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TWO-DIMENSIONAL HELMHOLTZ RESONATOR WITH TWO CLOSE POINT-LIKE WINDOWS: REGULARIZATION FOR THE NEUMANN CASE

A. G. BELOLIPETSKAIA, A. A. BOITSEV, S. FASSARI, I. Y. POPOV, AND F. RINALDI

Abstract. Explicitly solvable model for two-dimensional Helmholtz resonator with two close point-like windows is constructed. The model is based on the theory of self-adjoint extensions of symmetric operators. Limiting procedure is studied for the case where the distance between the windows tends to zero. A regularization is suggested.

Побудовано явно розв'язувану модель для двовимірного резонатора Гельмгольца з двома близькими точковими вікнами. Модель базується на теорії самоспряжених розширень симетричних операторів. Вивчено процедуру граничного переходу для випадку, коли відстань між вікнами прямує до нуля, та запропоновано регуляризацію.

1. Introduction

Devices operating on the principle of the Helmholtz resonator, according to Junger, could have been used by the ancient Greeks [10]. Today Helmholtz resonators are actively used in acoustics [9, 7, 2, 23], aircraft construction [3, 6, 27], hydraulics [11] and optics [5]. Also the Helmholtz resonator can be the basis for metamaterials [13, 15] and can even be used to measure the volume of liquids and solids [26]. But it was not until 1860 that Helmholtz formulated the first mathematical theory of Helmholtz resonators [9]. One of the directions in the study of Helmholtz resonators is the study of resonances that arise in many problems of mathematics, physics and technology [4, 12, 14, 16, 20, 24, 25]. Although the study of the resonances of the Helmholtz resonator was carried out by Rayleigh [22], it is still relevant. In this article, the mathematical description will be based on the meromorphic continuation of the Green's function. The poles of this meromorphic continuation will contain physical information: the real part of the pole is the oscillation frequency, the imaginary part of the pole is the decay rate [28]. The model of the theory of operator extension with Neumann boundary conditions will be used as a model of the Helmholtz resonator. This model has been described in [21, 17, 18, 19].

2. Problem setting and model construction

2.1. **Construction of a symmetric operator.** From here on we consider $\mathbb{R}^2$ to be the main space, the elements of the considered space we denote by

$$x = (x_1, x_2), \quad x_i \in \mathbb{R}, \quad i \in \{1, 2\}.$$

Consider a smooth unbounded curve $\gamma : I \to \mathbb{R}^2$ with the support Γ that is

$$\Gamma = \gamma(I)$$

dividing the main space into two unbounded open sets Ω^{in} and Ω^{ex}, i.e.,

$$\partial\Omega^{in} = \partial\Omega^{ex} = \Gamma, \quad \Omega^{in} \cup \Omega^{ex} \cup \Gamma = \mathbb{R}^2.$$

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Let us introduce the notations

$$\Omega = \Omega^{in} \oplus \Omega^{ex}, \quad \partial\Omega = \Gamma.$$

Now we consider Laplace operators $-\Delta^{ex}$ and $-\Delta^{in}$ in $L_2(\Omega^{ex})$ and in $L_2(\Omega^{in})$, respectively. We consider Laplace equation with Neumann boundary conditions on $\partial\Omega^{ex,in}$, namely

$$\Delta^{in,ex} f \equiv 0, \quad \left.\frac{\partial f}{\partial n}\right|_{\partial\Omega^{in,ex}} = 0,$$

where n denotes the external unit normal vector to $\partial\Omega^{in,ex}$. Let

$$x^i = (x^i_1, x^i_2), \quad i \in \{1,2\},$$

be two different points on $\Gamma = \partial\Omega^{ex} = \partial\Omega^{in}$, i.e., $x^i \in \Gamma$, $i \in \{0,1\}$. Let us restrict the considered operators onto the set of smooth functions that vanish in a neighbourhood of x^1 and x^2. It is well known that the closures $-\Delta_0^{ex}$ and $-\Delta_0^{in}$ of the considered operators are symmetric operators with deficiency indices $(2,2)$.

Thus, we can consider a symmetric operator

$$\Delta_0 = \Delta_0^{in} \oplus \Delta_0^{ex}$$

acting in Ω with the described boundary conditions on $\partial\Omega$ having deficiency indices $(4,4)$.

Let us describe the domain of the adjoint operator $-\Delta^*$. Using the Neumann formula, we conclude that the domain $\mathfrak{D}(-\Delta^*)$ consists of the functions u having the form

$$u = \begin{pmatrix} u^{ex} \\ u^{in} \end{pmatrix} = \begin{pmatrix} u_0^{ex} \\ u_0^{in} \end{pmatrix} + a_0^{ex} \begin{pmatrix} G^{ex}(x, x^0, k_0) \\ 0 \end{pmatrix} + a_0^{in} \begin{pmatrix} 0 \\ G^{in}(x, x^0, k_0) \end{pmatrix} + \\ + a_1^{ex} \begin{pmatrix} G^{ex}(x, x^1, k_0) \\ 0 \end{pmatrix} + a_1^{in} \begin{pmatrix} 0 \\ G^{in}(x, x^1, k_0) \end{pmatrix},$$

where $u_0^{in,ex} \in \mathfrak{D}(\Delta_0^{in,ex})$ and $G^{in,ex}$ are corresponding free Green functions, k_0^2 is a regular point for the operators $\Delta_0^{in,ex}$. Let us find the conditions for which the considered operator is self-adjoint.

2.2. **Boundary form construction.** Now we start with examining the boundary form for the introduced operator. It is clear that the boundary form for the operator $-\Delta^*$ reduces to the expression

$$(-\Delta^* u, v) - (u, -\Delta^* v) = \lim_{\varepsilon \to 0+0} \int\limits_{\Omega^{ex} \setminus (B_\varepsilon(x^0) \cup B_\varepsilon(x^1))} (-\Delta^* u^{ex} \overline{v^{ex}} + u^{ex} \Delta^* \overline{v^{ex}})\, dx + \\ + \lim_{\varepsilon \to 0+0} \int\limits_{\Omega^{in} \setminus (B_\varepsilon(x^0) \cup B_\varepsilon(x^1))} \left(-\Delta^* u^{in} \overline{v^{in}} + u^{in} \Delta^* \overline{v^{in}}\right) dx,$$

where $B_\varepsilon(x^i)$ is a ball of radius ε centered at x^i, $i \in \{0,1\}$. Using the asymptotics for the Green's function, we can obtain the following relation:

$$(-\Delta^* u, v) - (u, -\Delta^* v) = \sum_{i=0,1} (a_i^{ex} \widetilde{b}_i^{ex} - b_i^{ex} \widetilde{a}_i^{ex} + a_i^{in} \widetilde{b}_i^{in} - b_i^{in} \widetilde{a}_i^{in}),$$

where $a_i^{ex}, a_i^{in}, b_i^{ex} = u_0^{ex}(x^i), b_i^{in} = u_0^{in}(x^i)$ are the coefficients corresponding to the function u and $\widetilde{a}_i^{ex}, \widetilde{a}_i^{in}, \widetilde{b}_i^{ex} = v_0^{ex}(x^i), \widetilde{b}_i^{in} = v_0^{in}(x^i)$ are the coefficients corresponding to the function v, $i \in \{0,1\}$.

It is clear that any self-adjoint extension can be given by equating the latter expression to zero, that is,

$$\sum_{i=0,1} (a_i^{ex} \widetilde{b}_i^{ex} - b_i^{ex} \widetilde{a}_i^{ex} + a_i^{in} \widetilde{b}_i^{in} - b_i^{in} \widetilde{a}_i^{in}) = 0.$$

We choose the following condition:

$$a_i^{ex} = -a_i^{in}, \quad b_i^{ex} = b_i^{in}, \quad i \in \{0,1\}.$$

2.3. **Construction of an extension.** To construct an extension, we find the form of the coefficients of $\alpha_{+/-}^{ex/in}, \beta_{+/-}^{ex/in}$ of the Green function, which is written in the following form:

$$\begin{aligned}
G(x,y,k) = &\begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \alpha_+^{ex}\begin{pmatrix} G^{ex}(x,x^0,k) + G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} \\
&+ \alpha_-^{ex}\begin{pmatrix} G^{ex}(x,x^0,k) - G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} + \alpha_+^{in}\begin{pmatrix} 0 \\ G^{in}(x,x^0,k) + G^{in}(x,x^1,k) \end{pmatrix} \\
&+ \alpha_-^{in}\begin{pmatrix} 0 \\ G^{in}(x,x^0,k) - G^{in}(x,x^1,k) \end{pmatrix}, \qquad x \in \Omega^{ex}, \\
G(x,y,k) = &\begin{pmatrix} 0 \\ G^{in}(x,y,k) \end{pmatrix} + \beta_+^{ex}\begin{pmatrix} G^{ex}(x,x^0,k) + G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} \\
&+ \beta_-^{ex}\begin{pmatrix} G^{ex}(x,x^0,k) - G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} + \beta_+^{in}\begin{pmatrix} 0 \\ G^{in}(x,x^0,k) + G^{in}(x,x^1,k) \end{pmatrix} \\
&+ \beta_-^{in}\begin{pmatrix} 0 \\ G^{in}(x,x^0,k) - G^{in}(x,x^1,k) \end{pmatrix}, \qquad x \in \Omega^{in}.
\end{aligned}$$

Theorem 2.1. *The coefficients $\alpha_{+/-}^{ex/in}, \beta_{+/-}^{ex/in}$ from the formulas above have the following form:*

$$\begin{aligned}
\alpha_+^{ex} &= (G^{ex}(x^0,y,k)\cdot(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k) - g_1^{ex} - g_1^{in}) \\
&\quad + G^{ex}(x^1,y,k)\cdot(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k) - g_0^{ex} - g_0^{in}))\cdot g^{-1}, \\
\alpha_-^{ex} &= (-G^{ex}(x^0,y,k)\cdot(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k) + g_1^{ex} + g_1^{in}) \\
&\quad + G^{ex}(x^1,y,k)\cdot(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k) + g_0^{ex} + g_0^{in}))\cdot g^{-1}, \\
&\qquad \alpha_+^{in} = -\alpha_+^{ex}, \qquad \alpha_-^{in} = -\alpha_-^{ex}, \\
\beta_+^{ex} &= (G^{in}(x^0,y,k)\cdot(-G^{ex}(x^0,x^1,k) - G^{in}(x^0,x^1,k) + g_1^{ex} + g_1^{in}) \\
&\quad + G^{in}(x^1,y,k)\cdot(-G^{ex}(x^0,x^1,k) - G^{in}(x^0,x^1,k) + g_0^{ex} + g_0^{in}))\cdot g^{-1}, \\
\beta_-^{ex} &= (G^{in}(x^0,y,k)\cdot(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k) + g_1^{ex} + g_1^{in}) \\
&\quad - G^{in}(x^1,y,k)\cdot(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k) - g_0^{ex} - g_0^{in}))\cdot g^{-1}, \\
&\qquad \beta_+^{in} = -\beta_+^{ex}, \qquad \beta_-^{in} = -\beta_-^{ex},
\end{aligned}$$

where $g = (g_0^{ex} + g_0^{in})\cdot(g_1^{ex} + g_1^{in}) - (G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))^2$.

Proof. At first consider the case where $x \in \Omega^{ex}$. The Green function can be written as follows:

$$\begin{aligned}
G(x,y,k) &= \begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \alpha_0^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) \\ 0 \end{pmatrix} + \alpha_0^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) \end{pmatrix} \\
&\quad + \alpha_1^{ex} \begin{pmatrix} G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} + \alpha_1^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^1,k) \end{pmatrix} \\
&= \begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \alpha_0^{ex} \begin{pmatrix} G^{ex}(x,x^0,k_0) \\ 0 \end{pmatrix} + \alpha_0^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k_0) \end{pmatrix} \\
&\quad + \alpha_1^{ex} \begin{pmatrix} G^{ex}(x,x^1,k_0) \\ 0 \end{pmatrix} + \alpha_1^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^1,k_0) \end{pmatrix} \\
&\quad + \alpha_0^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) - G^{ex}(x,x^0,k_0) \\ 0 \end{pmatrix} \\
&\quad + \alpha_0^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) - G^{in}(x,x^0,k_0) \end{pmatrix} \\
&\quad + \alpha_1^{ex} \begin{pmatrix} G^{ex}(x,x^1,k) - G^{ex}(x,x^1,k_0) \\ 0 \end{pmatrix} \\
&\quad + \alpha_1^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^1,k) - G^{in}(x,x^1,k_0) \end{pmatrix}.
\end{aligned}$$

From this expansion we immediately get that

$$\alpha_i^{ex} = a_i^{ex}, \quad \alpha_i^{in} = a_i^{in}, \quad i \in \{0,1\},$$

therefore $\alpha_i^{ex} = -\alpha_i^{in}$.

Now let us find the values of the coefficients b_i^{ex}, b_i^{in}. It is clear that they can be obtained from the following expressions:

$$\begin{aligned}
b_0^{ex} &= G^{ex}(x^0,y,k) + \alpha_0^{ex} \left(G^{ex}(x,x^0,k) - G^{ex}(x,x^0,k_0)\right)\big|_{x=x^0} + \alpha_1^{ex} G^{ex}(x^0,x^1,k), \\
b_1^{ex} &= G^{ex}(x^1,y,k) + \alpha_1^{ex} \left(G^{ex}(x,x^1,k) - G^{ex}(x,x^1,k_0)\right)\big|_{x=x^1} + \alpha_0^{ex} G^{ex}(x^1,x^0,k), \\
b_1^{in} &= \alpha_0^{in} \left(G^{in}(x,x^0,k) - G^{in}(x,x^0,k_0)\right)\big|_{x=x^0} + \alpha_1^{in} G^{in}(x^0,x^1,k), \\
b_2^{in} &= \alpha_1^{in} \left(G^{in}(x,x^1,k) - G^{in}(x,x^1,k_0)\right)\big|_{x=x_2} + \alpha_0^{in} G^{in}(x^1,x^0,k).
\end{aligned}$$

Let us introduce the following auxiliary notations:

$$g_i^{in,ex} = \left(G^{in,ex}(x,x^i,k) - G^{in,ex}(x,x^i,k_0)\right)\big|_{x=x^i}, \quad i \in \{0,1\}.$$

The solution to the system of equations $b_0^{ex} = b_1^{ex}, b_0^{in} = b_1^{in}$ are the following expressions:

$$\begin{aligned}
\alpha_0^{ex} &= \frac{-G^{ex}(x^0,y,k)(g_1^{ex} + g_1^{in}) + G^{ex}(x^1,y,k)(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))}{(g_0^{ex} + g_0^{in})(g_1^{ex} + g_1^{in}) - (G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))^2}, \\
\alpha_1^{ex} &= \frac{-G^{ex}(x^1,y,k)(g_0^{ex} + g_0^{in}) + G^{ex}(x^0,y,k)(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))}{(g_0^{ex} + g_0^{in})(g_1^{ex} + g_1^{in}) - (G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))^2}.
\end{aligned}$$

Let us now rewrite $G(x,y,k)$ for $x \in \Omega^{ex}$ in another way,

$$\begin{aligned}
G(x,y,k) &= \begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \alpha_+^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) + G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} \\
&\quad + \alpha_-^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) - G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} + \alpha_+^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) + G^{in}(x,x^1,k) \end{pmatrix} \\
&\quad + \alpha_-^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) - G^{in}(x,x^1,k) \end{pmatrix},
\end{aligned}$$

where

$$\alpha_+^{ex} = \alpha_0^{ex} + \alpha_1^{ex}, \quad \alpha_-^{ex} = \alpha_0^{ex} - \alpha_1^{ex}, \quad \alpha_+^{in} = -\alpha_+^{ex}, \quad \alpha_-^{in} = -\alpha_-^{ex}.$$

Thus, the coefficients $\alpha_{+/-}^{ex/in}$ are obtained. Now consider the case where $x \in \Omega^{in}$. We have,

$$\begin{aligned}
G(x,y,k) &= \begin{pmatrix} 0 \\ G^{in}(x,y,k) \end{pmatrix} + \beta_0^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) \\ 0 \end{pmatrix} + \beta_0^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) \end{pmatrix} \\
&\quad + \beta_1^{ex} \begin{pmatrix} G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} + \beta_1^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^1,k) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ G^{in}(x,y,k) \end{pmatrix} + \beta_0^{ex} \begin{pmatrix} G^{ex}(x,x^0,k_0) \\ 0 \end{pmatrix} + \beta_0^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k_0) \end{pmatrix} \\
&\quad + \beta_1^{ex} \begin{pmatrix} G^{ex}(x,x^1,k_0) \\ 0 \end{pmatrix} + \beta_1^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^1,k_0) \end{pmatrix} \\
&\quad + \beta_0^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) - G^{ex}(x,x^0,k_0) \\ 0 \end{pmatrix} + \beta_0^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) - G^{in}(x,x^0,k_0) \end{pmatrix} \\
&\quad + \beta_1^{ex} \begin{pmatrix} G^{ex}(x,x^1,k) - G^{ex}(x,x^1,k_0) \\ 0 \end{pmatrix} + \beta_1^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^1,k) - G^{in}(x,x^1,k_0) \end{pmatrix}.
\end{aligned}$$

It is clear that $\beta_i^{ex} = -\beta_i^{in}$. Now we can find the value of the coefficients b_i^{ex}, b_i^{in},

$$\begin{aligned}
b_0^{ex} &= \beta_0^{ex}(G^{ex}(x,x^0,k) - G^{ex}(x,x^0,k_0))|_{x=x^0} + \beta_1^{ex} G^{ex}(x^0,x^1,k), \\
b_1^{ex} &= \beta_1^{ex}(G^{ex}(x,x^1,k) - G^{ex}(x,x^1,k_0))|_{x=x^1} + \beta_0^{ex} G^{ex}(x^1,x^0,k), \\
b_0^{in} &= G^{in}(x^0,y,k) + \beta_0^{in}(G^{in}(x,x^0,k) - G^{in}(x,x^0,k_0))|_{x=x^0} + \beta_1^{in} G^{in}(x^0,x^1,k), \\
b_1^{in} &= G^{in}(x^1,y,k) + \beta_1^{in}(G^{in}(x,x^1,k) - G^{in}(x,x^1,k_0))|_{x=x^1} + \beta_0^{in} G^{in}(x^1,x^0,k).
\end{aligned}$$

A solution to the system of equations $b_0^{ex} = b_1^{ex}, b_0^{in} = b_1^{in}$ are the following expressions:

$$\begin{aligned}
\beta_0^{ex} &= \frac{G^{in}(x^0,y,k)(g_1^{ex} + g_1^{in}) - G^{in}(x^1,y,k)(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))}{(g_0^{ex} + g_0^{in})(g_1^{ex} + g_1^{in}) - (G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))^2}, \\
\beta_1^{ex} &= \frac{G^{in}(x^1,y,k)(g_0^{ex} + g_0^{in}) - G^{in}(x^0,y,k)(G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))}{(g_0^{ex} + g_0^{in})(g_1^{ex} + g_1^{in}) - (G^{ex}(x^0,x^1,k) + G^{in}(x^0,x^1,k))^2}.
\end{aligned}$$

Now we wish to rewrite $G(x,y,k)$ for $x \in \Omega^{in}$ in the following way:

$$\begin{aligned}
G(x,y,k) &= \begin{pmatrix} 0 \\ G^{in}(x,y,k) \end{pmatrix} + \beta_+^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) + G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} \\
&\quad + \beta_-^{ex} \begin{pmatrix} G^{ex}(x,x^0,k) - G^{ex}(x,x^1,k) \\ 0 \end{pmatrix} + \beta_+^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) + G^{in}(x,x^1,k) \end{pmatrix} \\
&\quad + \beta_-^{in} \begin{pmatrix} 0 \\ G^{in}(x,x^0,k) - G^{in}(x,x^1,k) \end{pmatrix},
\end{aligned}$$

where

$$\beta_+^{ex} = \beta_0^{ex} + \beta_1^{ex}, \quad \beta_-^{ex} = \beta_0^{ex} - \beta_1^{ex}, \quad \beta_+^{in} = -\beta_+^{ex}, \quad \beta_-^{in} = -\beta_-^{ex}.$$

Thus, the theorem is proved. □

Corollary 2.1. *For a better understanding of the result given by the theorem above, consider a special case, where the regions Ω^{ex}, Ω^{in} are half-planes and the two points are given by $\mp x^0 = (\mp x_0, 0)$, $x_0 > 0$. The formulas turn out to be as follows:*

$$\alpha_+^{ex} = \frac{1}{\frac{2}{\pi} \ln \frac{k}{k_0} - 2G(-x^0,x^0,k)} (G^{ex}(-x^0,y,k) + G^{ex}(x^0,y,k)),$$

$$\alpha_-^{ex} = \frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0} + 2G(-x^0, x^0, k)}(G^{ex}(-x^0, y, k) - G^{ex}(x^0, y, k)),$$
$$\beta_+^{ex} = \frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0} + 2G(-x^0, x^0, k)}(G^{in}(-x^0, y, k) + G^{in}(x^0, y, k)),$$
$$\beta_-^{ex} = \frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0} - 2G(-x^0, x^0, k)}(G^{in}(-x^0, y, k) - G^{in}(x^0, y, k)).$$

3. Regularization of the Green function and existence of the limit as $x^0 \to 0$

As a consequence of the considerations in the previous section, the external Green function can be expressed as

$$G(x,y,k) = \begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \alpha_-^{ex}\begin{pmatrix} G^{ex}_{sym}(x,x^0,k) \\ 0 \end{pmatrix} + \alpha_-^{ex}\begin{pmatrix} G^{ex}_{asym}(x,x^0,k) \\ 0 \end{pmatrix}$$
$$+ \alpha_+^{in}\begin{pmatrix} 0 \\ G^{in}_{sym}(x,x^0,k) \end{pmatrix} + \alpha_-^{in}\begin{pmatrix} 0 \\ G^{in}_{asym}(x,x^0,k) \end{pmatrix}$$

with

$$G^{ex}_{sym}(x,x^0,k) = G^{ex}(x,x^0,k) + G^{ex}(x,-x^0,k),$$
$$G^{ex}_{asym}(x,x^0,k) = G^{ex}(x,x^0,k) - G^{ex}(x,-x^0,k).$$

Of course, the internal Green function is defined in a similar way. Taking account of the fact that $\alpha_+^{in} = -\alpha_+^{ex}$ and $\alpha_-^{in} = -\alpha_+^{ex}$ from theorem 2.1, it can be rewritten as follows:

$$G(x,y,k) = \begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \alpha_+^{ex}\begin{pmatrix} G^{ex}_{sym}(x,x^0,k) \\ -G^{in}_{sym}(x,x^0,k) \end{pmatrix} + \alpha_-^{ex}\begin{pmatrix} G^{ex}_{asym}(x,x^0,k) \\ -G^{in}_{asym}(x,x^0,k) \end{pmatrix}.$$

By setting $\Gamma(x^0;k) = G^{ex}(-x^0,x^0,k) + G^{in}(-x^0,x^0,k)$, the coefficients $\alpha_+^{ex}, \alpha_-^{ex}$ can be written as

$$\alpha_+^{ex} = \left[G^{ex}(-x^0,y,k)\left(\frac{2}{\pi}\ln\frac{k}{k_0} + 2\Gamma(x^0,k)\right) + G^{ex}(x^0,k)\left(\frac{2}{\pi}\ln\frac{k}{k_0} + 2\Gamma(x^0,k)\right)\right]$$
$$\times\left[\frac{4}{\pi^2}\left(\ln\frac{k}{k_0}\right)^2 - 4\Gamma^2(x^0,k)\right]^{-1},$$

that can easily be rewritten as

$$\alpha_+^{ex} = \frac{G^{ex}_{sym}(x^0,y,k)}{\frac{2}{\pi}\ln\frac{k}{k_0} - 2\Gamma(x^0,k)}.$$

Similarly,

$$\alpha_-^{ex} = \frac{G^{ex}_{asym}(x^0,y,k)}{\frac{2}{\pi}\ln\frac{k}{k_0} + 2\Gamma(x^0,k)}.$$

Hence, the expression of the external Green function can be recast as

$$G(x,y,k) = \begin{pmatrix} G^{ex}(x,y,k) \\ 0 \end{pmatrix} + \frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0} - 2\Gamma(x^0,k)}\begin{pmatrix} G^{ex}_{sym}(x,x^0,k)G^{ex}_{sym}(x^0,y,k) \\ -G^{in}_{sym}(x,x^0,k)G^{ex}_{sym}(x^0,y,k) \end{pmatrix}$$
$$+ \frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0} + 2\Gamma(x^0,k)}\begin{pmatrix} G^{ex}_{asym}(x,x^0,k)G^{ex}_{asym}(x^0,y,k) \\ -G^{in}_{asym}(x,x^0,k)G^{ex}_{asym}(x^0,y,k) \end{pmatrix}.$$

As this situation is quite reminiscent of the one encountered in [8], the denominators in $\alpha_{+/-}^{ex}$ can be rewritten as:

$$\frac{2}{\pi}\ln\frac{k}{k_0} - 2\Gamma(x^0,k) = 2\left[\frac{1}{\pi}\ln k + \beta^{-1}(k_0) - \Gamma(x^0,k)\right],$$

and

$$\frac{2}{\pi}\ln\frac{k}{k_0}+2\Gamma(x^0,k)=2\left[\frac{1}{\pi}\ln k+\beta^{-1}(k_0)+\Gamma(x^0,k)\right],$$

where $\beta^{-1}(k_0)=-\frac{1}{\pi}\ln k_0$. At this stage it is crucial to notice that $\Gamma(x^0,k)$ behaves like $-\frac{\ln(kx^0)+\gamma}{\pi}$ for small values of x^0 with γ denoting the well-known Euler-Mascheroni constant (see [1]), which would lead to a negatively divergent lowest eigenvalue as $x_0\to 0_+$. Therefore, mimicking what was done in [8] to regularise this singular behaviour, the coupling β must be regularized as follows:

$$\beta^{-1}(k_0,x^0)=-\frac{1}{\pi}\ln k_0+\Gamma(x^0,k_0).$$

As a consequence of this regularization, the denominator of the second term in the right-hand side of the external Green function would read

$$2\left[\frac{1}{\pi}\ln k+\beta^{-1}(k_0,x^0)-\Gamma(x^0,k)\right]=2\left[\frac{1}{\pi}\ln k-\frac{1}{\pi}\ln k_0+\Gamma(x^0,k_0)-\Gamma(x^0,k)\right],$$

which behaves like

$$\frac{2}{\pi}\ln\frac{k}{k_0}+\frac{2}{\pi}\left[\ln(x^0k)-\ln(x^0k_0)\right]=\frac{4}{\pi}\ln\frac{k}{k_0}=\frac{2}{\pi}\ln\frac{k^2}{k_0^2}$$

in a proximity of $x^0=0_+$. The latter fact implies the convergence of the regularized Green function

$$\begin{aligned}G^{reg}_{x^0}(x,y,k)&=\begin{pmatrix}G^{ex}(x,y,k)\\0\end{pmatrix}\\&\quad+\frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0}+2\Gamma(x^0,k_0)-2\Gamma(x^0,k)}\begin{pmatrix}G^{ex}_{sym}(x,x^0,k)G^{ex}_{sym}(x^0,y,k)\\-G^{in}_{sym}(x,x^0,k)G^{ex}_{sym}(x^0,y,k)\end{pmatrix}\\&\quad+\frac{1}{\frac{2}{\pi}\ln\frac{k}{k_0}+2\Gamma(x^0,k_0)+2\Gamma(x^0,k)}\begin{pmatrix}G^{ex}_{asym}(x,x^0,k)G^{ex}_{asym}(x^0,y,k)\\-G^{in}_{asym}(x,x^0,k)G^{ex}_{asym}(x^0,y,k)\end{pmatrix}\end{aligned}$$

to

$$G_0(x,y,k)=\begin{pmatrix}G^{ex}(x,y,k)\\0\end{pmatrix}+\frac{1}{\frac{4}{\pi}\ln\frac{k}{k_0}}\begin{pmatrix}G^{ex}_{sym}(x,0,k)G^{ex}_{sym}(0,y,k)\\-G^{in}_{sym}(x,0,k)G^{ex}_{sym}(0,y,k)\end{pmatrix}.$$

3.1. **Regularization: the general case.** For a further research, it is more convenient to introduce the following auxiliary variables:

$$\begin{aligned}\Delta&=(g_0^{ex}+g_0^{in})(g_1^{ex}+g_1^{in})-(G^{ex}(x^0,x^1,k)+G^{in}(x^0,x^1,k))^2,\\ \Delta_{\alpha_+}&=G^{ex}(x^0,y,k)\cdot(G^{ex}(x^0,x^1,k)+G^{in}(x^0,x^1,k)-g_1^{ex}-g_1^{in})\\&\quad+G^{ex}(x^1,y,k)\cdot(G^{ex}(x^0,x^1,k)+G^{in}(x^0,x^1,k)-g_0^{ex}-g_0^{in}),\\ \Delta_{\alpha_-}&=G^{ex}(x^0,y,k)\cdot(-G^{ex}(x^0,x^1,k)-G^{in}(x^0,x^1,k)-g_1^{ex}-g_1^{in})\\&\quad+G^{ex}(x^1,y,k)\cdot(G^{ex}(x^0,x^1,k)+G^{in}(x^0,x^1,k)+g_0^{ex}+g_0^{in}),\\ \Delta_{\beta_+}&=G^{in}(x^0,y,k)\cdot(-G^{ex}(x^0,x^1,k)-G^{in}(x^0,x^1,k)+g_1^{ex}+g_1^{in})\\&\quad+G^{in}(x^1,y,k)\cdot(-G^{ex}(x^0,x^1,k)-G^{in}(x^0,x^1,k)+g_0^{ex}+g_0^{in}),\\ \Delta_{\beta_-}&=G^{in}(x^0,y,k)\cdot(G^{ex}(x^0,x^1,k)+G^{in}(x^0,x^1,k)+g_1^{ex}+g_1^{in})\\&\quad+G^{in}(x^1,y,k)\cdot(-G^{ex}(x^0,x^1,k)-G^{in}(x^0,x^1,k)-g_0^{ex}-g_0^{in}).\end{aligned}$$

Note that using the new variables, one can obtain the following expressions for the coefficients $\alpha^{ex}_{+/-},\beta^{ex}_{+/-}$:

$$\alpha^{ex}_+=\frac{\Delta_{\alpha_+}}{\Delta},\quad \alpha^{ex}_-=\frac{\Delta_{\alpha_-}}{\Delta},$$

$$\beta_+^{ex} = \frac{\Delta_{\beta_+}}{\Delta}, \quad \beta_-^{ex} = \frac{\Delta_{\beta_-}}{\Delta}.$$

Theorem 3.1. *In a vicinity of the point $x_0 \in \partial\Omega$, the coefficients $\alpha_{+/-}^{ex}, \beta_{+/-}^{ex}$ take the following form:*

$$\alpha_+^{ex} = \frac{2G^{ex}(x^0, y, k)}{\frac{2}{\pi}\ln|x - x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0) + 2g^{ex} + 2g^{in}},$$

$$\beta_+^{ex} = \frac{-2G^{in}(x^0, y, k)}{\frac{2}{\pi}\ln|x - x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0) + 2g^{ex} + 2g^{in}},$$

$$\alpha_-^{ex} = 0, \quad \beta_-^{ex} = 0.$$

Proof. We will use the following asymptotics for the functions $G^{in,ex}(x, x^0, k)$ in a vicinity of the point $x^0 \in \partial\Omega$. For $x \in \Omega^{in}$ we have

$$G^{in}(x, x^0, k) = \frac{1}{\pi}\ln|x - x^0|^{-1} + c^{in}(x^0, k_0) + (k^2 - k_0^2)\sum_n \frac{|\varphi_n(x^0)|^2}{(\lambda_n - k^2)(\lambda_n - k_0^2)} + v^{in}(x, x^0, k)$$

for $x \in \Omega^{ex}$ we have

$$G^{ex}(x, x^0, k) = \frac{1}{\pi}\ln|x - x^0|^{-1} + c^{ex}(x^0, k_0) + (k^2 - k_0^2)\int_{\mathbb{R}^2} \frac{|\varphi(x_0, |s|, \nu)|^2}{(s^2 - k^2)(s^2 - k_0^2)} d^2 s + v^{ex}(x, x^0, k),$$

where $v^{in,ex} \in \mathfrak{D}(\overline{\Delta_0^{in,ex}})$, $v^{in,ex} = o(|x - x^0|^0)$, λ_n are eigenvalues for the operator Δ^{in}, φ_n are the corresponding normalized eigenfunctions, $\varphi(x^0, |s|, \nu)$ are normalized to δ-function scattered waves, corresponding to the point $|s|^2$ of the continuous spectrum of the operator Δ^{ex}, k_0^2 is some fixed regular value of the spectral parameter of the operator $\Delta = \Delta^{in} \oplus \Delta^{ex}$.

For convenience, let us denote the following expressions for $g^{in,ex}$:

$$g^{in} = (k^2 - k_0^2)\sum_n \frac{|\varphi_n(x^0)|^2}{(\lambda_n - k^2)(\lambda_n - k_0^2)},$$

$$g^{ex} = (k^2 - k_0^2)\int_{\mathbb{R}^2} \frac{|\varphi(x^0, |s|, \nu)|^2}{(s^2 - k^2)(s^2 - k_0^2)} d^2 s.$$

Then, in a vicinity of the point $x^0 \in \partial\Omega$, $g_i^{ex,in}$ behave in the following manner:

$$g_i^{in} = g^{in}, \quad g_i^{ex} = g^{ex}.$$

To investigate the behavior of the coefficients $\alpha_{+/-}, \beta_{+/-}$ in a vicinity of the point $x_0 \in \partial\Omega$, we investigate the behavior of the coefficients $\Delta_{\alpha_{+/-}}, \Delta_{\beta_{+/-}}$ in a vicinity of the point $x_0 \in \partial\Omega$,

$$\Delta_{\alpha_+} = 2G^{ex}(x^0, y, k) \cdot \left(\frac{2}{\pi}\ln|x - x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0)\right),$$

$$\Delta_{\beta_+} = -2G^{in}(x^0, y, k) \cdot \left(\frac{2}{\pi}\ln|x - x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0)\right),$$

$$\Delta_{\alpha_-} = 0, \quad \Delta_{\beta_-} = 0.$$

Thus, we obtain that in a vicinity of the point $x_0 \in \partial\Omega$, the coefficients $\alpha_{+/-}, \beta_{+/-}$ take the following form:

$$\alpha_+^{ex} = \frac{2G^{ex}(x^0, y, k)}{\frac{2}{\pi}\ln|x-x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0) + 2g^{ex} + 2g^{in}},$$

$$\beta_+^{ex} = \frac{-2G^{in}(x^0, y, k)}{\frac{2}{\pi}\ln|x-x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0) + 2g^{ex} + 2g^{in}},$$

$$\alpha_-^{ex} = 0, \quad \beta_-^{ex} = 0.$$

Thus, the theorem is proved. □

Corollary 3.1. *The equation describing resonances is $\Delta = 0$. Then, in a vicinity of the point $x^0 \in \partial\Omega$, the equation $\Delta = 0$ is equivalent to the combination of the following two equations:*

$$\frac{2}{\pi}\ln|x-x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0) = 0,$$

$$\frac{2}{\pi}\ln|x-x^0|^{-1} + c^{ex}(x^0, k_0) + c^{in}(x^0, k_0) + 2g^{ex} + 2g^{in} = 0.$$

To consider a regularization when the distance between point-like windows approaches 0, one should deal with the equation $\Delta = 0$. We may consider the surface to be smooth. The regularization is based on the existence of a relation between k_0 related to the windows width and the distance between the windows, $|x^0 - x^1|$. The main term of the asymptotics in k_0, for $k_0 \to \infty$, is determined by the first term of Δ, by $g_{0,1}^{in,ex}$. The main term of the asymptotics with respect to the distance $|x^0 - x^1|$, when $|x^0 - x^1| \to 0$, is determined by the second term of Δ. Thus, for the regularization, we choose $|x^0 - x^1|$ corresponding to $k_0/2$ in accordance with the equation $\Delta = 0$. We are led to an energy-dependent (k-dependent) condition.

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A. G. Belolipetskaia: annabel1502@mail.ru
Department of Higher Mathematics, ITMO University, 197101 St. Petersburg, Russian Federation

A. A. Boitsev: boitsevanton@gmail.com
Department of Higher Mathematics, ITMO University, 197101 St. Petersburg, Russian Federation

S. Fassari: silvestro.fassari@uva.es
Department of Higher Mathematics, ITMO University, 197101 St. Petersburg, Russian Federation, CERFIM, PO Box 1132, CH-6601 Locarno, Switzerland, Univ. degli Studi Guglielmo Marconi, Via Plinio 44, I-00193 Rome, Italy

I. Y. Popov: popov1955@gmail.com
Department of Higher Mathematics, ITMO University, 197101 St. Petersburg, Russian Federation

F. Rinaldi: f.rinaldi@unimarconi.it
Univ. degli Studi Guglielmo Marconi, Via Plinio 44, I-00193 Rome, Italy, Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Naples, Italy

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