Methods of Functional Analysis and Topology
Vol. 28 (2022), no. 2, pp. 105–109
https://doi.org/10.31392/MFAT-npu26_2.2022.02

A NOTE ON PENCIL OF BOUNDED LINEAR OPERATORS ON NON-ARCHIMEDEAN BANACH SPACES

AZIZ BLALI, ABDELKHALEK EL AMRANI, AND JAWAD ETTAYB

Abstract. We give a characterization of the essential spectrum for (A, B), where A is a closed linear operator and B is a bounded linear operator, by means of Fredholm operators on a Banach space of countable type over $\mathbb{Q}_p$.

За допомогою фредгольмових операторів на банаховому просторі зліченого типу над $\mathbb{Q}_p$ надано характеристику істотного спектра для (A, B), де A — замкнеґний лінійний оператор, а B — обмежений.

1. Introduction and Preliminaries

Throughout this paper, X and Y are non-archimedean (n.a) Banach spaces over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y. When, $X = Y$, we have $\mathcal{L}(X,Y) = \mathcal{L}(X)$. If $A \in \mathcal{L}(X)$, $N(A)$ and $R(A)$ denote the kernel and the range of A, respectively. For more details, we refer to [3, 7]. X is said to be of countable type if there is a countable set in X whose linear hull is dense. Recall that an unbounded linear operator $A : D(A) \subseteq X \to Y$ is said to be closed if for all $(x_n) \subset D(A)$ such that $\|x_n - x\| \to 0$ and $\|Ax_n - y\| \to 0$ as $n \to \infty$, for some $x \in X$ and $y \in Y$, then $x \in D(A)$ and $y = Ax$. The collection of all closed linear operators from X into Y is denoted by $\mathcal{C}(X,Y)$. When $X = Y$, if $A \in \mathcal{L}(X)$ and B is unbounded linear operator, then $A + B$ is closed if and only if B is closed [3]. For more details on non-archimedean operators theory, we refer to [2, 3, 7]. There are many interesting works on pseudospectra in the classical Banach space, see [4, 9].

Definition 1 ([3])**.** Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $\mathbb{K}$. We define $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N},\ x_i \in \mathbb{K},\ \textit{and}\ \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\},$$

and it is equipped with the norm

$$(\forall x \in \mathbb{E}_\omega) : x = (x_i)_i,\ \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|x_i|).$$

Remark 1. (1) [3, Example 2.21] The space $\big(\mathbb{E}_\omega,\ \|\cdot\|\big)$ is a non-archimedean Banach space.

(2) If $\langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \longrightarrow \mathbb{K}$, is defined by

$$(x, y) \longmapsto \sum_{i=0}^{\infty} x_i y_i \omega_i,$$

where $x = (x_i)_i$ and $y = (y_i)_i$, then the space $\Big(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle\Big)$ is called a p-adic (or non-archimedean) Hilbert space.

(2) The orthogonal basis $\{e_i,\ i \in \mathbb{N}\}$ is called the canonical basis of $\mathbb{E}_\omega$, where for all $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

In the next definition, X and Y are two vector spaces over $\mathbb{K}$.

2020 *Mathematics Subject Classification.* 47S10.
Keywords. Non-archimedean Banach spaces, spectrum, essential spectrum.

Definition 2 ([6]). We say that $A \in \mathcal{L}(X,Y)$ has an index when both $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim \Big(Y/R(A)\Big)$ are finite. In this case, the index of the linear operator A is defined as $ind(A) = \alpha(A) - \beta(A)$.

Definition 3 ([6]). Let $A \in \mathcal{L}(X,Y)$. A is said to be upper semi-Fredholm operator if

$$\alpha(A) \text{ is finite and } R(A) \text{ is closed.}$$

The set of all upper semi-Fredholm operators is denoted by $\Phi_+(X,Y)$.

Definition 4 ([6]). Let $A \in \mathcal{L}(X,Y)$. A is said to be lower semi-Fredholm operator if

$$\beta(A) \text{ is finite.}$$

The set of all lower semi-Fredholm operators is denoted by $\Phi_-(X,Y)$.

The set of all Fredholm operators is defined by

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

Let X be a non-archimedean Banach space over $\mathbb{K}$. A subset A of X is said to be compactoid if for every $\varepsilon > 0$, there is a finite subset B of X such that $A \subset B_\varepsilon(0) + C_0(B)$, where $B_\varepsilon(0) = \{x \in X : \|x\| \leq \varepsilon\}$ and $C_0(B)$ is an absolutely convex hull of X, i.e.

$$C_0(B) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n : n \in \mathbb{N},\ \lambda_1, \cdots, \lambda_n \in B_K,\ x_1, \cdots, x_n \in B\}.$$

We have the following definition [7, page 142], for more details, see [7].

Definition 5 ([7]). Let X and Y be two non-archimedean Banach spaces over $\mathbb{K}$. A linear map $A : X \to Y$ is said to be compact if $A(B_X)$ is compactoid in Y, where $B_X = \{x \in X : \|x\| \leq 1\}$.

We denote by $\mathcal{K}(X,Y)$, the set of all compact operators from X into Y.

Definition 6 ([7]). Let $T \in \mathcal{L}(X,Y)$. T is called an operator of finite rank if $R(A)$ is a finite dimensional subspace of Y.

Theorem 1 ([7]). *Let $T \in \mathcal{L}(X,Y)$. Then T is compact if, and only if, for every $\varepsilon > 0$, there exists an $S \in \mathcal{L}(X,Y)$ such that $R(S)$ is finite-dimensional and $\|T - S\| < \varepsilon$.*

Definition 7 ([3]). Let X be a non-archimedean Banach space and let $T \in \mathcal{L}(X)$. T is said to be completely continuous, if there exists a sequence of finite rank linear operators (T_n) such that $\|T_n - T\| \to 0$ as $n \to \infty$.

The collection of completely continuous linear operators on X is denoted by $\mathcal{C}_c(X)$.

Remark 2 ([7]).
- (i) In a non-archimedean Banach space X, we do not have the relationship between $\mathcal{C}_c(X)$ and $\mathcal{K}(X)$ as in the classical case. J. P. Serre has proved that those concepts coincide, when $\mathbb{K}$ is locally compact.
- (ii) If $\mathbb{K}$ is locally compact, then all completely continuous linear operators on X are compact on X.
- (iii) If $\mathbb{K}$ is locally compact, then T is compact if and only if $T(B_X)$ has compact closure.

Theorem 2 ([6]). *Suppose that $\mathbb{K}$ is spherically complete. Then, for each $T \in \Phi(X,Y)$ and $K \in \mathcal{K}(X,Y)$, $T + K \in \Phi(X,Y)$ and $ind(T+K) = ind(T)$.*

Lemma 1 ([8]). *If $x_1^*, \cdots, x_n^*$ are linearly independent vectors in X^*, then there are vectors $x_1, \cdots, x_n$ in X, such that*

$$x_j^*(x_k) = \delta_{j,k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \qquad 1 \leq j,k \leq n. \tag{1.1}$$

Moreover, if $x_1, \cdots, x_n$ are linearly independent vectors in X, then there are vectors $x_1^*, \cdots, x_n^*$ in X^* such that (1.1) holds.

Theorem 3 ([5]). *Assume that X, Y are non-archimedean Banach spaces. Let $A : D(A) \subseteq X \to Y$ be a surjective closed linear operator. Then A is an open map.*

When the domain of A is dense in X, the adjoint operator A^* of A is defined as usual. Specifically, the operator $A^* : D(A^*) \subseteq Y^* \to X^*$ satisfies

$$\langle Ax,\ y^* \rangle = \langle x,\ A^* y^* \rangle$$

for all $x \in D(A)$, $y^* \in D(A^*)$. As in the classic case, the following property is an immediate consequence of the definition.

Proposition 1 ([5]). *Let A be a linear operator with dense domain. Then A^* is a closed linear operator.*

Proposition 2 ([5]). *Let A be a linear operator with dense domain. Then the following statement holds:*

$$R(A)^{\perp} = \ker(A^*) \cong (Y/\overline{R(A)})^*.$$

Theorem 4 ([7]). *Let X be a non-archimedean Banach space. For any nonzero $x \in X$, there exists $x^* \in X^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.*

Definition 8 ([1]). Let $A, B \in \mathcal{L}(X)$ such that $A \neq B$ and B be a non-null operator. The Fredholm spectrum of a pair (A, B) is defined by

$$\sigma_F(A, B) = \{\lambda \in \mathbb{K} : A - \lambda B \notin \Phi(X)\}. \tag{1.2}$$

The Fredholm resolvent is defined by $\rho_F(A, B) = \mathbb{K} \backslash \sigma_F(A, B)$.

Definition 9 ([1]). Let $A, B \in \mathcal{L}(X)$ such that $A \neq B$ and B be a non-null operator. The essential spectrum of a pencil of bounded linear operators (A, B) is defined by

$$\sigma_e(A, B) = \{\lambda \in \mathbb{K} : A - \lambda B \textit{ is not a Fredholm operator of index } 0\}.$$

Proposition 3 ([1]). *Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $A, B \in \mathcal{L}(X)$ be such that B is a non null operator. Then, the Fredholm resolvent $\rho_F(A, B)$ is open.*

Theorem 5 ([1]). *Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $A - \lambda B \in \Phi(X)$ and $K \in \mathcal{K}(X)$. Then $A + K - \lambda B \in \Phi(X)$.*

Remark 3. We set $\sigma_1(A, B) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K, B)$.

The following result gives a characterization of the essential spectrum for (A, B), where $A \neq B$ by means of a Fredholm operator on a Banach space of countable type over $\mathbb{Q}_p$.

Theorem 6. *Let X be (n.a) Banach space of countable type over $\mathbb{Q}_p$. Let $B \in \mathcal{L}(X)$ and $A \in \mathcal{C}(X)$, $\lambda \in \mathbb{K}$ be such that A^*, B^* exist, and $N\Big((A - \lambda B)^*\Big) = R(A - \lambda B)^{\perp}$. Then*

$$\lambda \notin \sigma_1(A, B) \quad \textit{if and only if} \quad A - \lambda B \in \Phi(X) \quad \textit{and} \quad ind(A - \lambda B) = 0.$$

Proof. Let $\lambda \notin \sigma_1(A, B)$, then there exists $K \in \mathcal{K}(X)$ such that $\lambda \in \rho(A + K, B)$. Hence

$$A + K - \lambda B \in \Phi(X)$$

and

$$ind(A + K - \lambda B) = 0.$$

The operator $A - \lambda B$ can be written in the form

$$A - \lambda B = A + K - \lambda B - K.$$

Since $K \in \mathcal{K}(X)$, using Theorem 2, we have

$$A - \lambda B \in \Phi(X)$$

and

$$ind(A-\lambda B)=0.$$

Conversely, let $\lambda \in \mathbb{Q}_p$ such that $A-\lambda B \in \Phi(X)$ and $ind(A-\lambda B)=0$. Put $\alpha(A-\lambda B)=\beta(A-\lambda B)=n$. Let $\{x_1,\cdots,x_n\}$ be a basis for $N(A-\lambda B)$ and $\{y_1^*,\cdots,y_n^*\}$ be a basis for $R(A-\lambda B)^{\perp}$. By Lemma 1, there are functionals $x_1^*,\cdots,x_n^*$ in X^* $\big($ X^* is the dual space of $X\big)$ and elements $y_1,\cdots,y_n$ in X such that

$$x_j^*(x_k)=\delta_{j,k} \quad \text{and} \quad y_j^*(y_k)=\delta_{j,k}, \qquad 1\leq j,\ k\leq n,$$

where $\delta_{j,k}=0$ if $j\neq k$ and $\delta_{j,k}=1$ if $j=k$. Consider an operator $F\colon X\to X$ defined by

$$x\longmapsto \sum_{i=1}^{n} x_i^*(x)y_i.$$

It is easy to see that F is a linear operator and $D(F)=X$. In fact, for all $x\in X$,

$$\begin{aligned}\|Fx\| &= \|\sum_{i=1}^{n} x_i^*(x)y_i\|\\ &\leq \max_{1\leq i\leq n}\|x_i^*(x)y_i\|\\ &\leq \max_{1\leq i\leq n}(\|x_i^*\|\|y_i\|)\|x\|.\end{aligned}$$

Moreover, $R(F)$ is contained in a finite dimensional subspace of X. So, F is a finite rank operator, hence F is a compact operator. We demonstrate that

$$N(A-\lambda B)\cap N(F)=\{0\}, \tag{1.3}$$

and

$$R(A-\lambda B)\cap R(F)=\{0\}. \tag{1.4}$$

Let $x\in N(A-\lambda B)\cap N(F)$, hence

$$x=\sum_{i=1}^{n}\alpha_i x_i, \qquad \alpha_1,\cdots,\alpha_n\in\mathbb{Q}_p.$$

Then for all $1\leq j\leq n$, $x_j^*(x)=\sum_{i=1}^{n}\alpha_i\delta_{i,j}=\alpha_j$. On the other hand, if $x\in N(F)$, then $Fx=0$, so

$$\sum_{j=1}^{n} x_j^*(x)y_j=0.$$

Therefore, we have for all $1\leq j\leq n$, $x_j^*(x)=0$. Hence $x=0$. Consequently,

$$N(A-\lambda B)\cap N(F)=\{0\}.$$

Let $y\in R(A-\lambda B)\cap R(F)$, then $y\in R(A-\lambda B)$ and $y\in R(F)$. Let $y\in R(F)$, we have

$$y=\sum_{i=1}^{n}\alpha_i y_i, \qquad \alpha_1\cdots\alpha_n\in\mathbb{Q}_p.$$

Then for all $1\leq j\leq n$, $y_j^*(y)=\sum_{i=1}^{n}\alpha_i\delta_{i,j}=\alpha_j$. On the other hand, if $y\in R(A-\lambda B)$, then for all $1\leq j\leq n$, $y_j^*(y)=0$. Thus $y=0$. Therefore,

$$R(A-\lambda B)\cap R(F)=0.$$

On the other hand, F is a compact operator, hence we deduce from Theorem 2, $A-\lambda B+F\in\Phi(X)$ and $ind(A+F-\lambda B)=0$. Thus

$$\alpha(A+F-\lambda B)=\beta(A+F-\lambda B). \tag{1.5}$$

If $x \in N(A+F-\lambda B)$, then $(A-\lambda B)x = -Fx$ in $R(A-\lambda B) \cap R(F)$. It follows from (1.4) that $(A-\lambda B)x = -Fx = 0$, hence $x \in N(A-\lambda B) \cap N(F)$ and from (1.3), we have $x = 0$. Thus $\alpha(A+F-\lambda B) = 0$, it follow from (1.5), $R(A+F-\lambda B) = X$. Consequently, $A-\lambda B+F$ is invertible and we conclude that $\lambda \notin \sigma_1(A,B)$. □

By Theorem 6, we conclude that $\sigma_e(A,B) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A+K,B)$.

References

[1] R. Ameziane Hassani, A. El Amrani, A. Blali, and J. Ettayb, Pseudospectra and essential Pseudospectra of a Pencil of bounded linear operators on non archimedean Banach spaces, Filomat, to appear.

[2] A. Ammara, A. Bouchekouaa, and A. Jeribi, Pseudospectra in a Non-Archimedean Banach Space and Essential Pseudospectra in E_ω, Filomat **33** (2019), no. 12, 3961-3976, doi.org/10.2298/FIL1912961A.

[3] T. Diagana and F. Ramaroson, Non-archimedean Operators Theory, Springer, 2016, doi.org/10.1007/978-3-319-27323-5.

[4] Jos L. M. Van Dorsselaer, Pseudospectra for matrix pencils and stability of equilibria, BIT **37** (1997), no. 4, 833-845, doi.org/10.1007/BF02510354.

[5] H. R. Henriquez, H. Samuel Navarro, and G. Jose Aguayo, Closed linear operators between nonarchimedean Banach spaces, Indag. Mathem., N.S., **16** (2) (2005), 201-214, doi.org/10.1016/S0019-3577(05)80023-0.

[6] C. Perez-Garcia and S. Vega, Perturbation theory of p-adic Fredholm and semi-Fredholm operators, Indag. Mathem., N.S., **15** (2004), no. 1, 115-128, doi.org/10.1016/S0019-3577(04)90009-2.

[7] A. C. M. van Rooij, Non-Archimedean functional analysis. Monographs and Textbooks in Pure and Applied Math., 51. Marcel Dekker, Inc., New York, 1978.

[8] M. Schechter, Principles of Functional Analysis. Graduate Studies in Mathematics, vol. 36 American Mathematical Society, Providence, 2002.

[9] L. N. Trefethen and M. Embree, Spectra and pseudospectra. The behavior of nonnormal matrices and operators. Princeton University Press, Princeton, 2005, doi.org/10.2307/j.ctvzxx9kj.

Aziz Blali: aziz.blali@usmba.ac.ma
Department of Mathematics, University of Sidi Mohamed Ben Abdellah, ENS, Fez, Morocco

Abdelkhalek El Amrani: abdelkhalek.elamrani@usmba.ac.ma
Department of Mathematics and Computer Science, University of Sidi Mohamed Ben Abdellah, Faculty of Sciences Dhar El Mahraz, Fez, Morocco

Jawad Ettayb: jawad.ettayb@usmba.ac.ma
Department of Mathematics and Computer Science, University of Sidi Mohamed Ben Abdellah, Faculty of Sciences Dhar El Mahraz, Fez, Morocco

Received 18/01/2022; Revised 10/02/2022