\theta_1$ and $\kappa_1 + \kappa_2 = \kappa$. In the case $\kappa_1 + \kappa_2 < \kappa$, the product $\Sigma = \Sigma_2 \circ \Sigma_1$ of passive κ-admissible realizations of θ_1 and θ_2 is a passive realization of θ, but it is not κ-admissible. Consequently, factorizations with the index condition $\kappa_1 + \kappa_2 = \kappa$ are the most interesting.

For ordinary Schur function θ, the factorization $\theta = \theta_2\theta_1$, where θ_1 and θ_2 are ordinary Schur functions, is *regular*, if $\Sigma_\theta = \Sigma_{\theta_2} \circ \Sigma_{\theta_1}$ is a simple conservative realization of θ whenever Σ_{θ_1} and Σ_{θ_2} are simple conservative realizations of θ_1 and θ_2 [17]. Instead of the system theoretical definition above, an equivalent function theoretical definition of regularity that goes back to [39, Chapter VII] and [17] is often used. That definition is generalized to a Pontryagin state space case in [9, Section 8]. It is needed only in the last results of this paper, and it is stated in Definition 4.3. In the standard case Khanh [25] defines $(+)$-*regular* and $(-)$-*regular* factorizations by using modified and strengthened versions of the function theoretical definition, see again Definition 4.3. He shows that if the factorization is $(+)$-regular ($(-)$-regular), $\Sigma_{\theta_2} \circ \Sigma_{\theta_1}$ is observable conservative (controllable) whenever Σ_{θ_1} and Σ_{θ_2} are. Khanh's definition, as well as the function theoretical definition of regularity depends strongly on the functional model of Sz.-Nagy and Foias from [39], which is Hilbert space specific. In the standard Hilbert state space case, the characterizations of regularity that uses the canonical unitary de Branges–Rovnyak model, are derived in [12], see also [13, Section 7]. The general definitions of regularity and $(\pm)$-regularity, based on the de Branges–Rovnyak models instead of Sz.-Nagy and Foias, which covers also the classes $\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ where $\mathcal{U}$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index, can be stated. The notation follows [2, Chapter 1], especially, $\theta_2\mathcal{H}(\theta_1)$ means the space generated by the kernel $K'(w,z)=\theta_2(z)K_{\theta_1}(w,z)\theta_2^*(w)$, where K_{θ_1} is the Schur kernel (1.1) for θ_1.

Definition 3.2. Let $\theta=\theta_2\theta_1\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\theta_1\in\mathbf{S}_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$ and $\theta_2\in\mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y})$. Then the factorization $\theta=\theta_2\theta_1$ is called:

- (I) $(+)$-regular if the following two conditions hold:
  - (a) $\mathcal{H}(\theta)=\theta_2\mathcal{H}(\theta_1)\oplus\mathcal{H}(\theta_2)$;
  - (b) the mapping $h_1\mapsto\theta_2h_1$ is an isometry from $\mathcal{H}(\theta_1)$ to $\theta_2\mathcal{H}(\theta_1)$;
- (II) $(-)$-regular if the following two conditions hold:
  - (a) $\mathcal{H}(\theta^\#)=\theta_1^\#\mathcal{H}(\theta_2^\#)\oplus\mathcal{H}(\theta_1^\#)$;
  - (b) the mapping $h_2\mapsto\theta_1^\#h_2$ is an isometry from $\mathcal{H}(\theta_2^\#)$ to $\theta_1^\#\mathcal{H}(\theta_2^\#)$;
- (III) regular if the following two conditions hold:
  - (a) $\mathcal{D}(\theta)=R_1^\#\mathcal{D}(\theta_2)\oplus R_2\mathcal{D}(\theta_1)$, where
$$R_1(z)=\begin{pmatrix} I & 0\\ 0 & \theta_1(z)\end{pmatrix},\quad R_2(z)=\begin{pmatrix}\theta_2(z) & 0\\ 0 & I\end{pmatrix};$$
  - (b) the mappings
$$\begin{pmatrix}h_1\\ k_1\end{pmatrix}\mapsto\begin{pmatrix}\theta_2h_1\\ k_1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}h_2\\ k_2\end{pmatrix}\mapsto\begin{pmatrix}h_2\\ \theta_1^\#k_2\end{pmatrix}\tag{3.37}$$
    are isometries from $\mathcal{D}(\theta_1)$ to $R_2\mathcal{D}(\theta_1)$ and $\mathcal{D}(\theta_2)$ to $R_1^\#\mathcal{D}(\theta_2)$, respectively.

In general, the regularity, $(+)$-regularity and $(-)$-regularity are rather strong properties, see [39, Chapter VII, Proposition 3.5] and Example 3.5 and the discussion that follows it.

Theorem 3.3. *Let*
$$\Sigma_1=(A_1,B_1,C_1,D_1;\mathcal{X}_1,\mathcal{U},\mathcal{Y}_1;\kappa_1),\qquad\Sigma_2=(A_2,B_2,C_2,D_2;\mathcal{X}_2,\mathcal{Y}_1,\mathcal{Y};\kappa_2)$$
be passive realizations of $\theta_1\in\theta_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$ *and* $\theta_2\in\mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{U})$*, respectively. Denote* $\theta=\theta_2\theta_1$.

- (i) *If* Σ_1 *and* Σ_2 *are observable co-isometric, then the product* $\Sigma=\Sigma_2\circ\Sigma_1$ *is observable co-isometric if and only if the factorization* $\theta=\theta_2\theta_1$ *is* $(+)$*-regular;*
- (ii) *If* Σ_1 *and* Σ_2 *are controllable isometric, then the product* $\Sigma=\Sigma_2\circ\Sigma_1$ *is controllable isometric if and only if the factorization* $\theta=\theta_2\theta_1$ *is* $(-)$*-regular;*
- (iii) *If* Σ_1 *and* Σ_2 *are simple conservative, then the product* $\Sigma=\Sigma_2\circ\Sigma_1$ *is simple conservative if and only if the factorization* $\theta=\theta_2\theta_1$ *is regular.*

Moreover, if the factorization is $((\pm)$*-*$)$*regular, then* $\theta=\theta_2\theta_1\in\mathbf{S}_{\kappa_1+\kappa_2}(\mathcal{U},\mathcal{Y})$.

Proof. The parts (i) and (ii) with the additional condition that the incoming and outgoing spaces are Hilbert spaces, are proved in [29, Theorems 3.6 and 3.7]. The same proofs can be applied word by word in the general case. Therefore, only the proof of part (iii) will be given.

Since simple conservative realizations are unitarily similar, it can be assumed in part (iii) that Σ_1 and Σ_2 are canonical unitary realizations defined similarly as in (2.13).

Then, it follows from Lemma 2.2 that the identities (3.35) and (3.36) in Lemma 3.1 are equivalent to

$$\theta_2(z)h_1(z) = -h_2(z) \qquad \text{for all} \quad z \in \Omega, \tag{3.38}$$

$$\theta_1^{\#}(z)k_2(z) = -k_1(z) \qquad \text{for all} \quad z \in \Omega, \tag{3.39}$$

where $\binom{h_1}{k_1} \in \mathcal{D}(\theta_1)$, $\binom{h_2}{k_2} \in \mathcal{D}(\theta_2)$, and Ω is some sufficiently small symmetric neighbourhood of the origin. Suppose that the realization $\Sigma = \Sigma_2 \circ \Sigma_1$ is simple conservative. Then, $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where κ is the negative index of the state space $\mathcal{X}_1 \oplus \mathcal{X}_2$ of Σ, and therefore $\kappa = \kappa_1 + \kappa_2$. Then, [2, Theorem 4.1.5] and the discussion therein show that the spaces $R_2\mathcal{D}(\theta_1)$ and $R_1^{\#}\mathcal{D}(\theta_2)$ are contained contractively in $\mathcal{D}(\theta)$, and the mappings in (3.37) are partial isometries. Since Σ is simple, it follows from Lemma 3.1 that the pair of equations consisting of (3.38) and (3.39) has only the trivial solution. Since

$$R_2(z)\begin{pmatrix} h_1(z) \\ k_1(z) \end{pmatrix} = \begin{pmatrix} \theta_2(z)h_1(z) \\ k_1(z) \end{pmatrix}, \qquad R_1^{\#}(z)\begin{pmatrix} h_2(z) \\ k_2(z) \end{pmatrix} = \begin{pmatrix} h_2(z) \\ \theta_1^{\#}(z)k_2(z) \end{pmatrix}, \tag{3.40}$$

it can be deduced that the mappings in (3.37) have only the trivial kernel, and therefore they are isometries. Moreover, $R_1^{\#}\mathcal{D}(\theta_2) \cap R_2\mathcal{D}(\theta_1) = \{0\}$, and it follows from the results of [2, Theorem 1.5.3] that $\mathcal{D}(\theta) = R_1^{\#}\mathcal{D}(\theta_2) \oplus R_2\mathcal{D}(\theta_1)$, and necessity is proved.

Assume then that the factorization $\theta = \theta_2\theta_1$ is regular. From the condition (III) (a) in Definition 3.2 and the identities in (3.40) it follows that the equations (3.38) and (3.39) can hold only if $h_2 \equiv 0$ and $k_1 \equiv 0$. But in that case, it follows from the condition (III) (b) in Definition 3.2 that $h_1 \equiv 0$ and $k_2 \equiv 0$ also. That is, the pair of equations consisting of (3.38) and (3.39) have only the trivial solution. Then by Lemma 3.1 the system Σ is simple, and the sufficiency is proved. □

Theorem 3.4. *Suppose $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$ and $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y})$, where $\mathcal{U}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ are Pontryagin spaces with the same negative index. Let*

$$\Sigma_1 = (A_1,B_1,C_1,D_1;\mathcal{X}_1,\mathcal{U},\mathcal{Y}_1;\kappa_1), \quad \Sigma_2 = (A_2,B_2,C_2,D_2;\mathcal{X}_2,\mathcal{Y}_1,\mathcal{U};\kappa_2) \tag{3.41}$$

be κ-admissible observable (controllable, simple, minimal) passive realizations of θ_1 and θ_2, respectively. If the factorization $\theta = \theta_2\theta_1$ is $(+)$-regular ($(-)$-regular, regular, $(+)$- and $(-)$-regular), then the product $\Sigma = \Sigma_2 \circ \Sigma_1 = (A,B,C,D;\mathcal{X},\mathcal{Y},\mathcal{U};\kappa_1+\kappa_2)$ is a κ-admissible observable (controllable, simple, minimal) passive realization of θ.

Proof. The factorization $\theta = \theta_2\theta_1$ is $(+)$-regular if and only if the factorization $\theta^{\#} = \theta_1^{\#}\theta_2^{\#}$ is $(-)$-regular. Therefore it suffices to prove the claims involving $(-)$-regular and regular factorizations, since the other claims then follow by duality. Both cases will be proved simultaneously, the expressions in brackets refer to the regular case. Let the realizations in (3.41) be controllable [simple] passive, and let

$$\Sigma_1' = (A_1',B_1',C_1',D_1;\mathcal{X}_1',\mathcal{U},\mathcal{Y}_1;\kappa_1), \quad \Sigma_2' = (A_2',B_2',C_2',D_2;\mathcal{X}_2',\mathcal{Y}_1,\mathcal{U};\kappa_2)$$

be isometric controllable [simple conservative] realizations of θ_1 and θ_2, respectively. The existence of such realizations is guaranteed by Lemma 2.1. Let $Z_1 : \mathcal{X}' \to \mathcal{X}_1$ and $Z_2 : \mathcal{X}' \to \mathcal{X}_2$ be contractive mappings with dense ranges such that

$$Z_k(I - zA_k')^{-1}B_k' = (I - zA_k)^{-1}B_k \quad \left[\text{and} \quad Z_k(I - zA_k^{'*})^{-1}C_k^{*'} = (I - zA_k^{*'n})^{-1}C_k^{*'}\right],$$

for $k = 1,2$, and for all $z \in \Omega$ in a sufficiently small symmetric neighbourhood Ω of the origin. The existence of such mappings is guaranteed by Lemma 2.7. Suppose that (3.36) [and (3.35)] hold for some $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$ and for all $z \in \Omega$. One obtains

$$\begin{aligned} &\theta_1^{\#}(z)B_2^*(I - zA_2^*)^{-1}x_2 = -B_1^*(I - zA_1^*)^{-1}x_1 \\ \iff \quad &\theta_1^{\#}(z)B_2^{'*}(I - zA_2^{'*})^{-1}Z_2^*x_2 = -B_1^{'*}(I - zA_1^{'*})^{-1}Z_1^*x_1 \end{aligned}$$

[and

$$\begin{aligned}&\theta_2(z)C_1(I-zA_1)^{-1}x_1=-C_2(I-zA_2)^{-1}x_2\\ \iff\quad &\theta_2(z)C_1'(I-zA_1')^{-1}Z_1^*x_1=-C_2'(I-zA_2')^{-1}Z_2^*x_2 \quad].\end{aligned}$$

Since the factorization $\theta_2\theta_1$ is $(-)$-regular [regular], the system $\Sigma'=\Sigma_2'\circ\Sigma_1'$ is controllable isometric [simple conservative] by Theorem 3.3, and it follows from Lemma 3.1 that $Z_1^*x_1=0$ and $Z_2^*x_2=0$. Since Z_1 and Z_2 have dense ranges, the adjoints Z_1^* and Z_2^* have only the trivial kernels, and therefore $x_1=0$ and $x_2=0$. It follows then from Lemma 3.1 that $\Sigma=\Sigma_2\circ\Sigma_1$ is controllable [simple]. □

The converse to Theorem 3.4 is not true. A counterexample below provides a product of observable passive systems that is observable, but the corresponding factorization is not $(+)$-regular.

Example 3.5. Define

$$\theta_1(z)\equiv\begin{pmatrix}0 & \frac{1}{\sqrt{2}}\end{pmatrix},\qquad \theta_2(z)=\frac{1-\bar\alpha z}{z-\alpha},\quad \alpha\in\mathbb{D}\setminus\{0\},\qquad \theta(z)=\theta_2(z)\theta_1(z). \tag{3.42}$$

By combining [29, Example 3.8] and Theorem 3.3, it follows that $\theta_1\in\mathbf{S}(\mathbb{C}^2,\mathbb{C})$, where $\mathbb{C}$ and $\mathbb{C}^2$ are considered with the usual inner products, $\theta_2\in\mathbf{S}_1(\mathbb{C},\mathbb{C})$, $\theta\in\mathbf{S}_1(\mathbb{C}^2,\mathbb{C})$, and the factorization $\theta_2\theta_1$ is $(-)$-regular but not $(+)$-regular. Define $\mathcal{X}_1=\{0\}$, $A_1=0$, $B_1=0$, $C_1=0$, and $D_1=\begin{pmatrix}0 & \frac{1}{\sqrt{2}}\end{pmatrix}$. Then $\Sigma_1=(A_1,B_1,C_1,D_1;\mathcal{X}_1,\mathbb{C}^2,\mathbb{C};0)$ clearly is a minimal passive κ-admissible realization of θ_1. Let $\Sigma_2=(A_2,B_2,C_2,D_2;\mathcal{X}_2,\mathbb{C},\mathbb{C};1)$ be any minimal passive κ-admissible realization of θ_2, and consider the product $\Sigma=\Sigma_2\circ\Sigma_1=(A,B,C,D;\mathcal{X}_1\oplus\mathcal{X}_2,\mathbb{C}^2,\mathbb{C};1)$, where the operators are defined as in (3.33). The system Σ is a passive κ-admissible realization of θ, and since the factorization is $(-)$-regular, it follows from Theorem 3.4 that it is controllable. Moreover, since Σ_2 is observable, it holds

$$\overline{\operatorname{span}}\,\{A^{n*}C^*:n\in\mathbb{N}_0\}=\overline{\operatorname{span}}\left\{\begin{pmatrix}0\\ A_2^{n*}C_2^*\end{pmatrix}:n\in\mathbb{N}_0\right\}=\begin{pmatrix}0\\ \mathcal{X}_2\end{pmatrix}=\mathcal{X}_1\oplus\mathcal{X}_2,$$

and therefore Σ is observable and thus minimal.

The functions θ_1 and θ_2 in Example 3.5 are rational functions. Since the product Σ of two minimal systems Σ_1 and Σ_2 with the finite dimensional state spaces is minimal, the factorization $\theta=\theta_2\theta_1$ is *minimal* in a sense of rational matrix functions; see [15, Theorem 8.23 and p. 163]. Thus Example 3.5 illustrates also that the minimality of the factorization in the sense of rational matrix functions does not imply $(+)$-regularity nor $(-)$-regularity. However, by combining Theorem 3.4 and [15, Theorem 8.23 and p. 163], it follows that if the factorization $\theta_2\theta_1$ of two rational matrix functions is both $(+)$-regular and $(-)$-regular, then it is also minimal.

The factorization in Example 3.5 is the so-called left *Kreĭn-Langer factorization*. Indeed, in the case where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, it is known from [27] that $\theta\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ can be represented as

$$\theta(z)=\theta_r(z)B_r^{-1}(z)=B_l^{-1}(z)\theta_l(z) \tag{3.43}$$

where $\theta_r,\theta_l\in\mathbf{S}(\mathcal{U},\mathcal{Y})$, and the functions $B_r\in\mathbf{S}_\kappa(\mathcal{U},\mathcal{U})$ and $B_l\in\mathbf{S}_\kappa(\mathcal{Y},\mathcal{Y})$ are *Blaschke products* of degree κ. The factorizations in (3.43) are called the right and the left Kreĭn-Langer factorizations of θ, respectively. For details about Blaschke products and Kreĭn-Langer factorizations in the operator-valued case, see, for instance, [2, Chapter 4 and Appendix]. By combining Theorem 3.8 and [29, Theorem 3.9], it follows that in general, the right Kreĭn-Langer factorization, which is essentially unique, is always $(+)$-regular and the essentially unique left Kreĭn-Langer factorization is always $(-)$-regular.

It is not immediately clear from the definition, or from Theorem 3.3 and Theorem 3.4, that regularity is by no means a weaker property than a (+)- or (−)-regularity. However, it is, since the next lemma shows that if the factorization is (+)-regular or (−)-regular, it is also regular.

Proposition 3.6. *Let $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$, $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y})$ and $\theta = \theta_2\theta_1 \in \mathbf{S}_{\kappa}(\mathcal{U},\mathcal{Y})$, where $\kappa = \kappa_1 + \kappa_2$, and let*

$$\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1), \qquad \Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$$

be simple conservative realizations of θ_1 and θ_2, respectively. Then the factorization $\theta = \theta_2\theta_1$ is (+)-regular ((−)-regular) if and only if it holds $\mathcal{X}^o = \mathcal{X}_1^o \oplus \mathcal{X}_2^o$ ($\mathcal{X}^c = \mathcal{X}_1^c \oplus \mathcal{X}_2^c$), where $\mathcal{X}^o, \mathcal{X}_1^o$ and $\mathcal{X}_2^o$ ($\mathcal{X}^c, \mathcal{X}_1^c$ and $\mathcal{X}_2^c$) are observable (controllable) subspaces of $\Sigma_2 \circ \Sigma_1$, Σ_1 and Σ_2, respectively. Moreover, if the factorization is (+)-regular or (−)-regular, then it is also regular.

Proof. It suffices to prove the claims for (+)-regular factorizations, since the other claims follow by duality. The realizations Σ_1 and Σ_2 are κ-admissible, so according to Remark 2.4, the spaces $(\mathcal{X}_1^o)^\perp$ and $(\mathcal{X}_2^o)^\perp$ are Hilbert subspaces. By applying Lemma 2.3, the systems Σ_1 and Σ_2 can be represented as

$$T_{\Sigma_1} = \begin{pmatrix} A_{11} & A_{12} & B_{11} \\ 0 & A_{1,o} & B_{1,o} \\ 0 & C_{1,o} & D_1 \end{pmatrix} : \begin{pmatrix} (\mathcal{X}_1^o)^\perp \\ \mathcal{X}_1^o \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} (\mathcal{X}_1^o)^\perp \\ \mathcal{X}_1^o \\ \mathcal{Y}_1 \end{pmatrix}, \tag{3.44}$$

$$T_{\Sigma_2} = \begin{pmatrix} A_{21} & A_{22} & B_{21} \\ 0 & A_{2,o} & B_{2,o} \\ 0 & C_{2,o} & D_2 \end{pmatrix} : \begin{pmatrix} (\mathcal{X}_2^o)^\perp \\ \mathcal{X}_2^o \\ \mathcal{Y}_1 \end{pmatrix} \to \begin{pmatrix} (\mathcal{X}_2^o)^\perp \\ \mathcal{X}_2^o \\ \mathcal{Y} \end{pmatrix}, \tag{3.45}$$

where the restrictions

$$\Sigma_{1,o} = (A_{1,o}, B_{1,o}, C_{1,o}, D_1; \mathcal{X}_1^o, \mathcal{U}, \mathcal{Y}_1; \kappa_1), \qquad \Sigma_{2,o} = (A_{2,o}, B_{2,o}, C_{2,o}, D_2; \mathcal{X}_2^o, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$$

are observable co-isometric. It follows from [2, Theorem 2.4.1] that for $k = 1, 2$, the operator A_{k1} is either the zero operator or an isometry in $(\mathcal{X}_k^o)^\perp$ such that the identity

$$\lim {A_{k1}^*}^n x = 0 \tag{3.46}$$

holds for every $x \in (\mathcal{X}_k^o)^\perp$. Consider $\Sigma = \Sigma_2 \circ \Sigma_1 = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \kappa_1 + \kappa_2)$. The system Σ is a κ-admissible realization of $\theta = \theta_2\theta_1$. A computation using the representations (3.44) and (3.45) gives

$$T_\Sigma = \begin{pmatrix} \begin{pmatrix} A_{11} & 0 & A_{12} & 0 \\ 0 & A_{21} & B_{21}C_{1,o} & A_{22} \\ 0 & 0 & A_{1,o} & 0 \\ 0 & 0 & B_{2,o}C_{1,o} & A_{2,o} \end{pmatrix} & \begin{pmatrix} B_{11} \\ B_{21}D_1 \\ B_{1,o} \\ B_{2,o}D_1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & D_2C_{1,o} & C_{2,o} \end{pmatrix} & D_2D_1 \end{pmatrix} : \begin{pmatrix} \begin{pmatrix} (\mathcal{X}_1^o)^\perp \\ (\mathcal{X}_2^o)^\perp \\ \mathcal{X}_1^o \\ \mathcal{X}_2^o \end{pmatrix} \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} (\mathcal{X}_1^o)^\perp \\ (\mathcal{X}_2^o)^\perp \\ \mathcal{X}_1^o \\ \mathcal{X}_2^o \end{pmatrix} \\ \mathcal{Y} \end{pmatrix}.$$

Since T_Σ is unitary, the restriction $\operatorname{res}_{\mathcal{X} \to \mathcal{X}_1^o \oplus \mathcal{X}_2^o} \Sigma = \Sigma_{2,o} \circ \Sigma_{1,o} := \Sigma_o'$ is co-isometric, and according to Theorem 3.3, it is observable if and only if the factorization $\theta_2\theta_1$ is (+)-regular. Suppose $\mathcal{X}^o = \mathcal{X}_1^o \oplus \mathcal{X}_2^o$. Then by Lemma 2.3, Σ_o' is an observable restriction Σ_o of Σ, so the factorization is (+)-regular.

Suppose then that the factorization is (+)-regular. Then Σ_o' is observable co-isometric. Let $x \in (\mathcal{X}^o)^\perp$. The identity (2.5) shows that $CA^n x = 0$ for $n = 0, 1, 2, \ldots$, and an easy calculation shows that $P_{\mathcal{X}_1^o \oplus \mathcal{X}_2^o} x = 0$. Therefore $(\mathcal{X}^o)^\perp \subset (\mathcal{X}_1^o)^\perp \oplus (\mathcal{X}_2^o)^\perp$, which implies $\mathcal{X}_1^o \oplus \mathcal{X}_2^o \subset \mathcal{X}^o$, since the orthocomplement of $(\mathcal{X}_1^o)^\perp \oplus (\mathcal{X}_2^o)^\perp$ in $\mathcal{X}$ is $\mathcal{X}_1^o \oplus \mathcal{X}_2^o$, Since $\mathcal{X}^o \subset \mathcal{X}_1^o \oplus \mathcal{X}_2^o$ holds in general by (2.5) and (3.33), one deduces $\mathcal{X}_1^o \oplus \mathcal{X}_2^o = \mathcal{X}^o$.

Assume still that the factorization is (+)-regular. By Theorem 3.3, to prove that the factorization is also regular, it suffices to prove that $\Sigma_2 \circ \Sigma_1$ is simple. It follows from the identities (2.9) and (2.16) that the Hilbert space $(\mathcal{X}^s)^\perp$ is a subspace of $(\mathcal{X}^o)^\perp = (\mathcal{X}_1^o)^\perp \oplus (\mathcal{X}_2^o)^\perp$, $C(\mathcal{X}^s)^\perp = \{0\}$, $B^*(\mathcal{X}^s)^\perp = \{0\}$ and $(\mathcal{X}^s)^\perp$ is both A-invariant and A^*-invariant. Since the system operator T_Σ is unitary, it can be deduced that

$$T_\Sigma\restriction_{(\mathcal{X}^s)^\perp} = A\restriction_{(\mathcal{X}^s)^\perp} = (A_{11}\oplus A_{21})\restriction_{(\mathcal{X}^s)^\perp}$$

is a unitary operator. But it follows from (3.46) that

$$\lim_{n\to\infty}\begin{pmatrix} A_{11} & 0 \\ 0 & A_{21}\end{pmatrix}^{*n}\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = 0$$

for every $x_1\oplus x_2 \in (\mathcal{X}^s)^\perp \subset (\mathcal{X}_1^o)^\perp \oplus (\mathcal{X}_2^o)^\perp$, which implies $(\mathcal{X}^s)^\perp = \{0\}$, and thus Σ is a simple conservative system. □

In particular, it follows from Proposition 3.6 that for ordinary Schur functions, the definition of $(\pm)$-regularity is equivalent to the definition given by Khanh; see [25, Theorems 1 and 2]. Indeed, Khanh's definition will be stated and applied for special cases considered in Section 4.

Proposition 3.7. *Suppose that $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, where $\theta_1 \in S_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$ and $\theta_2 \in S_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y})$. If the factorization $\theta_2\theta_1$ is regular ($(+)$-regular, $(-)$-regular), every κ-admissible conservative (co-isometric, isometric) realization Σ of θ can be represented as a product of the form $\Sigma = \Sigma_2\circ\Sigma_1$, where Σ_1 and Σ_2 are κ-admissible conservative (co-isometric, isometric) realizations of θ_1 and θ_2. If desired, Σ_1 and Σ_2 can be chosen such that one of them is simple (observable, controllable).*

Proof. Suppose that the factorization is regular, and let

$$\begin{gathered}\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\kappa_1+\kappa_2),\\ \Sigma_1' = (A_1',B_1',C_1',D_1';\mathcal{X}_1',\mathcal{U},\mathcal{Y}_1;\kappa_1), \qquad \Sigma_2' = (A_2',B_2',C_2',D_2';\mathcal{X}_2',\mathcal{U},\mathcal{Y}_1;\kappa_2)\end{gathered} \tag{3.47}$$

be κ-admissible conservative realizations of θ,θ_1 and θ_2, respectively, such that Σ_1' and Σ_2' are simple. By Lemma 2.3, the system operator T of Σ can be represented as

$$T = \begin{pmatrix}\begin{pmatrix} A_0 & 0 \\ 0 & A_s\end{pmatrix} & \begin{pmatrix} 0 \\ B_s\end{pmatrix} \\ \begin{pmatrix} 0 & C_s\end{pmatrix} & D\end{pmatrix} : \begin{pmatrix}\begin{pmatrix}(\mathcal{X}^s)^\perp \\ \mathcal{X}^s\end{pmatrix} \\ \mathcal{U}\end{pmatrix} \to \begin{pmatrix}\begin{pmatrix}(\mathcal{X}^s)^\perp \\ \mathcal{X}^s\end{pmatrix} \\ \mathcal{Y}\end{pmatrix}, \tag{3.48}$$

where $(\mathcal{X}^s)^\perp$ is a Hilbert subspace, and $\Sigma_s = (A_s,B_s,C_s,D;\mathcal{X}^s,\mathcal{U},\mathcal{Y};\kappa_1+\kappa_2)$ is a simple conservative restriction of Σ. Theorem 3.3 shows that

$$\Sigma_2'\circ\Sigma_1' = (\hat{A},\hat{B},\hat{C},D;\mathcal{X}_1'\oplus\mathcal{X}_2',\mathcal{U},\mathcal{Y};\kappa_1+\kappa_2)$$

is simple conservative, and thus unitarily similar with Σ_s. Therefore there exists a unitary operator $U:\mathcal{X}_1'\oplus\mathcal{X}_2'\to\mathcal{X}^s$ such that $\hat{A} = U^{-1}A_sU$, $\hat{B} = U^{-1}B_s$ and $\hat{C} = C_sU$. Define

$$\mathcal{X}_1^s = U\mathcal{X}_1', \qquad \mathcal{X}_2^s = U\mathcal{X}_2'.$$

Then $U_1 = U\restriction\mathcal{X}_1' \to \mathcal{X}_1^s$ and $U_2 = U\restriction\mathcal{X}_2'\to\mathcal{X}_2^s$ are unitary operators. Since $\hat{A}\mathcal{X}_2'\subset\mathcal{X}_2'$, also $A\mathcal{X}_2^s\subset\mathcal{X}_2^s$. Define

$$A_{sk} = U_kA_1'U_k^{-1}, \quad B_{sk} = U_kB_k', \quad C_{sk} = C_k'U_k^{-1}, \quad D_k = D_k', \qquad k=1,2.$$

and then

$$
\begin{aligned}
&\mathcal{X}_1 = (\mathcal{X}^s)^\perp \oplus \mathcal{X}_1^s, \qquad \mathcal{X}_2 = \mathcal{X}_2^s,\\
&A_1 = \begin{pmatrix} A_{0k} & 0 \\ 0 & A_{sk} \end{pmatrix} : \begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}_1^s \end{pmatrix}, \qquad A_2 = A_{s2},\\
&B_1 = \begin{pmatrix} 0 \\ B_{s1} \end{pmatrix} : \mathcal{U} \to \begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}_1^s \end{pmatrix}, \qquad B_2 = B_{s2},\\
&C_1 = \begin{pmatrix} 0 & C_{sk} \end{pmatrix} : \begin{pmatrix} (\mathcal{X}^s)^\perp \\ \mathcal{X}_1^s \end{pmatrix} \to \mathcal{Y}_1, \qquad C_2 = C_{s2},\\
&\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1), \qquad \Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}_1; \kappa_2).
\end{aligned} \tag{3.49}
$$

Calculations show that Σ_1, Σ_2 and $\Sigma_2 \circ \Sigma_1 = \Sigma$ are conservative κ-admissible realizations of θ_1, θ_2 and $\theta = \theta_2\theta_1$, respectively. Moreover, Σ_2 is simple. Thus $\Sigma_2 \circ \Sigma_1$ is a desired representation. By choosing $\mathcal{X}_1 = \mathcal{X}_1^s$ and $\mathcal{X}_2 = (\mathcal{X}^s)^\perp \oplus \mathcal{X}_2^s$ in (3.49) and then the operators in a corresponding way, one deduces that Σ_1 is simple instead of Σ_2, but all the other results remains the same.

In the case where the factorization is $(+)$-regular ($(-)$-regular), and the systems are co-isometric (isometric), one has to consider the observable (controllable) restriction of the form (2.14) ((2.15)) instead of (3.48), and to include appropriate restrictions and projections in (3.49). Otherwise the proofs are analogous, and so other details will be omitted. □

An observable co-isometric realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is optimal by Proposition 2.8, and therefore the product of two observable co-isometric realizations is optimal if the factorization is $(+)$-regular. In general, $(\pm)$-regularity of the factorization affects that how optimality and *-optimality are preserved under the product of systems, see Theorem 3.8 below. In the Hilbert space setting for ordinary Schur functions, such properties are studied by Khanh in [25], Hang in [24] and by Khanh and Hang in [26]. Their results do not cover the following theorem.

Theorem 3.8. *Suppose that $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, where $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U}, \mathcal{Y}_1)$ and $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1, \mathcal{Y})$, and let*

$$\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \kappa_1), \qquad \Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \kappa_2)$$

be optimal realizations of θ_1 and θ_2, respectively. If the factorization $\theta_2\theta_1$ is $(+)$-regular, then $\Sigma = \Sigma_2 \circ \Sigma_1$ is an optimal realization of θ.

Proof. Let the systems in (3.47) be observable co-isometric realizations of θ_1 and θ_2, respectively. By Proposition 2.8, they are optimal, and so $E_{\mathcal{X}_j}\left(\sum_{k=0}^n A_j^k B_j u_k\right) = E_{\mathcal{X}_j'}\left(\sum_{k=0}^n A_j'^k B_j' u_k\right)$ for every $n \in \mathbb{N}_0$ and $\{u_k\}_{k=0}^n \subset \mathcal{U}$ or $\{u_k\}_{k=0}^n \subset \mathcal{Y}_1$. The polarization identity then gives

$$\left\langle \sum_{k=0}^{n_1} A_j^k B_j u_k, \sum_{k=0}^{n_2} A_j^k B_j f_k \right\rangle_{\mathcal{X}_j} = \left\langle \sum_{k=0}^{n_1} A_j'^k B_j' f_k, \sum_{k=0}^{n_2} A_j'^k B_j' f_k \right\rangle_{\mathcal{X}_j'}, u_k, f_k \in \mathcal{U} \text{ or } u_k, f_k \in \mathcal{Y}_1. \tag{3.50}$$

Denote the products $\Sigma_2 \circ \Sigma_1 = \Sigma$ and $\Sigma_2' \circ \Sigma_1' = \Sigma'$ by

$$\Sigma = (A, B, C, D; \mathcal{X}_1 \oplus \mathcal{X}_2; \mathcal{U}; \mathcal{Y}; \kappa_1 + \kappa_2), \quad \Sigma = (A', B', C', D; \mathcal{X}_1' \oplus \mathcal{X}_2'; \mathcal{U}; \mathcal{Y}; \kappa_1 + \kappa_2).$$

By induction, we have

$$\begin{aligned} A^k B &= \begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix}^k \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix} \\ &= \begin{pmatrix} A_1^k & 0 \\ A_2^{k-1}B_2C_1 + A_2^{k-2}B_2C_1A_1 + \cdots + B_2C_1A_1^{k-1} & A_2^k \end{pmatrix} \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix} \\ &= \begin{pmatrix} A_1^k B_1 \\ A_2^{k-1}B_2C_1B_1 + A_2^{k-2}B_2C_1A_1B_1 + \cdots + B_2C_1A_1^{k-1}B_1 + A_2^kB_2D_1 \end{pmatrix} \end{aligned} \tag{3.51}$$

for every $k \in \mathbb{N}_0$ with the obvious interpretation for the cases $n = 0$ or $n = 1$. Denote

$$\alpha(k) = A_2^{k-1}B_2C_1B_1 + A_2^{k-2}B_2C_1A_1B_1 + \cdots + B_2C_1A_1^{k-1}B_1,$$
$$\alpha'(k) = {A'_2}^{k-1}B'_2C'_1B'_1 + {A'_2}^{k-2}B'_2C'_1A'_1B'_1 + \cdots + B'_2C'_1{A'_1}^{k-1}B'_1.$$

Since the transfer functions of Σ_j and Σ'_j coincide, $C_jA_j^kB_j = C'_j{A'_j}^kB'_j$ for every $k \in \mathbb{N}_0$. By applying the identity (3.50), one deduces $\langle \alpha(k)u_k, \alpha(j)u_j \rangle_{\mathcal{X}_2} = \langle \alpha'(k)u_k, \alpha'(j)u_j \rangle_{\mathcal{X}'_2}$, and then

$$\left\langle \sum_{k=0}^{M} \alpha(k)u_k, \sum_{j=0}^{N} \alpha(j)u_j \right\rangle_{\mathcal{X}_2} = \left\langle \sum_{k=0}^{M} \alpha'(k)u_k, \sum_{j=0}^{N} \alpha'(j)u_j \right\rangle_{\mathcal{X}'_2}, \tag{3.52}$$

$$\left\langle \sum_{k=0}^{M} \alpha(k)u_k, \sum_{j=0}^{N} A_2^jB_2u_j \right\rangle_{\mathcal{X}_2} = \left\langle \sum_{k=0}^{M} \alpha'(k)u_k, \sum_{j=0}^{N} {A'_2}^jB'_2u_j \right\rangle_{\mathcal{X}'_2}. \tag{3.53}$$

The identities (3.50)–(3.53) then yield

$$\begin{aligned} &E_{\mathcal{X}_1\oplus\mathcal{X}_2}\left(\sum_{k=0}^{n} A^kBu_k\right) = E_{\mathcal{X}_1\oplus\mathcal{X}_2}\left(\begin{pmatrix} \sum_{k=0}^{n} A_1^kB_1u_k \\ \sum_{k=0}^{n}\alpha(k)u_k + \sum_{k=0}^{n} A_2^kB_2D_1u_k \end{pmatrix}\right) = E_{\mathcal{X}_1}\left(\sum_{k=0}^{n} A_1^kB_1u_k\right) \\ &\quad + E_{\mathcal{X}_2}\left(\sum_{k=0}^{n}\alpha(k)u_k\right) + 2\Re\left\langle \sum_{k=0}^{n}\alpha(k)u_k, \sum_{k=0}^{n} A_2^kB_2D_1u_k \right\rangle_{\mathcal{X}_2} + E_{\mathcal{X}_2}\left(\sum_{k=0}^{n} A_2^kB_2D_1u_k\right) \\ &= E_{\mathcal{X}'_1}\left(\sum_{k=0}^{n} {A'_1}^kB'_1u_k\right) + E_{\mathcal{X}'_2}\left(\sum_{k=0}^{n}\alpha'(k)u_k\right) + 2\Re\left\langle \sum_{k=0}^{n}\alpha'(k)u_k, \sum_{k=0}^{n} {A'_2}^kB'_2D_1u_k \right\rangle_{\mathcal{X}'_2} \\ &\quad + E_{\mathcal{X}'_2}\left(\sum_{k=0}^{n} {A'_2}^kB'_2D_1u_k\right) = E_{\mathcal{X}'_1\oplus\mathcal{X}'_2}\left(\sum_{k=0}^{n} {A'}^kB'u_k\right). \end{aligned} \tag{3.54}$$

Since the factorization is $(+)$-regular, the product Σ' is observable co-isometric by Theorem 3.3, and therefore optimal by Proposition 2.8. It follows from the identity (3.54) that Σ is also optimal. □

A *-optimal realization must be observable by definition, and therefore Proposition 3.8 has no straightforward symmetric counterparts for $(-)$-regular factorizations and *-optimal systems. However, a realization Σ of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is optimal minimal if and only if the dual Σ^* is *-optimal minimal realization of $\theta^\#$ [31, Theorem 3.5], and therefore for minimal systems, symmetric results can be obtained. In the Hilbert space settings for ordinary Schur functions, the following result, obtained by a different approach and different, but equivalent, definitions of $(+)$-regularity and $(-)$-regularity, is due to Khanh [25, Proposition 6].

Corollary 3.9. *If a factorization $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ is both $(+)$-regular and $(-)$-regular, then the product $\Sigma = \Sigma_2 \circ \Sigma_1$, where Σ_1 and Σ_1 are optimal (*-optimal) minimal realizations of θ_1 and θ_2, respectively, is an optimal (*-optimal) minimal realization of θ.*

Proof. For optimal systems, the result follows by combining Theorems 3.4 and 3.8. Since $\Sigma = \Sigma_2 \circ \Sigma_1$ is *-optimal minimal if and only if $\Sigma^* = \Sigma_1^* \circ \Sigma_2^*$ is optimal minimal, the claim involving *-optimality follows by duality. □

4. Invariant subspaces and factorizations

As it can be seen from (3.33), for every product representation $\Sigma_2 \circ \Sigma_1$ of the system Σ, there is a corresponding invariant Pontryagin subspace of the main operator of Σ. The converse is also true. For isometric (co-isometric, conservative) system Σ, a product representation corresponding to a given invariant Pontryagin subspace of the main operator can be constructed by using the method of [2, Theorem 1.2.2]. For a passive system Σ, one may consider an isometric *embedding*, see [31, Section 2], $\widetilde{\Sigma}$ of Σ, construct a product representation of $\widetilde{\Sigma}$, and then choose suitable blocks to get a product representation of Σ. By applying the method of [2, Theorem 1.2.2] to canonical co-isometric realization of $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$ and $(+)$-regular factorizations, one obtains a correspondence of the backward shift invariant regular subspaces of the generalized de Branges–Rovnyak spaces and $(+)$-regular factorizations. Similar results hold for regular ($(-)$-regular) factorizations and invariant Pontryagin subspaces of the main operator of the canonical unitary (isometric) realizations. For ordinary Schur functions and regular factorization, this is stated in [13, Theorem 6.1]; see also [17].

Theorem 4.1. *The space $\mathcal{H}$ is a backward shift invariant Pontryagin subspace of $\mathcal{H}(\theta)$, where $\theta \in \mathbf{S}_\kappa(\mathcal{U}, \mathcal{Y})$, if and only if there exists a $(+)$-regular factorization $\theta_2\theta_1$ of θ such that $U\mathcal{H}(\theta_2) = \mathcal{H}$, where U is a unitary operator. The negative indices of $\mathcal{H}^\perp$ and $\mathcal{H}$ coincide with the indices of the generalized Schur functions θ_1 and θ_2, respectively.*

Proof. Let Σ be a canonical observable co-isometric realization of θ, given in (2.12). The main operator of Σ is a backward shift, and if $\mathcal{H}$ is a backward shift invariant Pontryagin subspace with negative index κ', then $\mathcal{H}^\perp$ is a Pontryagin subspace with the negative index $\kappa'' = \kappa - \kappa'$. By applying [2, Theorem 1.2.2] one obtains a product representation $\Sigma_2 \circ \Sigma_1 = \Sigma$, where $\Sigma_1 = (T_{\Sigma_1}; \mathcal{H}^\perp, \mathcal{U}; \mathcal{Y}_1; \kappa'')$ and $\Sigma_2 = (T_{\Sigma_2}; \mathcal{H}, \mathcal{Y}_1; \mathcal{Y}; \kappa')$ are co-isometric; the details and notations are as in [2, Theorem 1.2.2]. Since Σ is observable, so are Σ_1 and Σ_2, and since T_{Σ_1} and T_{Σ_2} both are co-isometric, $\mathcal{Y}_1$ has the same negative index as $\mathcal{U}$ and $\mathcal{Y}$. Then by Lemma 2.1, the transfer functions θ_1 and θ_2 of Σ_1 and Σ_2, respectively, belong to $\mathbf{S}_{\kappa''}(\mathcal{U}, \mathcal{Y}_1)$ and $\mathbf{S}_{\kappa'}(\mathcal{Y}_1, \mathcal{Y})$, respectively. Moreover, the the factorization is $(+)$-regular by Theorem 3.3. The system Σ_2 is unitarily similar with the canonical co-isometric realization of θ_2, and therefore $U\mathcal{H}(\theta_2) = \mathcal{H}$, where U is the unitary similarity from $\mathcal{H}(\theta_2) \to \mathcal{H}$.

Suppose then that there exists a $(+)$-regular factorization $\theta_2\theta_1$ of θ. Let Σ_1 and Σ_2 be canonical co-isometric realizations of θ_1 and θ_2. Then $\Sigma_2 \circ \Sigma_1$ is a co-isometric observable realization of θ, and therefore it is unitarily similar with the canonical co-isometric realization of θ. Denote the unitary similarity mapping from $\mathcal{H}(\theta_1) \oplus \mathcal{H}(\theta_2)$ onto $\mathcal{H}(\theta)$ by U. Since the state space $\mathcal{H}(\theta_2)$ of Σ_2 is a regular invariant subspace of the main operator of $\Sigma_2 \circ \Sigma_1$, by applying the properties (2.10) of the unitary similarity, one deduces that $\mathcal{H} := U\mathcal{H}(\theta_2)$ is an invariant regular subspace of the main operator of the canonical co-isometric realization of θ. That is, $\mathcal{H}$ is a backward shift invariant regular subspace of $\mathcal{H}(\theta)$, $\operatorname{ind}_- \mathcal{H}^\perp = \operatorname{ind}_- \mathcal{H}(\theta_1)$ and $\operatorname{ind}_- \mathcal{H} = \operatorname{ind}_- \mathcal{H}(\theta_2)$, and those indices coincide with the indices of θ_1 and θ_2, respectively. □

If an ordinary scalar valued $\theta \in \mathbf{S}(\mathbb{C})$ is not an extreme point of the unit ball of H^∞, i.e., there exists an outer function φ such that $|\theta|^2 + |\varphi|^2 = 1$ almost everywhere on the unit circle $\mathbb{T}$, Sarason proved in [34] that the space $\mathcal{H}$ is a backward shift invariant subspace of $\mathcal{H}(\theta)$ if and only if $\mathcal{H} = \mathcal{H}(\theta) \cap \mathcal{H}(u)$ or $\mathcal{H} = \mathcal{H}(\theta)$, where u is an inner function. If θ is an inner function, the same result holds by Beurling's Theorem. In a case of a non-inner extreme point, Suárez characterized backward shift invariant subspaces of $\mathcal{H}(\theta)$ in [38]. In that case, the characterization is more complicated.

For certain functions, there is no difference between $(+)$-regular ($(-)$-regular, $(+)$-and $(-)$-regular) and regular factorizations. This happen when the right (left, right and left) defect function φ_θ (ψ_θ) of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ is the zero function. For an ordinary Schur function, φ_θ (ψ_θ) is a maximal outer (co-outer) minorant such that

$$\varphi_\theta^*(\zeta)\varphi_\theta(\zeta) \leq I - \theta^*(\zeta)\theta(\zeta), \qquad (\ \psi_\theta(\zeta)\psi_\theta^*(\zeta) \leq I - \theta(\zeta)\theta^*(\zeta)\ ) \qquad \text{for a.e.} \quad \zeta \in \mathbb{T}.$$

For generalized Schur functions, see the definition and properties in [31, Section 4]. An ordinary Schur function $\theta \in \mathbf{S}(\mathcal{U},\mathcal{Y})$ is an extreme point of the unit ball of $H^\infty(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ if and only if $\varphi_\theta \equiv 0$ or $\psi_\theta \equiv 0$ [40, Chapter 2], see also [14, Section 7].

Proposition 4.2. *Let $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ such that $\varphi_\theta \equiv 0$ ($\psi_\theta \equiv 0$, $\varphi_\theta \equiv 0$ and $\psi_\theta \equiv 0$). Then a regular factorization of θ is $(+)$-regular ($(-)$-regular, $(+)$- and $(-)$-regular).*

Proof. Since $\varphi_\theta \equiv 0$ ($\psi_\theta \equiv 0$, $\varphi_\theta \equiv 0$ and $\psi_\theta \equiv 0$), a simple conservative realization Σ of θ is observable (controllable, minimal) [31, Theorem 4.8]. If the factorization $\theta = \theta_2\theta_1$, is regular, then by Proposition 3.7, the system Σ has a corresponding product representation $\Sigma = \Sigma_2 \circ \Sigma_1$. By [2, Theorem 1.2.1], Σ_1 and Σ_2 are observable (controllable, minimal) conservative, so it follows from Theorem 3.3 that the factorization $\theta = \theta_2\theta_1$ is $(+)$-regular ($(-)$-regular, $(+)$- and $(-)$-regular). ☐

From a point of a view of invariant Pontryagin subspaces, it is possible to consider $(+)$-regular and $(-)$-regular factorizations as regular factorizations of functions whose defect functions are zeros. To this end, let $\theta_2\theta_1$ be a $(+)$-regular factorization of $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, and let $\mathcal{H}$ be a backward shift invariant subspace of $\mathcal{H}(\theta)$ as described in Theorem 4.1. By applying Julia embedding techniques from [31, Section 4], one can construct a Hilbert space $\mathcal{Y}'$ and a function $\Theta = \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \in \mathbf{S}_\kappa\left(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y}'\right)$ such that Θ has an observable conservative realization whose state space is $\mathcal{H}(\theta)$ and the main operator is the backward shift. Such a realization must be unitarily similar to the canonical unitary realization of Θ, and by using the corresponding unitary similarity mapping U, one obtains a Pontryagin subspace $U\mathcal{H}$ of $\mathcal{D}(\Theta)$, which is invariant under the main operator of the canonical unitary realization. Then there exists a corresponding regular, or what is now the same thing, $(+)$-regular factorization $\Theta_2\Theta_1 = \begin{pmatrix} \theta_2' \\ \varphi' \end{pmatrix}\Theta_1$ of Θ. In fact, $\theta_2'\Theta_1$ is then another $(+)$-regular factorization of θ, possibly different from $\theta_2\theta_1$, corresponding to the invariant Pontryagin subspace $\mathcal{H}$.

For any $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, the strong radial limit values

$$\theta(\zeta) := \lim_{r \to 1^-} \theta(r\zeta) \tag{4.55}$$

exist and are contractive, with respect to the indefinite inner product of $\mathcal{U}$ and $\mathcal{Y}$, for a.e. $\zeta \in \mathbb{T}$ [30, Theorem 2.8]. Suppose now that $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. A function $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ belongs to the class of *generalized inner (co-inner, bi-inner) functions* $\mathbf{I}_\kappa(\mathcal{U},\mathcal{Y})$ ($\mathbf{I}_\kappa^*(\mathcal{U},\mathcal{Y})$, $\mathbf{U}_\kappa(\mathcal{U},\mathcal{Y})$), if the limit values (4.55) are isometric (co-isometric, unitary) for a.e. $\zeta \in \mathbb{T}$. If $\kappa = 0$, these classes consist of ordinary inner, co-inner and bi-inner functions. If θ_1, θ_2 and $\theta = \theta_2\theta_1$ are ordinary Schur functions, it is known, see [39, Proposition 3.3], that if θ_2 is an inner function or θ_1 is a co-inner function, then the factorization $\theta = \theta_2\theta_1$ is regular. Corresponding results for generalized Schur, inner and co-inner functions is true, if one assumes the index condition $\kappa = \kappa_1 + \kappa_2$ For ordinary Schur functions, even more is true; if θ_2 is inner (θ_1 is co-inner), the factorization is, besides regular, also $(+)$-regular ($(-)$-regular). These results are immediate, if one uses Arov et al.'s definition of regular factorization for generalized Schur functions, and

Khanh's definition of (+)-regular factorizations of ordinary Schur functions. To this end, let $L^2(\mathcal{U})$ denote the Hilbert space of $\mathcal{U}$-valued, where $\mathcal{U}$ is a separable Hilbert space, measurable functions with the square integrable norm on the unit circle. The space $H^2(\mathcal{U})$ is a subspace of $L^2(\mathcal{U})$ that consists of functions with vanishing negative Fourier coefficients, see the details from [39, Chapter V]. For $\theta \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$, define

$$\Delta_\theta(\zeta) = (I_{\mathcal{U}} - \theta^*(\zeta)\theta(\zeta))^{1/2}, \qquad \zeta \in \mathbb{T}.$$

Definition 4.3. Let $\mathcal{U}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ be Hilbert spaces, and let $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$, $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y})$ and $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ such that $\kappa = \kappa_1 + \kappa_2$. Define a linear relation

$$V : \overline{\Delta_\theta L^2(\mathcal{U})} \to \overline{\begin{pmatrix} \Delta_{\theta_2} L^2(\mathcal{Y}_1) \\ \Delta_{\theta_1} L^2(\mathcal{U}) \end{pmatrix}},$$

by setting

$$V\left(\Delta_\theta(\zeta)u(\zeta)\right) = \begin{pmatrix} \Delta_{\theta_2}(\zeta)\theta_1(\zeta)u(\zeta) \\ \Delta_{\theta_1}(\zeta)u(\zeta) \end{pmatrix}, \qquad u \in L^2(\mathcal{U}).$$

The factorization $\theta = \theta_2\theta_1$ is called regular in a sense of Brodskiĭ and Sz.-Nagy and Foias, if V has a unitary extension. If $\kappa_1 = \kappa_2 = \kappa = 0$, define a linear relation

$$V^+ : \overline{\Delta_\theta H^2(\mathcal{U})} \to \overline{\begin{pmatrix} \Delta_{\theta_2} H^2(\mathcal{Y}_1) \\ \Delta_{\theta_1} H^2(\mathcal{U}) \end{pmatrix}},$$

by setting

$$V^+\left(\Delta_\theta(\zeta)u(\zeta)\right) = \begin{pmatrix} \Delta_{\theta_2}(\zeta)\theta_1(\zeta)u(\zeta) \\ \Delta_{\theta_1}(\zeta)u(\zeta) \end{pmatrix}, \qquad u \in H^2(\mathcal{U}).$$

The factorization $\theta = \theta_2\theta_1$ is called (+)-regular in a sense of Khanh, if V^+ has a unitary extension.

By combining Theorem 3.3, Proposition 3.6, [25, Theorems 1] and [9, Theorem 8.1], it follows that the definitions above, in the settings where they both can be applied, are equivalent to those given in Definition 3.2.

Proposition 4.4. *Let $\mathcal{U}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ be Hilbert spaces, and let $\theta_1 \in \mathbf{S}_{\kappa_1}(\mathcal{U},\mathcal{Y}_1)$, $\theta_2 \in \mathbf{S}_{\kappa_2}(\mathcal{Y}_1,\mathcal{Y})$ and $\theta = \theta_2\theta_1 \in \mathbf{S}_\kappa(\mathcal{U},\mathcal{Y})$ such that $\kappa = \kappa_1 + \kappa_2$. If θ_2 (θ_1) is a generalized inner function (co-inner function), then the factorization $\theta_2\theta_1 = \theta$ is regular, and if $\kappa_1 = \kappa_2 = \kappa = 0$, then it is also (+)-regular ((−)-regular).*

Proof. The factorization $\theta = \theta_2\theta_1$ is regular ((+)-regular) if and only if the factorization $\theta^\# = \theta_1^\#\theta_2^\#$ is regular ((−)-regular), and θ_2 is a generalized inner function if and only if $\theta_2^\#$ is a generalized co-inner function. Therefore, it suffices to prove the claims concerning the case where θ_2 is a generalized inner function. When it is, $\Delta_{\theta_2}(\zeta) = 0$ and $\Delta_\theta(\zeta) = \Delta_{\theta_1}(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Then $\overline{\Delta_{\theta_2} L^2(\mathcal{Y}_1)}$ and $\overline{\Delta_{\theta_2} H^2(\mathcal{Y}_1)}$ are zero spaces, and V and V^+ are identity relations, so they do have unitary extensions, and the claims follow. □

Note that the last statement of Proposition 4.4 does not hold for generalized Schur functions, as it already breaks down in a case where $\kappa_1 = 0$ and $\kappa = \kappa_2 = 1$. Indeed, in Example 3.42, the left factor θ_2 of the factorization $\theta_2\theta_1$ is a generalized bi-inner function, while the factorization is not (+)-regular. Consequently, Khanh's definition of (+)-regular factorization of ordinary Schur functions cannot be applied verbatim for generalized Schur functions in a consistent way with respect to the results of this paper, while Sz.-Nagy and Foiaş's definition of regular factorization can, if the index condition is included.

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