A NOTE ON PENCIL OF BOUNDED LINEAR OPERATORS ON NON-ARCHIMEDEAN BANACH SPACES

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Abstract. We give a characterization of the essential spectrum for \((A, B)\), where \(A\) is a closed linear operator and \(B\) is a bounded linear operator, by means of Fredholm operators on a Banach space of countable type over \(\Bbb{Q}_p\).

Introduction and Preliminaries

Throughout this paper, \(X\) and \(Y\) are non-archimedean (n.a) Banach spaces over a (n.a) non trivially complete valued field \(\Bbb{K}\) with valuation \(\| \cdot \|\), \(\mathcal{L}(X, Y)\) denotes the set of all bounded linear operators from \(X\) into \(Y\). When, \(X = Y\), we have \(\mathcal{L}(X, Y) = \mathcal{L}(X)\). If \(A \in \mathcal{L}(X), N(A)\) and \(R(A)\) denote the kernel and the range of \(A\), respectively. For more details, we refer to [3, 7]. \(X\) is said to be of countable type if there is a countable set in \(X\) whose linear hull is dense. Recall that an unbounded linear operator \(A : D(A) \subseteq X \rightarrow Y\) is said to be closed if for all \((x_n) \subset D(A)\) such that \(\|x_n - x\| \rightarrow 0\) and \(\|Ax_n - y\| \rightarrow 0\) as \(n \rightarrow \infty\), for some \(x \in X\) and \(y \in Y\), then \(x \in D(A)\) and \(y = Ax\). The collection of all closed linear operators from \(X\) into \(Y\) is denoted by \(\mathcal{C}(X, Y)\). When \(X = Y\), if \(A \in \mathcal{L}(X)\) and \(B\) is unbounded linear operator, then \(A + B\) is closed if and only if \(B\) is closed [3]. For more details on non-archimedean operators theory, we refer to [2, 3, 7]. There are many interesting works on pseudospectra in the classical Banach space, see [4, 9].

Definition 1 ([3]). Let \(\omega = (\omega_i)_i\) be a sequence of non-zero elements of \(\Bbb{K}\). We define \(\mathbb{E}_\omega\) by

\[\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N}, x_i \in \mathbb{K}, \text{ and } \lim_{i \to \infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\}\]

and it is equipped with the norm

\[(\forall x \in \mathbb{E}_\omega) : x = (x_i)_i, \quad \|x\| = \sup_{i \in \mathbb{N}} |\omega_i|^{\frac{1}{2}}|x_i|\]

Remark 1.

1. [3, Example 2.21] The space \((\mathbb{E}_\omega, \| \cdot \|)\) is a non-archimedean Banach space.

2. If \((\cdot, \cdot) : \mathbb{E}_\omega \times \mathbb{E}_\omega \rightarrow \mathbb{K}\), is defined by

\[(x, y) \mapsto \sum_{i=0}^{\infty} x_i y_i \omega_i\]

where \(x = (x_i)_i\) and \(y = (y_i)_i\), then the space \((\mathbb{E}_\omega, \| \cdot \|, (\cdot, \cdot))\) is called a \(p\)-adic (or non-archimedean) Hilbert space.

2. The orthogonal basis \(\{e_i, i \in \mathbb{N}\}\) is called the canonical basis of \(\mathbb{E}_\omega\), where for all \(i \in \mathbb{N}\), \(|e_i| = |\omega_i|^{\frac{1}{2}}\).

In the next definition, \(X\) and \(Y\) are two vector spaces over \(\Bbb{K}\).

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**Definition 2** ([6]). We say that \( A \in \mathcal{L}(X, Y) \) has an index when both \( \alpha(A) = \dim N(A) \) and \( \beta(A) = \dim \left( Y/R(A) \right) \) are finite. In this case, the index of the linear operator \( A \) is defined as \( \text{ind}(A) = \alpha(A) - \beta(A) \).

**Definition 3** ([6]). Let \( A \in \mathcal{L}(X, Y) \). \( A \) is said to be upper semi-Fredholm operator if
\[ \alpha(A) \text{ is finite and } R(A) \text{ is closed.} \]
The set of all upper semi-Fredholm operators is denoted by \( \Phi_+(X, Y) \).

**Definition 4** ([6]). Let \( A \in \mathcal{L}(X, Y) \). \( A \) is said to be lower semi-Fredholm operator if
\[ \beta(A) \text{ is finite.} \]
The set of all lower semi-Fredholm operators is denoted by \( \Phi_-(X, Y) \).

The set of all Fredholm operators is defined by
\[ \Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y). \]
Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \). A subset \( A \) of \( X \) is said to be compactoid if for every \( \varepsilon > 0 \), there is a finite subset \( B \) of \( X \) such that \( A \subset B(0) + C_0(B) \), where \( B(0) = \{ x \in X : ||x|| \leq \varepsilon \} \) and \( C_0(B) \) is an absolutely convex hull of \( X \), i.e.
\[ C_0(B) = \{ \lambda_1 x_1 + \cdots + \lambda_n x_n : n \in \mathbb{N}, \lambda_1, \cdots, \lambda_n \in B_K, x_1, \cdots, x_n \in B \}. \]

We have the following definition [7, page 142], for more details, see [7].

**Definition 5** ([7]). Let \( X \) and \( Y \) be two non-archimedean Banach spaces over \( \mathbb{K} \). A linear map \( A : X \to Y \) is said to be compact if \( A(B_X) \) is compactoid in \( Y \), where \( B_X = \{ x \in X : ||x|| \leq 1 \} \).

We denote by \( K(X, Y) \), the set of all compact operators from \( X \) into \( Y \).

**Definition 6** ([7]). Let \( T \in \mathcal{L}(X, Y) \). \( T \) is called an operator of finite rank if \( R(A) \) is a finite dimensional subspace of \( Y \).

**Theorem 1** ([7]). Let \( T \in \mathcal{L}(X, Y) \). Then \( T \) is compact if, and only if, for every \( \varepsilon > 0 \), there exists an \( S \in \mathcal{L}(X, Y) \) such that \( R(S) \) is finite-dimensional and \( ||T - S|| < \varepsilon \).

**Definition 7** ([3]). Let \( X \) be a non-archimedean Banach space and let \( T \in \mathcal{L}(X) \). \( T \) is said to be completely continuous, if there exists a sequence of finite rank linear operators \( (T_n) \) such that \( ||T_n - T|| \to 0 \) as \( n \to \infty \).

The collection of completely continuous linear operators on \( X \) is denoted by \( \mathcal{C}_c(X) \).

**Remark 2** ([7]).
(i) In a non-archimedean Banach space \( X \), we do not have the relationship between \( \mathcal{C}_c(X) \) and \( K(X) \) as in the classical case. J. P. Serre has proved that those concepts coincide, when \( \mathbb{K} \) is locally compact.
(ii) If \( \mathbb{K} \) is locally compact, then all completely continuous linear operators on \( X \) are compact on \( X \).
(iii) If \( \mathbb{K} \) is locally compact, then \( T \) is compact if and only if \( T(B_X) \) has compact closure.

**Theorem 2** ([6]). Suppose that \( \mathbb{K} \) is spherically complete. Then, for each \( T \in \Phi(X, Y) \) and \( K \in K(X, Y) \), \( T + K \in \Phi(X, Y) \) and \( \text{ind}(T + K) = \text{ind}(T) \).

**Lemma 1** ([8]). If \( x_1, \cdots, x_n \) are linearly independent vectors in \( X^* \), then there are vectors \( x_1, \cdots, x_n \) in \( X \), such that
\[ x_j^*(x_k) = \delta_{j,k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \quad 1 \leq j, k \leq n. \quad (1.1) \]
Moreover, if \( x_1, \cdots, x_n \) are linearly independent vectors in \( X \), then there are vectors \( x_1^*, \cdots, x_n^* \) in \( X^* \) such that (1.1) holds.
**Theorem 3** ([5]). Assume that $X$, $Y$ are non-archimedean Banach spaces. Let $A : D(A) \subseteq X \to Y$ be a surjective closed linear operator. Then $A$ is an open map.

When the domain of $A$ is dense in $X$, the adjoint operator $A^*$ of $A$ is defined as usual. Specifically, the operator $A^* : D(A^*) \subseteq Y^* \to X^*$ satisfies

\[
\langle Ax, y^* \rangle = \langle x, A^* y^* \rangle
\]

for all $x \in D(A)$, $y^* \in D(A^*)$. As in the classic case, the following property is an immediate consequence of the definition.

**Proposition 1** ([5]). Let $A$ be a linear operator with dense domain. Then $A^*$ is a closed linear operator.

**Proposition 2** ([5]). Let $A$ be a linear operator with dense domain. Then the following statement holds:

\[
R(A)^\bot = \ker(A^*) \cong (Y/R(A))^*.
\]

**Theorem 4** ([7]). Let $X$ be a non-archimedean Banach space. For any nonzero $x \in X$, there exists $x^* \in X^*$ such that $x^* (x) = 1$ and $\|x^*\| = \|x\|^{-1}$.

**Definition 8** ([11]). Let $A, B \in \mathcal{L}(X)$ such that $A \neq B$ and $B$ be a non-null operator. The Fredholm spectrum of a pair $(A, B)$ is defined by

\[
\sigma_F(A, B) = \{ \lambda \in \mathbb{K} : A - \lambda B \notin \Phi(X) \}.
\]

The Fredholm resolvent is defined by $\rho_F(A, B) = \mathbb{K}\setminus\sigma_F(A, B)$.

**Definition 9** ([11]). Let $A, B \in \mathcal{L}(X)$ such that $A \neq B$ and $B$ be a non-null operator. The essential spectrum of a pencil of bounded linear operators $(A, B)$ is defined by

\[
\sigma_e(A, B) = \{ \lambda \in \mathbb{K} : A - \lambda B \text{ is not a Fredholm operator of index } 0 \}.
\]

**Proposition 3** ([11]). Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $A, B \in \mathcal{L}(X)$ be such that $B$ is a non-null operator. Then, the Fredholm resolvent $\rho_F(A, B)$ is open.

**Theorem 5** ([11]). Suppose that $\mathbb{K} = \mathbb{Q}_p$. Let $A - \lambda B \in \Phi(X)$ and $K \in \mathcal{K}(X)$. Then $A + K - \lambda B \in \Phi(X)$.

**Remark 3.** We set $\sigma_1(A, B) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K, B)$.

The following result gives a characterization of the essential spectrum for $(A, B)$, where $A \neq B$ by means of a Fredholm operator on a Banach space of countable type over $\mathbb{Q}_p$.

**Theorem 6.** Let $X$ be a (n.a) Banach space of countable type over $\mathbb{Q}_p$. Let $B \in \mathcal{L}(X)$ and $A \in \mathcal{C}(X)$, $\lambda \in \mathbb{K}$ be such that $A^*$, $B^*$ exist, and $N \left( (A - \lambda B)^* \right) = R(A - \lambda B)^\bot$. Then

\[
\lambda \notin \sigma_1(A, B) \quad \text{if and only if} \quad A - \lambda B \in \Phi(X) \quad \text{and} \quad \text{ind}(A - \lambda B) = 0.
\]

**Proof.** Let $\lambda \notin \sigma_1(A, B)$, then there exists $K \in \mathcal{K}(X)$ such that $\lambda \in \rho(A + K, B)$. Hence $A + K - \lambda B \in \Phi(X)$ and

\[
\text{ind}(A + K - \lambda B) = 0.
\]

The operator $A - \lambda B$ can be written in the form

\[
A - \lambda B = A + K - \lambda B - K.
\]

Since $K \in \mathcal{K}(X)$, using Theorem 2, we have

\[
A - \lambda B \in \Phi(X).
\]
and

\[ ind(A - \lambda B) = 0. \]

Conversely, let \( \lambda \in \mathbb{Q}_p \) such that \( A - \lambda B \in \Phi(X) \) and \( ind(A - \lambda B) = 0 \). Put \( \alpha(A - \lambda B) = \beta(A - \lambda B) = n \). Let \( \{x_1, \ldots, x_n\} \) be a basis for \( N(A - \lambda B) \) and \( \{y_1, \ldots, y_n\} \) be a basis for \( R(A - \lambda B) \). By Lemma 1, there are functionals \( x_1^*, \ldots, x_n^* \) in \( X^* \) (\( X^* \) is the dual space of \( X \)) and elements \( y_1, \ldots, y_n \) in \( X \) such that

\[ x_j^*(x_k) = \delta_{j,k} \quad \text{and} \quad y_j^*(y_k) = \delta_{j,k}, \quad 1 \leq j, k \leq n, \]

where \( \delta_{j,k} = 0 \) if \( j \neq k \) and \( \delta_{j,k} = 1 \) if \( j = k \). Consider an operator \( F: X \rightarrow X \) defined by

\[ x \mapsto \sum_{i=1}^{n} x_i^*(x) y_i. \]

It is easy to see that \( F \) is a linear operator and \( D(F) = X \). In fact, for all \( x \in X \),

\[ \|Fx\| = \| \sum_{i=1}^{n} x_i^*(x) y_i \| \leq \max_{1 \leq i \leq n} \| x_i^*(x) y_i \| \leq \max_{1 \leq i \leq n} (\| x_i^* \| \| y_i \| ) \| x \|. \]

Moreover, \( R(F) \) is contained in a finite dimensional subspace of \( X \). So, \( F \) is a finite rank operator, hence \( F \) is a compact operator. We demonstrate that

\[ N(A - \lambda B) \cap N(F) = \{0\}, \quad (1.3) \]

and

\[ R(A - \lambda B) \cap R(F) = \{0\}. \quad (1.4) \]

Let \( x \in N(A - \lambda B) \cap N(F) \), hence

\[ x = \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_1, \ldots, \alpha_n \in \mathbb{Q}_p. \]

Then for all \( 1 \leq j \leq n \), \( x_j^*(x) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j \). On the other hand, if \( x \in N(F) \), then \( Fx = 0 \), so

\[ \sum_{j=1}^{n} x_j^*(x) y_j = 0. \]

Therefore, we have for all \( 1 \leq j \leq n \), \( x_j^*(x) = 0 \). Hence \( x = 0 \). Consequently,

\[ N(A - \lambda B) \cap N(F) = \{0\}. \]

Let \( y \in R(A - \lambda B) \cap R(F) \), then \( y \in R(A - \lambda B) \) and \( y \in R(F) \). Let \( y \in R(F) \), we have

\[ y = \sum_{i=1}^{n} \alpha_i y_i, \quad \alpha_1 \ldots \alpha_n \in \mathbb{Q}_p. \]

Then for all \( 1 \leq j \leq n \), \( y_j^*(y) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j \). On the other hand, if \( y \in R(A - \lambda B) \), then for all \( 1 \leq j \leq n \), \( y_j^*(y) = 0 \). Thus \( y = 0 \). Therefore,

\[ R(A - \lambda B) \cap R(F) = 0. \]

On the other hand, \( F \) is a compact operator, hence we deduce from Theorem 2, \( A - \lambda B + F \in \Phi(X) \) and \( ind(A + F - \lambda B) = 0 \). Thus

\[ \alpha(A + F - \lambda B) = \beta(A + F - \lambda B). \quad (1.5) \]
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If \( x \in N(A + F - \lambda B) \), then \((A - \lambda B)x = -Fx \) in \( R(A - \lambda B) \cap R(F) \). It follows from (1.4) that \((A - \lambda B)x = -Fx = 0\), hence \( x \in N(A - \lambda B) \cap N(F) \) and from (1.3), we have \( x = 0 \). Thus \( \alpha(A + F - \lambda B) = 0 \), it follow from (1.5), \( R(A + F - \lambda B) = X \). Consequently, \( A - \lambda B + F \) is invertible and we conclude that \( \lambda \notin \sigma_1(A, B) \).

By Theorem 6, we conclude that \( \sigma_e(A, B) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K, B) \).

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