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MOMENTS AND AN INTEGRAL REPRESENTATION FOR THE NON-SYMMETRIC DUNKL-CLASSICAL FORM

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Abstract. In this paper, we determine the moments and an integral representation for the non-symmetric Dunkl-classical form.

Знайдено моменти та інтегральне представлення для несиметричної класичної за Данклом форми.

1. Introduction and Preliminaries results

The moments and integral representation play an important role in various topics of orthogonal polynomials, their applications are also extended to other domains such as statistics and probability theory (see [10, 13]). The links between them and classical orthogonal polynomials are extensively studied by many authors (see [11, 12]).

Note that the classical continous orthogonal polynomial families (Hermite, Laguerre, Bessel and Jacobi) are very much related to probability theory and statistics (see [10]). The measures of the Hermite, Laguerre and Jacobi polynomials are the normal, the Gamma and the Beta distributions, respectively.

A monic orthogonal polynomial sequence (MOPS, for short) $\{P_n\}_{n\geq 0}$ is called Dunkl-classical or T_μ-classical polynomial sequence (the associated linear functional is called Dunkl-classical or T_μ-classical linear functional) if $\{T_\mu P_n\}_{n\geq 1}$ is an orthogonal polynomial sequence, where T_μ is the Dunkl operator [9] : $T_\mu = D + 2\mu H_{-1}$, $\mu > -\frac{1}{2}$, D (resp. H_{-1}) denotes the derivative operator $D = \frac{d}{dx}$ (resp. the Hahn operator given by $(H_{-1}f)(x) = \frac{f(x)-f(-x)}{2x}$).

Y. Ben Cheikh and his coworker [3] introduced the notion of Dunkl-classical orthogonal polynomials and proved that the only symmetric Dunkl-classical orthogonal polynomials are the generalized Hermite polynomials and the generalized Gegenbauer polynomials. Later on, non-symmetric Dunkl-classical orthogonal polynomials have been studied as shown in [6], precisely the authors showed that the unique non-symmetric Dunkl-classical linear form is

$$\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}) = -\frac{2\alpha}{1+2\mu+2\alpha}(x-1)^{-1}\mathcal{G}(\alpha,\mu-\frac{1}{2}) + \delta_1$$

where $n+\alpha \neq 0$, $2\mu+2\alpha+2n+1 \neq 0$, $n \geq 0$ and $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is the generalized Gegenbauer form [1, 2].

Recently, a distributional Rodrigues formula for non-symmetric Dunkl-classical orthogonal polynomial sequences has been established [7], and the previous formula was employed to determine the recurrence coefficients of the second-order recurrence relation that non-symmetric T_μ-classical orthogonal sequences satisfy.

In this paper, we aim to determine the moments and an integral representation for non-symmetric Dunkl-classical form $\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})$.

Our study was initiated by giving some preliminary results needed for the sequel. By $\mathcal{P}$ we denote the vector space of polynomials with coefficients in $\mathbb{C}$, and $\mathcal{P}'$ denotes its

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dual. We call elements of $\mathcal{P}'$ linear forms. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u.

The left-multiplication of a linear form by a polynomial is defined by

$$\langle gu, f\rangle = \langle u, gf\rangle, \qquad f, g \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

The homothetic of a linear form is

$$\langle h_a u, f\rangle = \langle u, h_a f\rangle, \qquad f \in \mathcal{P}, \quad u \in \mathcal{P}', \quad a \in \mathbb{C} \setminus \{0\},$$

where

$$h_a f(x) = f(ax), \qquad f \in \mathcal{P}, \quad a \in \mathbb{C} \setminus \{0\}.$$

The derivative of a linear form u is the linear form Du such that

$$\langle Du, f\rangle = -\langle u, f'\rangle, \qquad f \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

The Dirac mass at the point $c \in \mathbb{C}$, denoted by δ_c, is the form defined by

$$\langle \delta_c, f\rangle = f(c),\ f \in \mathcal{P}.$$

The division of a linear form by a polynomial of first degree is given as

$$(x-c)^{-1}((x-c)u) = u - (u)_0\delta_c,\ u \in \mathcal{P}',\ c \in \mathbb{C}.$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$. The dual sequence associated to $\{P_n\}_{n\geq 0}$ is the sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, defined by

$$\langle u_n, P_m\rangle = \delta_{n,m}, \qquad n,\ m \geq 0,$$

where $\delta_{n,m}$ is the Kroneckers symbol.

A sequence $\{P_n\}_{n\geq 0}$ is called orthogonal (MOPS) if we can associate with it a form u ($(u)_0 = 1$) and a sequence of numbers $\{r_n\}_{n\geq 0}$ ($r_n \neq 0$, $n \geq 0$) such that [8]

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m}, \qquad n,\ m \geq 0.$$

In this case, the form u is called regular or quasi-definite.

Theorem 1.1 (Favard's Theorem [8]). *Let $\{P_n\}_{n\geq 0}$ be a monic polynomial sequence. Then $\{P_n\}_{n\geq 0}$ is orthogonal with respect to a quasi-definite linear form if and only if there exist two sequences of complex numbers $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$, such that $\gamma_n \neq 0$, $n \geq 1$ and it satisfies the three-term recurrence relation*

$$\begin{aligned} P_0(x) = 1, \qquad P_1(x) = x - \beta_0,& \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \qquad n \geq 0.& \end{aligned} \tag{1.1}$$

A form u is said to be symmetric if and only if $(u)_{2n+1} = 0$, $n \geq 0$, or equivalently, in (1.1), $\beta_n = 0$, $n \geq 0$.

Let us introduce the Dunkl's operator [9]:

$$T_\mu f(x) = f'(x) + \mu\frac{f(x) - f(-x)}{x}, \qquad \mu > -\frac{1}{2}, \quad f \in \mathcal{P}. \tag{1.2}$$

By transposition, we define the operator T_μ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows:

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle, \qquad f \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

In particular,

$$(T_\mu u)_n = -\mu_n (u)_{n-1}, \qquad n \geq 0,$$

with the convention $(u)_{-1} = 0$; where

$$\mu_n = n + \mu(1 - (-1)^n), \qquad n \geq 0.$$

Now, consider a MOPS $\{P_n\}_{n\geq 0}$ and let

$$P_n^{[1]}(x,\mu) = \frac{1}{\mu_{n+1}}(T_\mu P_{n+1})(x), \qquad \mu \neq -n - \frac{1}{2}, \quad n \geq 0.$$

Let us denote by $\{u_n^{[1]}\}_{n\geq 0}$, the dual sequence of $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$.

2. Main result

In the section, we describe the moments and an integral representation of $\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})$. For this we need the following results. In the beginning it's important to introduce some properties of the generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geq 0}$. They satisfy the three-term recurrence relation (1.1) in [2]:

$$\begin{aligned}\beta_n &= 0,\\ \gamma_{n+1} &= \frac{(n+1+\delta_n)(n+1+2\alpha+\delta_n)}{4(n+\alpha+\mu+\frac{1}{2})(n+\alpha+\mu+\frac{3}{2})}, \qquad n\geq 0.\\ \delta_n &= \mu(1+(-1)^n),\end{aligned}$$

The sequence $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geq 0}$ is orthogonal with respect to the form $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ satisfying the following integral representation and moments for all $f\in\mathcal{P}$:

$$\langle \mathcal{G}(\alpha,\mu-\frac{1}{2}),\, f\rangle = \frac{\Gamma(\alpha+\mu+\frac{3}{2})}{\Gamma(\alpha+1)\Gamma(\mu+\frac{1}{2})}\int_{-1}^{+1}|x|^{2\mu}(1-x^2)^{\alpha}f(x)\,dx, \qquad Re(\alpha)>-1,$$

$$(\mathcal{G}(\alpha,\mu-\frac{1}{2}))_{2n+1}=0,$$

$$(\mathcal{G}(\alpha,\mu-\frac{1}{2}))_{2n} = \frac{\Gamma(\alpha+\mu+\frac{3}{2})\Gamma(n+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(n+\alpha+\mu+\frac{3}{2})}, \qquad n\geq 0.$$

The following theorem was proved in [4]

Theorem 2.1. *Let $\{P_n\}_{n\geq 0}$ be a MPS orthogonal with respect to a linear form u_0. For $\mu\neq\frac{1}{2}$ and $\mu\neq 0$, the following statements are equivalent:*

(a) *The sequence $\{P_n\}_{n\geq 0}$ is Dunkl-classical.*

(b) *There exist a non-zero complex number K and three (monic) polynomials Φ, B and Ψ with $\deg\Phi\leq 2$, $\deg B\leq 3$ and $\deg\Psi = 1$ such that*

$$\Psi'(0)+\frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2[n]-n)+\frac{KB'''(0)}{3(1-4\mu^2)}\mu([n]-n)\neq 0,$$

and

$$T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big)+\frac{1-4\mu^2}{K}\Psi u_0 = 0,$$

with

$$x\Phi(x)u_0 = h_{-1}(B(x)u_0).$$

In our previous investigation [6], we proved that Theorem 2.1 can be successfully used to classify all Dunkl-classical linear forms. More particularly, it was shown that the unique non-symmetric Dunkl-classical linear form for $\mu\neq\frac{1}{2}$ and $\mu\neq 0$ is, up to a dilatation, the perturbed generalized Gegenbauer linear form $\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})$ satisfying:

$$T_\mu\Big((x^2-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})\Big)-\frac{1+2\mu}{\beta_0}(x-\beta_0)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})=0, \tag{2.3}$$

$$(x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}) = h_{-1}((x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})), \tag{2.4}$$

with the regularity conditions

$$\beta_0\notin\{0,1\}, \qquad 1+2\mu+\beta_0(n-2\mu[n])\neq 0, \quad n\geq 0,$$

since $(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_0=1$, $(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_1=\beta_0$, $(\mathcal{G}(\alpha,\mu-\frac{1}{2}))_0=1$ and the fact that

$$(x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})=-\frac{2\alpha}{1+2\mu+2\alpha}\,\mathcal{G}(\alpha,\mu-\frac{1}{2})$$

Then, we can deduce that $\beta_0=\frac{1+2\mu}{1+2\mu+2\alpha}$. Therefore (2.3) becomes

$$T_\mu\Big((x^2-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})\Big)-\Big((1+2\mu+2\alpha)x+1+2\mu\Big)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})=0, \tag{2.5}$$

2.1. **Moments expression.**

Proposition 2.2. *The moments of the non-symmetric Dunkl-classical form* $\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})$ *are given by*

$$\begin{aligned}&(\widetilde{\mathcal{G}}(\alpha,\mu-\tfrac{1}{2}))_0=1,\\ &(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+1}=(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+2}=\prod_{k=0}^{n}\frac{1+2\mu+2k}{1+2\mu+2\alpha+2k},\quad n\geq 0.\end{aligned} \tag{2.6}$$

Proof. From (2.5), we have

$$\langle T_\mu\Big((x^2-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})\Big)-\Big((1+2\mu+2\alpha)x+1+2\mu\Big)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}),\,x^n\rangle=0,\quad n\geq 0.$$

Using the definition (1.2), we obtain

$$(\mu_n+1+2\mu+2\alpha)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{n+1}-(1+2\mu)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_n-\mu_n(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{n-1}=0,\ n\geq 0.$$

with $(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{-1}=0$.

The changes of the indices $n\to 2n$ and $n\to 2n+1$ in the last equation give respectively

$$\begin{aligned}(2n+2\alpha+2\mu+1)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+1}&-(1+2\mu)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n}\\ &-2n(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n-1}=0,\quad n\geq 0.\end{aligned}$$

and

$$\begin{aligned}(2n+2\alpha+4\mu+2)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+2}&-(1+2\mu)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+1}\\ &-(2n+2\mu+1)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n}=0,\quad n\geq 0.\end{aligned} \tag{2.7}$$

On the other hand, from (2.4), we have

$$\langle (x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})-h_{-1}\Big((x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})\Big),\,x^n\rangle=0,\qquad n\geq 0.$$

Then

$$\Big(1-(-1)^n\Big)\Big((\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{n+1}-(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_n\Big)=0,\qquad n\geq 0.$$

Making the change of indice $n\to 2n+1$ in the last equation, we obtain

$$(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+2}=(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+1},\qquad n\geq 0.$$

Therefore, equation (2.7) becomes

$$(2n+2\alpha+2\mu+1)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n+2}=(2n+2\mu+1)(\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))_{2n},\quad n\geq 0. \tag{2.8}$$

Consequently by induction and using (2.8) we can prove (2.6). □

Corollary 2.3. *The form $\widetilde{\mathcal{G}}(\alpha, \mu - \frac{1}{2})$ satisfies the following relation*

$$(\widetilde{\mathcal{G}}(\alpha, \mu - \frac{1}{2}))_{2n+2} = \frac{\Gamma(\alpha + \mu + \frac{1}{2})\Gamma(n + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})\Gamma(n + \alpha + \mu + \frac{3}{2})}, \qquad n \geq 0,$$

where Γ is the gamma function.

Proof. The result follows immediately from the definition of the gamma function Γ and (2.6). ☐

2.2. **Integral representation.**

Proposition 2.4. *The form $\widetilde{\mathcal{G}}(\alpha, \mu - \frac{1}{2})$ has the following integral representation:*

$$\langle \widetilde{\mathcal{G}}(\alpha, \mu - \tfrac{1}{2}),\, f \rangle = \frac{\Gamma(\alpha + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})\Gamma(\alpha)} \int_{-1}^{+1} |x|^{2\mu} (1 - x^2)^{\alpha - 1}(1 + x) f(x)\, dx, \quad f \in \mathcal{P}, \quad Re(\alpha) > 0.$$

Proof. We look for a function U such that

$$\langle \widetilde{\mathcal{G}}(\alpha, \mu - \frac{1}{2}),\, f \rangle = \int_{-\infty}^{+\infty} U(x) f(x)\, dx,\ f \in \mathcal{P},$$

where we assume that the function U is absolutely continuous on $\mathbb{R}$ and it decays as fast as its derivative U'.

The relation $\langle T_\mu(\Phi\widetilde{\mathcal{G}}(\alpha, \mu - \frac{1}{2})) + \Psi\widetilde{\mathcal{G}}(\alpha, \mu - \frac{1}{2}),\, f(x) \rangle = 0$ implies

$$\int_{-\infty}^{+\infty} U(x)\Big(\Psi(x) f(x) - \Phi(x)(T_\mu f)(x)\Big)\, dx = 0, \qquad f \in \mathcal{P}.$$

Using the definition of T_μ, the last equation becomes

$$\int_{-\infty}^{+\infty} U(x)\Big(\Psi(x) f(x) - \Phi(x) f'(x) - 2\mu\Phi(x)(H_{-1} f)(x)\Big)\, dx = 0, \qquad f \in \mathcal{P}. \tag{2.9}$$

On the one hand, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} U(x)\Big(\Psi(x) f(x) - \Phi(x) f'(x)\Big)\, dx &= \int_{-\infty}^{+\infty} (U\Psi f)(x) - (U\Phi f)'(x) + (U\Phi)'(x) f(x)\, dx \\ &= \int_{-\infty}^{+\infty} \Big(U\Psi + (U\Phi)'\Big)(x) f(x) - (U\Phi f)'(x)\, dx. \end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}\int_{-\infty}^{+\infty} U(x)\Phi(x)(H_{-1} f)(x)\, dx &= \int_{-\infty}^{+\infty} \frac{U(x)\Phi(x)}{2x} f(x)\, dx - \int_{-\infty}^{+\infty} \frac{U(x)\Phi(x)}{2x} f(-x)\, dx \\ &= \int_{-\infty}^{+\infty} \frac{U(x)\Phi(x)}{2x} f(x)\, dx + \int_{-\infty}^{+\infty} \frac{U(-x)\Phi(-x)}{2x} f(x)\, dx \\ &= \int_{-\infty}^{+\infty} \frac{U(x)\Phi(x) + U(-x)\Phi(-x)}{2x} f(x)\, dx, \qquad f \in \mathcal{P}. \end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we get

$$\int_{-\infty}^{+\infty} \Big((U\Psi + (U\Phi)')(x) - \mu\, \frac{U(x)\Phi(x) + U(-x)\Phi(-x)}{x}\Big) f(x)\, dx - [(U\Phi f)]_{-\infty}^{+\infty} = 0, \qquad f \in \mathcal{P}.$$

Hence, from the assumptions on U, the following conditions hold:

$$\left\{\begin{array}{l} \displaystyle\int_{-\infty}^{+\infty}\Big((U\Psi+(U\Phi)')(x)-\mu\,\frac{U(x)\Phi(x)+U(-x)\Phi(-x)}{x}\Big)f(x)\,dx=0, \quad f\in\mathcal{P}, \\ {[(U\Phi f)]}_{-\infty}^{+\infty}=0, \quad f\in\mathcal{P}. \end{array}\right. \tag{2.12}$$

The first relation in (2.12) implies that

$$(U\Phi)'(x)+(U\Psi)(x)-\mu\,\frac{U(x)\Phi(x)+U(-x)\Phi(-x)}{x}=\lambda g(x), \qquad x\neq 0. \tag{2.13}$$

where w is arbitrary and g is a locally integrable function with rapid decay representing the null form

$$\int_{-\infty}^{+\infty}x^n g(x)\,dx=0, \qquad n\geq 0.$$

For example the function g was given by Stieltjes [14]

$$g(x)=\left\{\begin{array}{ll} 0, & x\leq 0, \\ e^{-x^{\frac14}}\sin x^{\frac14}, & x>0. \end{array}\right.$$

Using in (2.13) the fact that

$$\Phi(x)=x^2-1, \qquad \Psi(x)=-(1+2\mu+2\alpha)x-1-2\mu,$$

and taking $\lambda=0$, we get

$$\Big((x^2-1)U(x)\Big)'-\Big((1+2\mu+2\alpha)x-1-2\mu\Big)U(x)-\mu\,\frac{(x^2-1)(U(x)+U(-x))}{x}=0,$$

or equivalently

$$(x^3-x)U'(x)+\Big((1-2\mu-2\alpha)x+1+2\mu\Big)xU(x)-\mu(x^2-1)(U(x)+U(-x))=0. \tag{2.14}$$

On the other hand, from (2.4), we have

$$\langle (x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac12)-h_{-1}\Big((x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac12)\Big),\,f\rangle=0, \qquad f\in\mathcal{P},$$

or equivalently

$$\langle (x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac12),\,f(x)-f(-x)\rangle=0, \qquad f\in\mathcal{P}.$$

The last equation can be written as

$$\int_{-\infty}^{+\infty}(x-1)U(x)\Big(f(x)-f(-x)\Big)\,dx=0, \qquad f\in\mathcal{P}, \tag{2.15}$$

But by the change of variable $x\to -x$, we have

$$\int_{-\infty}^{+\infty}(x-1)U(x)f(-x)\,dx=-\int_{-\infty}^{+\infty}(x+1)U(-x)f(x)\,dx.$$

Then (2.15) becomes

$$\int_{-\infty}^{+\infty}\Big((x-1)U(x)+(x+1)U(-x)\Big)f(x)\,dx=0, \qquad f\in\mathcal{P}.$$

This implies

$$(x-1)U(x)+(x+1)U(-x)=\alpha h(x), \qquad \alpha\in\mathbb{C},$$

where $h\neq 0$ is a locally integrable function and rapid decay representing the null form.

Taking $\alpha=0$, the last equation becomes

$$(x-1)U(x)+(x+1)U(-x)=0.$$

Consequently,

$$(x^2-1)(U(x)+U(-x)) = 2(x-1)U(x). \tag{2.16}$$

Substituting (2.16) into (2.14), we obtain

$$(x^3-x)U'(x) + \Big((1-2\mu-2\alpha)x^2 + x + 2\mu\Big)U(x) = 0,$$

Thus

$$\begin{aligned}\frac{U'(x)}{U(x)} &= \frac{(2\alpha+2\mu-1)x^2 - x - 2\mu}{x(x-1)(x+1)}\\ &= \frac{2\mu(x^2-1)}{x(x-1)(x+1)} + \frac{2\alpha x}{(x-1)(x+1)} - \frac{1}{x-1}\\ &= \frac{2\mu}{x} + \frac{\alpha}{x-1} + \frac{\alpha}{x+1} - \frac{1}{x-1}\\ &= \frac{2\mu}{x} + \frac{\alpha}{x+1} + \frac{\alpha-1}{x-1}.\end{aligned}$$

Consequently,

$$U(x) = \begin{cases} K|x|^{2\mu}(1-x^2)^{\alpha-1}(1+x), & |x|<1,\\ 0, & |x|>1,\end{cases}$$

where K is a constant.

Taking into account the fact that $(u)_0 = 1$ and since $x \mapsto |x|^{2\mu}(1-x^2)^{\alpha-1}x$ is an odd function, we can easily determine K. In fact

$$\begin{aligned}1 = \langle u,\, 1\rangle &= K\int_{-1}^{1} |x|^{2\mu}(1-x^2)^{\alpha-1}(1+x)\,dx\\ &= K\int_{-1}^{1} |x|^{2\mu}(1-x^2)^{\alpha-1}\,dx\\ &= 2K\int_{0}^{1} |x|^{2\mu}(1-x^2)^{\alpha-1}\,dx\\ &= KB(\mu+\tfrac{1}{2},\alpha).\end{aligned}$$

where $B(p,q)$ is the beta function. Thus

$$K = \frac{1}{B(\mu+\frac{1}{2},\alpha)} = \frac{\Gamma(\alpha+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\alpha)}.$$

which completes the proof. □

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