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DIFFERENTIAL SYSTEMS WITH ALGEBRAIC AND NON-ALGEBRAIC LIMIT CYCLES EXPLICITLY GIVEN

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ABSTRACT. In this paper we give an explicit expression of invariant algebraic curves of a multi-parameter planar polynomial differential systems, then we prove that these systems are integrable and we introduce an explicit expression of a first integral. Moreover, we determine sufficient conditions for these systems to possess two limit cycles: one of them is algebraic and the other one, explicitly given, is shown to be non-algebraic. Concrete examples exhibiting the applicability of our results are introduced.

Надано явний вираз інваріантних алгебраїчних кривих багатопараметричної поліноміальної диференціальної системи на площині. Доводено, що ці системи є інтегровні, і наведено явний вираз дляй першого інтеграла. Більш того, отримано достатні умови для того, щоб ці системи мали два граничні цикли: один з яких є алгебраїчний, і доведено, що інший цикл, для якого отримано явний вираз, не є алгебраїчний. Надано конкретні приклади, які демонструють можливість застосування отриманих результатів.

1. Introduction

We consider the differential system

$$\begin{cases} x' = \dfrac{dx}{dt} = P(x,y), \\ \\ y' = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. Here, the degree of system (1.1) is denoted by $n = \max\{\deg P, \deg Q\}$. In the literature equivalent mathematical objects to refer to these planar differential systems appear as a vector field

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

A limit cycle of system (1.1) is an isolated periodic solution in the set of all its periodic solutions of system (1.1), and it is said to be algebraic if it is contained in the zero level set of a polynomial function [1, 11]. In 1900 Hilbert [10] in the second part of his 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane $\mathbb{R}^2$, even more difficult problem is to give an explicit expression of them [2, 3, 5, 4, 7, 9, 13]. This was one of the main problems in the qualitative theory of planar differential equations in the 20th century. To distinguish when a limit cycle is algebraic or not, usually, it is not easy. Thus, the well known limit cycle of the van der Pol differential system exhibited in 1926, was not proved until 1995 by Odani [12] that it was not algebraic. The van der Pol system can be written as a polynomial system (1.1) of degree 3, but its limit cycle is not known explicitly.

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System (1.1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant C^1 function $H:\Omega\to\mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1.1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt}=\frac{\partial H(x,y)}{\partial x}P(x,y)+\frac{\partial H(x,y)}{\partial y}Q(x,y)\equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H=h$ is the general solution of this equation, where h is an arbitrary constant.

Since for such vector fields the notion of integrability is based on the existence of a first integral [6, 8], the following question arises: Given the polynomial differential system (1.1), how to recognize if this polynomial differential system has a first integral, and how to compute it when it exists?

A curve $U(x,y)=0$, where $U(x,y)$ is a polynomial of degree m with real coefficients, is an invariant algebraic curve of system (1.1) if and only if there exists a polynomial $K=K(x,y)$ of degree at most $n-1$ satisfying

$$\frac{\partial U(x,y)}{\partial x}P(x,y)+\frac{\partial U(x,y)}{\partial y}Q(x,y)=K(x,y)\,U(x,y). \tag{1.2}$$

The curve $\Gamma=\left\{(x,y)\in\mathbb{R}^2 : U(x,y)=0\right\}$ is non-singular for system (1.1) if the equilibrium points of the system that satisfy

$$\begin{cases} x'=\dfrac{dx}{dt}=P(x,y)=0,\\ y'=\dfrac{dy}{dt}=Q(x,y)=0,\end{cases}$$

are not contained on the curve Γ.

The polynomial $K(x,y)$ is called the cofactor of $U(x,y)=0$, if the cofactor is identically zero, then $U(x,y)$ is a polynomial first integral for system (1.1). The corresponding cofactor of $U(x,y)$ is always polynomial whether $U(x,y)$ is algebraic or non-algebraic. If U is real, the curve $U(x,y)=0$ is an invariant under the flow of differential system (1.1) and the set $\left\{(x,y)\in\mathbb{R}^2,\ U(x,y)=0\right\}$ is formed by orbits of system (1.1). There are strong relationships between the integrability of system (1.1) and its number of invariant algebraic solutions.

In this paper we give an explicit expression of invariant algebraic curves, then we prove that these systems are integrable and we introduce an explicit expression of a first integral of a multi-parameter planar polynomial differential systems of the form

$$\begin{cases} x'=\dfrac{dx}{dt}=x+\left(x^2+y^2\right)^3\left(P_5(x,y)-x\left(x^2+y^2\right)^5 S_4(x,y)\right),\\[2ex] y'=\dfrac{dy}{dt}=y+\left(x^2+y^2\right)^3\left(Q_5(x,y)-y\left(x^2+y^2\right)^5 S_4(x,y)\right),\end{cases} \tag{1.3}$$

where

$P_5(x,y)=ax^5+bx^4y+(c-a)\,x^3y^2+(d-b)\,x^2y^3+nxy^4-my^5$,
$Q_5(x,y)=mx^5+ax^4y+(b+3m)\,x^3y^2+(c-a)\,x^2y^3+(3m-b+d)\,xy^4+ny^5$ and
$S_4(x,y)=(a+1)\,x^4+(b+m)\,x^3y+(c-a+2)\,x^2y^2+(2m-b+d)\,xy^3+(n+1)\,y^4$,

in which a, c, b, d, n and m are real constants.

Moreover, we determine sufficient conditions for a polynomial differential system to possess two limit cycles: one of them is algebraic and the other one is shown to be non-algebraic, explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

We define the trigonometric functions

$$A(\theta) = \frac{1}{4}a + \frac{1}{8}c + \frac{3}{8}n + \frac{1}{2}(a-n)(\cos 2\theta) + \frac{1}{4}(d+3m)(\sin 2\theta) + \frac{1}{8}(2a-c+n)(\cos 4\theta) + \frac{1}{8}(2b-d-m)(\sin 4\theta),$$

$$B(\theta) = \left(-1 - \frac{1}{8}c - \frac{3}{8}n - \frac{1}{4}a\right) + \frac{1}{2}(n-a)(\cos 2\theta) - \frac{1}{4}(d+3m)(\sin 2\theta) + \frac{1}{8}(c-2a-n)(\cos 4\theta) + \frac{1}{8}(d-2b+m)(\sin 4\theta),$$

$$G(\theta) = \exp\left(\frac{10}{m}\int_0^\theta (A(s) + 2B(s))\,ds\right),$$

$$F(\theta) = \frac{-10}{m}\int_0^\theta B(w)\exp\left(\frac{10}{m}\int_0^w (A(s)+2B(s))\,ds\right)dw.$$

2. Main result

Our first result on the critical point and the expression of invariant algebraic curves of the system (1.3) is the following.

Theorem 2.1. *Consider a multi-parameter planar polynomial differential system* (1.3), *then the following statements hold.*

1) If $m \neq 0$, *then the origin of coordinates* $O(0,0)$ *is the unique critical point at a finite distance. Moreover,* $O(0,0)$ *is star node.*

2) The curve $\Gamma_1 = \{(x,y) \in \mathbb{R}^2 : \left(x^2+y^2\right)^5 - 1 = 0\}$ *is an invariant algebraic curve of system* (1.3) *with the cofactor*

$$K(x,y) = -10\left(x^2+y^2\right)^5 ((a+1)x^{10} + (b+m)x^9y + (2a+c+5)x^8y^2 + (2b+d+5m)x^7y^3 + (3c+n+10)x^6y^4 + (3d+9m)x^5y^5 + (3c-2a+3n+10)x^4y^6 + (3d-2b+7m)x^3y^7 + (c-a+3n+5)x^2y^8 + (d-b+2m)xy^9 + (n+1)y^{10} + 1). \quad (2.4)$$

Moreover, the curve Γ_1 *is non-singular for system* (1.3).

Proof. **1)** We say that $A(x_0,y_0) \in \mathbb{R}^2$ is a critical point of system (1.3) if

$$\begin{cases} x_0 + \left(x_0^2+y_0^2\right)^3\left(P_5(x_0,y_0) - x_0(x_0^2+y_0^2)^5 S_4(x_0,y_0)\right) = 0, \\ y_0 + (x_0^2+y_0^2)^3\left(Q_5(x_0,y_0) - y_0(x_0^2+y_0^2)^5 S_4(x_0,y_0)\right) = 0. \end{cases}$$

It follows that $y_0\left(x_0^2+y_0^2\right)^3 P_5(x_0,y_0) - x_0\left(x_0^2+y_0^2\right)^3 Q_5(x_0,y_0) = -m\left(x_0^2+y_0^2\right)^6 = 0$, and $x_0 = 0, y_0 = 0$ is a unique solution of this equation. Thus the origin is the unique critical point at a finite distance.

Computing the Jacobian matrix of the system (1.3) evaluated at $O(0,0)$, we have

$$J = \left.\begin{pmatrix} \dfrac{\partial P}{\partial x}(x,y) & \dfrac{\partial P}{\partial y}(x,y) \\ \dfrac{\partial Q}{\partial x}(x,y) & \dfrac{\partial Q}{\partial y}(x,y) \end{pmatrix}\right|_{O(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix has repeated positive real eigenvalues $\lambda = 1 > 0$, so that $O(0,0)$ is a star node of system (1.3). This completes the proof of statement 1) of Theorem 2.1.

2) A computation shows that $U(x,y) = \left(x^2+y^2\right)^5 - 1$ satisfies the linear partial differential equation (1.2), the associated cofactor $K(x,y)$ is given by (2.4), then the curve $U(x,y) = 0$ is an invariant algebraic curve of system (1.3) with cofactor $K(x,y)$.

We have, the origin $O(0,0)$ is the unique critical point at a finite distance, consequently, the curve $\Gamma_1 = \left\{(x,y) \in \mathbb{R}^2 : \left(x^2+y^2\right)^5 - 1 = 0\right\}$ is non-singular for system (1.3). This completes the proof of statement (2) of Theorem 2.1. □

Our second result on the existence of a first integral and an explicit expression for two limit cycles of the system (1.3) is the following.

Theorem 2.2. *Consider a multi-parameter planar polynomial differential system* (1.3). *Then the following statements hold.*

1) The system (1.3) *has the first integral*

$$H(x,y) = \frac{G(\arctan \frac{y}{x}) - \left(\left(x^2+y^2\right)^5 - 1\right) F(\arctan \frac{y}{x})}{\left(x^2+y^2\right)^5 - 1}.$$

2) The system (1.3) *has an explicit limit cycle, given in the Cartesian coordinates by* (Γ_1), $\left(x^2+y^2\right)^5 - 1 = 0$.

3) If

$$\begin{aligned} &-8-c-3n-2a > 4|n-a| + 2|d+3m| + |c-2a-n| + |d-2b+m|, \\ &16+c+3n+2a > 4|n-a| + 2|d+3m| + |c-2a-n| + |d-2b+m|, \\ &m<0 \quad \text{and} \quad 2a-c+n \neq 0. \end{aligned} \tag{2.5}$$

then system (1.3) *has non-algebraic limit cycle* (Γ_2), *explicitly given in polar coordinates* (r,θ) *by the equation*

$$r(\theta, r_*) = \left(\frac{(G(2\pi)-1)(G(\theta)+F(\theta)) + F(2\pi)}{(G(2\pi)-1)F(\theta)+F(2\pi)}\right)^{\frac{1}{10}},$$

where

$$r_* = \left(\frac{F(2\pi)+G(2\pi)-1}{F(2\pi)}\right)^{\frac{1}{10}}.$$

Moreover, the non-algebraic limit cycle (Γ_2) *lies inside the algebraic limit cycle* (Γ_1).

Proof. In order to prove our results 1), 2), and 3) in Theorem 2.2, we write the polynomial differential system (1.3) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$. The system becomes

$$\begin{cases} r' = \dfrac{dr}{dt} = r + A(\theta) r^{11} + B(\theta) r^{21}, \\ \theta' = \dfrac{d\theta}{dt} = m r^{10}, \end{cases} \tag{2.6}$$

where the trigonometric functions $A(\theta)$ and $B(\theta)$ are given in Introduction.

We have that $m < 0$, then θ' is negative for all $t \in \mathbb{R}$, the orbits $(r(t), \theta(t))$ of system (2.6) have the opposite orientation with respect to those $(x(t), y(t))$ of system (1.3).

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = \frac{1}{m} r^{-9} + \frac{A(\theta)}{m} r + \frac{B(\theta)}{m} r^{11}. \tag{2.7}$$

Via the change of variables $\rho = r^{10}$, this equation (2.7) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = \frac{10}{m} + \frac{10A(\theta)}{m}\rho + \frac{10B(\theta)}{m}\rho^2. \tag{2.8}$$

This equation is integrable, since it possesses the particular solution $\rho = 1$.

By introducing the standard change of variables $z = \rho - 1$ we obtain the Bernoulli equation

$$\frac{dz}{d\theta} = \frac{10A(\theta) + 20B(\theta)}{m} z + \frac{10B(\theta)}{m} z^2. \tag{2.9}$$

Note that $z = 0$ is a solution of (2.9).

We assume that $z \neq 0$, by introducing the standard change of variables $y = \frac{1}{z}$ we obtain the linear equation

$$\frac{dy}{d\theta} = \frac{-10A(\theta) - 20B(\theta)}{m} y + \frac{-10B(\theta)}{m}. \tag{2.10}$$

The general solution of linear equation (2.10) is

$$y(\theta) = \frac{F(\theta) + \lambda}{G(\theta)},$$

where $\lambda \in \mathbb{R}$ and $F(\theta)$ and $G(\theta)$ are trigonometric functions given in Introduction.

Then the general solution of equation (2.9) is

$$z(\theta) = 0, \qquad z(\theta) = \frac{G(\theta)}{F(\theta) + \lambda},$$

where $\lambda \in \mathbb{R}$. The general solution of equation (2.8) is

$$\rho(\theta) = 1, \qquad \rho(\theta) = \frac{G(\theta) + F(\theta) + \lambda}{F(\theta) + \lambda},$$

where $\lambda \in \mathbb{R}$. Consequently, the general solution of (2.7) is

$$r^{10}(\theta) = 1, \qquad r(\theta) = \left(\frac{G(\theta) + F(\theta) + \lambda}{F(\theta) + \lambda}\right)^{\frac{1}{10}},$$

where $\lambda \in \mathbb{R}$.

From this solution we obtain a first integral in the variables (x, y) of the form

$$H(x, y) = \frac{G(\arctan \frac{y}{x}) - \left(\left(x^2 + y^2\right)^5 - 1\right) F(\arctan \frac{y}{x})}{\left(x^2 + y^2\right)^5 - 1}.$$

Hence, statement 1) of Theorem 2.2 is proved.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.3), in Cartesian coordinates are written as

$$\left(x^2 + y^2\right)^5 = \frac{G(\arctan \frac{y}{x}) + F(\arctan \frac{y}{x}) + h}{F(\arctan \frac{y}{x}) + h},$$

where $h \in \mathbb{R}$.

Note that system (1.3) has a periodic orbit if and only if equation (2.7) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (2.7) that fulfills $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$.

The solution $r(\theta, r_0)$ of the differential equation (2.7) such that $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = \left(\frac{\left(r_0^{10} - 1\right)(G(\theta) + F(\theta)) + 1}{\left(r_0^{10} - 1\right) F(\theta) + 1}\right)^{\frac{1}{10}},$$

where $r_0 = r(0)$.

We have a particular solution $\rho(\theta) = 1$ of the differential equation (2.7), and from this solution we obtain that $r^{10}(\theta, 1) = 1 > 0$, for all $\theta \in [0, \pi]$, is a particular solution of the differential equation (2.7).

This is an algebraic limit cycle for the differential systems (1.3), corresponding of course to an invariant algebraic curve $U(x,y)=0$.

More precisely, in that Cartesian coordinates, $r^2=x^2+y^2$ and $\theta=\arctan\left(\frac{y}{x}\right)$, and the curve (Γ_1) defined by this limit cycle is (Γ_1): $\left(x^2+y^2\right)^5-1=0$. Hence, statement 2) of Theorem 2.2 is proved.

A periodic solution of system (1.3) must satisfy the condition $r(2\pi,r_0)=r(0,r_0)$, which leads to a unique value $r_0=r_*$, given by

$$r_*=\left(\frac{F(2\pi)+G(2\pi)-1}{F(2\pi)}\right)^{\frac{1}{10}},$$

where

$$G(2\pi)=\exp\left(\int_0^{2\pi}\left(\frac{10A(s)+20B(s)}{m}\right)ds\right),$$
$$F(2\pi)=\int_0^{2\pi}\left(\frac{-10B(w)}{m}\exp\left(\int_0^{w}\left(\frac{10A(s)+20B(s)}{m}\right)ds\right)dw\right).$$

Here r_* is the intersection of the periodic orbit with the OX_+ axis.

After the substitution of this value r_* into $r(\theta,r_0)$ we obtain

$$r(\theta,r_*)=\left(1+\frac{(G(2\pi)-1)\,G(\theta)}{G(2\pi)\,F(\theta)+F(2\pi)-F(\theta)}\right)^{\frac{1}{10}}.$$

In what follows we prove that $r(\theta,r_*)>0$. Indeed,

$$\begin{aligned}F(2\pi)-F(\theta)&=\int_0^{2\pi}\frac{-10B(w)}{m}\exp\left(\int_0^{w}\left(\frac{10A(s)+20B(s)}{m}\right)ds\right)dw\\&\quad+\int_\theta^{0}\frac{-10B(w)}{m}\exp\left(\int_0^{w}\left(\frac{10A(s)+20B(s)}{m}\right)ds\right)dw\\&=\int_\theta^{2\pi}\frac{-10B(w)}{m}\exp\left(\int_0^{w}\left(\frac{10A(s)+20B(s)}{m}\right)ds\right)dw\end{aligned}$$

According to conditions (2.5) we see that $m(A(s)+2B(s))>0$ and $mB(w)<0$ for all $\theta\in(0,\pi)$. Then we have $F(2\pi)-F(\theta)>0$ and $G(2\pi)>1$, which ensures that r_* and $r(\theta,r_*)$ are well defined for all $\theta\in(0,\pi)$. Therefore we have $r_*>0$ and $r(\theta,r_*)>0$ for all $\theta\in[0,\pi]$ and the limit cycle does not pass through the equilibrium point $O(0,0)$ of system (1.3). This is the second limit cycle for the differential system (1.3), and we denote it by (Γ_2).

This limit cycle (Γ_2) is not algebraic, more precisely, in the Cartesian coordinates $r^2=x^2+y^2$ and $\theta=\arctan\left(\frac{y}{x}\right)$, the curve (Γ_2) defined by this limit cycle is (Γ_2): $L(x,y)=0$ where

$$L(x,y)=\left(x^2+y^2\right)^5-1+\frac{(G(2\pi)-1)\,G\left(\arctan\frac{y}{x}\right)}{G(2\pi)\,F\left(\arctan\frac{y}{x}\right)+F(2\pi)-F\left(\arctan\frac{y}{x}\right)}.$$

According to the conditions (2.5), we have

$$\begin{aligned}A(\theta)+2B(\theta)&=-2-\frac{1}{8}c-\frac{3}{8}n-\frac{1}{4}a+\left(\frac{1}{2}n-\frac{1}{2}a\right)(\cos 2\theta)+\left(-\frac{1}{4}d-\frac{3}{4}m\right)(\sin 2\theta)\\&\quad+\left(\frac{1}{8}c-\frac{1}{4}a-\frac{1}{8}n\right)(\cos 4\theta)+\left(\frac{1}{8}d-\frac{1}{4}b+\frac{1}{8}m\right)(\sin 4\theta)\neq 0,\end{aligned}$$

for all $\theta \in (0, \pi)$. Then

$$\int_0^{\arctan \frac{y}{x}} (A(s) + 2B(s))\, dw = -\frac{1}{16}b - \frac{3}{32}d - \frac{11}{32}m - \left(2 + \frac{1}{8}c + \frac{3}{8}n + \frac{1}{4}a\right) \arctan \frac{y}{x} + \frac{1}{4}(n-a) \sin 2\left(\arctan \frac{y}{x}\right) + \frac{1}{8}(d+3m)\cos 2\left(\arctan \frac{y}{x}\right) + \frac{1}{32}(c-2a-n)\sin 4\left(\arctan \frac{y}{x}\right) - \frac{1}{32}(d-2b+m)\cos 4\left(\arctan \frac{y}{x}\right).$$

Since to $2a - c + n \neq 0$, we have that the non-algebraic expression $\frac{5(c-2a-n)}{16m} \sin 4\left(\arctan \frac{y}{x}\right)$ appears in the

$$G\left(\arctan \frac{y}{x}\right) = \exp\left(\frac{10}{m}\int_0^{\arctan \frac{y}{x}} (A(s) + 2B(s))\, ds\right).$$

Hence the expression $G\left(\arctan \frac{y}{x}\right)$ is non-algebraic. Consequently, $L(x, y) = 0$ is non-algebraic, and therefore the curve (Γ_2): $L(x, y) = 0$ is non-algebraic, as well as the limit cycle.

In order to prove hyperbolicity of the limit cycle it is sufficient to prove that for the Poincaré return map, for more details see [6, Sect. 1.6]. A computation shows that

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0 = r_*} = \exp\left(\int_0^{2\pi}\left(\frac{10A(s) + 20B(s)}{-m}\right) ds\right) < 1.$$

Therefore the limit cycle (Γ_2) of the differential system (1.3) is hyperbolic.

According to conditions (2.5), we get

$$G(2\pi) = \exp\left(\int_0^{2\pi}\left(\frac{10A(s) + 20B(s)}{m}\right) ds\right) > 1,$$

$$F(2\pi) = \int_0^{2\pi}\left(\frac{-10B(w)}{m}\exp\left(\int_0^{w}\left(\frac{10A(s) + 20B(s)}{m}\right) ds\right) dw\right) > 0.$$

for all $\theta \in [0, \pi]$. Then we have $r_* = \left(1 + \frac{G(2\pi)-1}{F(2\pi)}\right)^{\frac{1}{10}} > 1$. Moreover,

$$r(\theta, r_*) = \left(1 + \frac{(G(2\pi) - 1)\, G(\theta)}{G(2\pi)\, F(\theta) + F(2\pi) - F(\theta)}\right)^{\frac{1}{10}} > 1,$$

which shows that the algebraic limit cycle (Γ_1) lies inside the non-algebraic limit cycle (Γ_2).

We conclude that system (1.3) has two limit cycles, the algebraic (Γ_1) lying inside the non-algebraic one (Γ_2). This completes the proof of statement 3) of Theorem 2.2. □

3. Examples

The following examples are given to illustrate our results.

Example 1. If we take $a = n = \frac{-15}{10}, b = \frac{19}{20}, c = -4, d = \frac{31}{10}$ and $m = -1$, then system (1.3) reads

$$\begin{cases} x' = x - \frac{1}{20}\left(x^2+y^2\right)^3\left(30x^5 - 19x^4y + 50x^3y^2 - 43x^2y^3 + 30xy^4 - 20y^5\right) \\ \quad + \frac{1}{20}x\left(x^2+y^2\right)^8\left(10x^4 + x^3y + 10x^2y^2 - 3xy^3 + 10y^4\right), \\ y' = y - \frac{1}{20}\left(x^2+y^2\right)^3\left(20x^5 + 30x^4y + 41x^3y^2 + 50x^2y^3 + 17xy^4 + 30y^5\right) \\ \quad + \frac{1}{20}y\left(x^2+y^2\right)^8\left(10x^4 + x^3y + 10x^2y^2 - 3xy^3 + 10y^4\right). \end{cases} \tag{3.11}$$

The system (3.11) has the first integral

$$H(x,y)=\frac{G(\arctan\frac{y}{x})-\left(\left(x^2+y^2\right)^5-1\right)F(\arctan\frac{y}{x})}{\left(x^2+y^2\right)^5-1},$$

where

$$G(\theta)=\exp\left(\frac{1}{16}+\frac{45}{8}\theta-\frac{1}{8}\cos(2\theta)+\frac{1}{16}\cos(4\theta)-\frac{5}{32}\sin(4\theta)\right),$$

$$F(\theta)=\int_0^\theta\left(\frac{\frac{35}{8}+\frac{5}{8}\cos 4s-\frac{1}{4}\sin 2s+\frac{1}{4}\sin 4s}{\exp\left(\frac{-1}{16}-\frac{45}{8}s+\frac{1}{8}\cos 2s-\frac{1}{16}\cos 4s+\frac{5}{32}\sin 4s\right)}\right)ds.$$

System (3.11) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1): $\left(x^2+y^2\right)^5-1=0$.

This system (3.11) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\left(\frac{(G(2\pi)-1)(G(\theta)+F(\theta))+F(2\pi)}{(G(2\pi)-1)F(\theta)+F(2\pi)}\right)^{\frac{1}{10}},$$

where $\theta\in\mathbb{R}$, $G(2\pi)\simeq 2.2348\times 10^{15}$ and $F(2\pi)\simeq 2.0167\times 10^{15}$.

The intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_*=\left(\frac{2.0167\times 10^{15}+2.2348\times 10^{15}-1}{2.0167\times 10^{15}}\right)^{\frac{1}{10}}\simeq 1.0774.$$

We conclude that system (3.11) has two limit cycles. Since $r_*=1.0774>1$, the algebraic one (Γ_1) lies inside the non-algebraic one (Γ_2).

Example 2. If we take $a=\frac{-16}{10}, b=\frac{24}{5}, c=-\frac{9}{2}, d=\frac{151}{10}, n=\frac{-14}{10}$, and $m=-5$, System (1.3) reads

$$\begin{cases} x' &= x-\frac{1}{10}\left(x^2+y^2\right)^3\left(16x^5-48x^4y+29x^3y^2-103x^2y^3+14xy^4-50y^5\right)\\ &\quad +\frac{1}{10}x\left(x^2+y^2\right)^8\left(6x^4+2x^3y+9x^2y^2-3xy^3+4y^4\right),\\ y' &= y-\frac{1}{10}\left(x^2+y^2\right)^3\left(50x^5+16x^4y+102x^3y^2+29x^2y^3+47xy^4+14y^5\right)\\ &\quad +\frac{1}{10}y\left(x^2+y^2\right)^8\left(6x^4+2x^3y+9x^2y^2-3xy^3+4y^4\right). \end{cases} \tag{3.12}$$

The system (3.12) has the first integral

$$H(x,y)=\frac{G(\arctan\frac{y}{x})-\left(\left(x^2+y^2\right)^5-1\right)F(\arctan\frac{y}{x})}{\left(x^2+y^2\right)^5-1},$$

where

$$G(\theta)=\exp\left(\frac{-1}{160}+\frac{41}{40}\theta-\frac{1}{40}\cos 2\theta+\frac{1}{32}\cos 4\theta-\frac{1}{10}\sin 2\theta-\frac{1}{160}\sin 4\theta\right)$$

$$F(\theta)=\int_0^\theta\left(\frac{\frac{39}{40}+\frac{1}{5}\cos 2s+\frac{1}{40}\cos 4s-\frac{1}{20}\sin 2s+\frac{1}{8}\sin 4s}{\exp\left(\frac{-1}{160}+\frac{41}{40}s-\frac{1}{40}\cos 2s+\frac{1}{32}\cos 4s-\frac{1}{10}\sin 2s-\frac{1}{160}\sin 4s\right)}\right)ds.$$

The system (3.11) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1): $\left(x^2+y^2\right)^5-1=0$.

This system (3.11) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\left(\frac{(G(2\pi)-1)(G(\theta)+F(\theta))+F(2\pi)}{(G(2\pi)-1)F(\theta)+F(2\pi)}\right)^{\frac{1}{10}},$$

where $\theta \in \mathbb{R}$, $G(2\pi) \simeq 626.57$, and $F(2\pi) \simeq 637.01$

The intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \left(\frac{637.01 + 626.57 - 1}{637.01} \right)^{\frac{1}{10}} \simeq 1.0708.$$

We conclude that system (3.11) has two limit cycles. Since $r_* = 1.0708 > 1$, the algebraic one (Γ_1) lies inside the non-algebraic one (Γ_2).

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