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# PROJECTIONLESS REAL $C^*$-ALGEBRAS

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Abstract. In this paper the projectionless real $C^*$-algebras are investigated. Following construction of [4] a real $C^*$-algebra is constructed, which is separable, simple, nuclear, nonunital, and contains no nonzero projections. It is proved that a real $C^*$-algebra is projectionless if and only if the enveloping $C^*$-algebra is projectionless. An example of a projectionless real Banach $^*$-algebra with the $C^*$-property is constructed, the complexification of which contains a non-trivial projection.

В роботі досліджено безпроекційні дійсні $C^*$-алгебри. Використовуючи результати [4], побудовано дійсну $C^*$-алгебру, яка є сепарабельною, простою, ядерною, неунітальною, і яка не містить ненульових проекторів. Доведено, що дійсна $C^*$-алгебра є безпроекційною тоді і тільки тоді, коли огортуюча $^*$-алгебра є безпроекційною. Побудовано приклад безпроекційної дійсної банахової $^*$-алгебру із властивістю $C^*$, комплексифікіція якої містить нетривіальний проектор.

## 1. Introduction.

A real or complex $C^*$-algebra is said to be *projectionless* if it contains no projections other than $\mathbb{1}$ (if present) and 0. It has long been an open question whether there exists a projectionless simple $C^*$-algebra. The problem of whether simple infinite-dimensional $C^*$-algebras with this property exist was posed in 1958 by Irving Kaplansky and the first example of one was published in 1981 by Bruce Blackadar (see [4, 6, 8]). In the paper [4] a projectionless simple separable nuclear nonunital $C^*$-algebra is constructed and in [10] a sufficient condition for a unital $C^*$-algebra to have no nontrivial projections is given.

In this paper the projectionless real $C^*$-algebras are investigated. It is proved that a real $C^*$-algebra is projectionless if and only if the enveloping $C^*$-algebra is projectionless. Following construction of [4] a real $C^*$-algebra is constructed, which is separable, simple, nuclear, nonunital, and contains no nonzero projections. An example of a projectionless real Banach $^*$-algebra with the $C^*$-property is constructed, the complexification of which contains a non-trivial projection.

## 2. Preliminaries.

Let $A$ be a Banach $^*$-algebra over the field $\mathbb{C}$. The algebra $A$ is called a $C^*$-*algebra* if $\|aa^*\| = \|a\|^2$ for any $a \in A$. A $C^*$-algebra $M$ is called a $W^*$-*algebra* if there exists a Banach space $M_*$, so-called a *predual* of $M$ such that $(M_*)^* = M$. Let $B(H)$ be the algebra of all bounded linear operators acting on a complex Hilbert space $H$ and let $M \subset B(H)$ be a $^*$-subalgebra. The subset $M' = \{a \in B(H) : ba = ab, \forall b \in M\}$ is called the *commutant* of $M$. It is easy to see that $M \subset M'' = M^{IV} = M^{VI} = \dots$ and $M' = M^{III} = M^V = \dots$, where $M'' = (M')'$. If $M = M''$, then it is called a *von Neumann algebra*. By the bicommutant theorem any von Neumann algebra is $W^*$-*algebra*, i.e. $M$ is weakly closed with $\mathbb{1} \in M$. The converse is also true. Therefore, $W^*$-algebras are also called von Neumann algebras.

Now, a real $^*$-subalgebra $R \subset B(H)$ with $\mathbb{1}$ is called a *real* $W^*$-*algebra* if it is weakly closed and $R \cap iR = \{0\}$. The smallest (complex) $W^*$-algebra $M$ containing $R$ coincides

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with its complexication $R+iR$, i.e. $M=R+iR$. It is known that $R$ generates a natural involutive (i.e. of order 2) *-antiautomorphism $\alpha_R$ of $M$, namely $\alpha_R(x+iy)=x^*+iy^*$, where $x+iy\in M$, $x,y\in R$. In this case $R=\{x\in M:\alpha_R(x)=x^*\}$. Conversely, given a $W^*$-algebra $M$ and any involutive *-antiautomorphism $\alpha$ on $M$, the set $(M,\alpha)=\{a\in M:\alpha(a)=a^*\}$ is a real $W^*$-algebra (see [2]).

A real Banach *-algebra $A$ is called a *real $C^*$-algebra* if $\|aa^*\|=\|a\|^2$ and the element $\mathbb{1}+aa^*$ is invertible for any $a\in A$. It is known that $A$ is a real $C^*$-algebra if and only if the norm on $A$ can be extended on the complexification $A^c=A+iA$ of $A$ so that $A^c$ is a $C^*$-algebra (see [11, 5.1.1]).

Let $A$ be a real or complex algebra. A subspace $I$ of an algebra $A$ is called an left ideal (resp. right ideal) if $xy\in I$ for all $x\in A$ and $y\in I$ (resp. for all $x\in I$ and $y\in A$). A left and right ideal is called a two-sided ideal or ideal. An algebra $A$ is said to be *simple* if it contains no non-trivial two-sided ideals and the multiplication operation is not zero (that is, there is some $a$ and some $b$ such that $ab\neq 0$). The second condition in the definition precludes the following situation; consider the algebra with the usual matrix operations:

$$\left\{\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \quad : \quad \alpha\in\mathbb{C}\right\}$$

This is a one-dimensional algebra in which the product of any two elements is zero. This condition ensures that the algebra has a minimal nonzero left ideal, which simplifies certain arguments.

It is easy to see that if the complexification $A+iA$ of a real algebra $A$ is simple, then a real algebra $A$ is also simple. But the converse is not true. We will show this below (see Example 3.6).

Now recall that [12] a real (resp. complex) $C^*$-algebra $A$ is *nuclear* if for all real (resp. complex) $C^*$-algebras $B$ the algebraic tensor product $A\otimes_{\mathbb{R}}B$ (resp. $A\otimes_{\mathbb{C}}B$) has a unique $C^*$-norm. It is known that (see [12, Proposition 2.]) a real $C^*$-algebra $A$ is nuclear if and only if the $C^*$-algebra $A+iA$ is nuclear.

## 3. Main results.

3.1. **A simple real $C^*$-algebra with no nontrivial projections.** Following the construction of the paper [4], we construct a projectionless real $C^*$-algebras as follows and herewith the similar moments we will skip, or briefly summarize.

The algebra $B$ will be the (unique) simple unital AF algebra whose ordered group $K_0(B)$ is isomorphic to the additive group of real algebraic numbers [5, 7]. B has the following properties:

(1) $B$ has a unique normalized trace $\tau$, which is faithful.
(2) If $p$ and $q$ are projections in $B$, then $p\sim q$ if and only if $\tau(p)=\tau(q)$.
(3) If $\lambda$ is a number with $0<\lambda<1$, then there is a projection $p\in B$ with $\tau(p)=\lambda$.
(4) If $p$ is any nonzero projection of $B$, then $pBp\cong B$.

The fact that $B$ satisfies (l)-(4) follows easily from the results of [6] and [7, III.2.11, III.2.12, III.3.4]

Let us define an algebra $A_1=A(\sigma_1)$ as the real $C^*$-algebra of continuous real functions $f:[0,1]\to B$ such that $f(1)=\sigma_1(f(0))$. Since $B$ is a real $C^*$-algebra (even is simple AF $C^*$-algebra), and the norm and the involution on $A_1$ are defined by $\|f\|=\sup_{0\leq t\leq 1}\|f(t)\|$ and $f^*(t)=f(t)^*$, respectively. We have

$$\|ff^*\|=\sup_{0\leq t\leq 1}\|(ff^*)(t)\| \quad = \quad \sup_{0\leq t\leq 1}\|f(t)f(t)^*\|=\sup_{0\leq t\leq 1}\|f(t)\|^2=\|f\|^2,$$

$$(\mathbb{1}_A+ff^*)(t) \quad = \quad \mathbb{1}_A(t)+(ff^*)(t)=\mathbb{1}+f(t)f^*(t),$$

therefore, the invertiblity of $\mathbb{1} + f(t)f^*(t)$ implies the invertiblity of $\mathbb{1}_A + ff^*$. Hence $A_1$ is a real $C^*$-algebra. Then the complexification $A_1^c = A_1 + iA_1$ of $A_1$ is a (complex) $C^*$-algebra. Similarly to [4, Proposition 1.1.] we have

**Proposition 3.1.** *$A(\sigma_1)$ is projectionless if and only if $\sigma_1(\mathbb{1}) \neq \mathbb{1}_1$, where $\mathbb{1}_1 = p_1$ is the unit of $p_1Bp_1$.*

The proof of Proposition 3.1 is carried out similarly to the proof of [4, Proposition 1.1]. Only here, it is necessary to clarify the following point. A trace $\tau$ is extended to the enveloping (complex) $C^*$-algebra $B^c = B + iB$ as $\overline{\tau}(a + ib) = \tau(a)$, since $a + ib \geq 0$ implies $a \geq 0$ (see [2, Corollary 1.1.4]). Since any projector of a real $C^*$-algebra $B$ is automatically a projector in the enveloping $C^*$-algebra $B^c$, according to [9, Lemma 1.8] for projectors $e, h \in B$, the condition $\|e - h\| < 1$ implies $e \sim h$, i.e., $\tau(e) = \tau(h)$.

An isomorphism $\sigma_1$ can be extended by linearity to an isomorphism $\overline{\sigma}_1$ between $B^c$ and $B_1^c = p_1Bp_1 + ip_1Bp_1$ as $\overline{\sigma}_1(x + iy) = \sigma_1(x) + i\sigma_1(y)$, $x, y \in p_1Bp_1$. Then it is directly shown that $A_1^c = A(\overline{\sigma}_1)$ and $A_1^c$ is the $C^*$-algebra of continuous functions $F : [0, 1] \to B^c$ with $F(1) = \overline{\sigma}_1(F(0))$. Moreover $A_1 = (A_1^c, \alpha) = \{x \in A_1^c : \alpha(x) = x^*\}$, where $\alpha$ is an involutive *-antiautomorphism of $A_1^c$, defined by $\alpha(a + ib) = a^* + ib^*$ $(a, b \in A_1)$. And by [4, Proposition 1.1.] we also have the fillowing.

**Proposition 3.2.** *$A(\overline{\sigma}_1)$ is projectionless if and only if $\overline{\sigma}_1(\mathbb{1}) \neq \mathbb{1}_1$.*

It is easily shown that $\overline{\sigma}_1(\mathbb{1}) \neq \mathbb{1}_1 \iff \sigma_1(\mathbb{1}) \neq \mathbb{1}_1$, therefore we have

**Proposition 3.3.** *$A(\sigma_1)$ is projectionless if and only if $A(\overline{\sigma}_1)$ is projectionless.*

Further, sequences of real $C^*$-algebras $A_2, A_3, \ldots$ and of $C^*$-algebras $A_2^c$, $A_3^c$, $\ldots$ are constructed similarly to the scheme for constructing the same sequence from [4]: suppose $A_2, A_3, \ldots, A_n$ have been defined, with $A_n = A(\sigma_n)$, where $\sigma_n$ is an isomorphism of $B$ onto $p_nBp_n$. $p_n \in B$ is a projection with $0 < \lambda_n = \tau(p_n) < 1$. Let $\mu = \lambda_n^{1/2}/(1+\lambda_n^{1/2})$ and let $q, r \in B$ be orthogonal projections with $\tau(q) = \mu$, $\tau(r) = \lambda_n(1 - \mu)$. Put $s = \mathbb{1} - q - r$. Choose a fixed isomorphism $\sigma_{n+1}$ of $B$ onto $(q + r)B(q + r)$ such that $\sigma_{n+1}(q) = r$. This is possible because

$$\tau(\sigma_{n+1}(q)) = [\mu + \lambda_n(1 - \mu)] = \lambda_n(1 - \mu) = \tau(r).$$

Analogously, this isomorphism can be extended by linearity to an isomorphism $\overline{\sigma}_{n+1}$ between $B^c$ and $(q + r)B^c(q + r)$. Set $A_{n+1} = A(\sigma_{n+1})$, $\sigma_{n+1}$ induces isomorphisms $\beta_1 : qBq \to rBr$ and $\beta_2 : (\mathbb{1} - q)B(\mathbb{1} - q) \to qBq$ by restriction. Let $\gamma : B \to qBq$ and $\delta : B \to (\mathbb{1} - q)B(\mathbb{1} - q)$ be arbitrary isomorphisms with $\delta(p) = r$. It is also possible since $\tau(\delta(p)) = \lambda_n(1 - \mu) = \tau(r)$. The linear extensions of these isomorphisms to the complexification of the corresponding algebras are denoted by $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\gamma}$, and $\overline{\delta}$, respectively.

By [4, Theorem 2.3] there is a pointwise-continuous path of automorphisms $\overline{\theta}_t$ $(0 \leq t \leq 1)$ of $qB^cq$ with

$$\overline{\theta}_0 = id \quad \text{and} \quad \overline{\theta}_1 = \overline{\beta}_2 \circ \overline{\delta} \circ \overline{\gamma}^{-1}.$$

In the proof of the theorem, passing from a real $C^*$-algebra to its complexification, for automorphisms $\overline{\theta}_t$ one can obtain the $\tilde{\alpha}$-invariance: $\overline{\theta}_t \circ \tilde{\alpha} = \tilde{\alpha} \circ \overline{\theta}_t$, where $\tilde{\alpha}$ is the involutive *-antiautomorphism of $qB^cq$, generating $qBq$, i.e., $qBq = (qB^cq, \tilde{\alpha})$. Let $w_t \in rB^cr$ $(0 \leq t < 1)$ be a continuous path of unitaries with $w_0 = r$ such that the family of inner *-automorphisms $Ad(w_t) = w_t \,.\, w_t^*$ converges pointwise to $\overline{\beta}_1 \circ \overline{\gamma} \circ \overline{\sigma}_n^{-1} \circ \overline{\delta}^{-1}|_{rB^cr}$ as $t \to 1$. Similarly, here we can also choose as: $Ad(w_t) \circ \hat{\alpha} = \hat{\alpha} \circ Ad(w_t)$, where $\hat{\alpha}$ is the involutive *-antiautomorphism of $rB^cr$, generating $rBr$, i.e., $rBr = (rB^cr, \hat{\alpha})$. Put $u_t = w_t + s$. Then $u_t$ is unitary in $(\mathbb{1} - q)B^c(\mathbb{1} - q)$, for which the inner *-automorphisms $\pi_t = Ad(u_t)$ $(0 \leq t < 1)$ are invariant under the corresponding involutive *-antiautomorphism. Now define $\overline{\phi}_n : A_n^c \to A_{n+1}^c$ as follows:

$$[\overline{\phi}_n(f)](t) = \begin{bmatrix} (\overline{\theta}_t \circ \overline{\gamma})[f(t/2)] & 0 & 0 \\ 0 & (\overline{\pi}_t \circ \overline{\delta})[f((t+1)/2)] & \\ 0 & & \end{bmatrix}, \qquad \text{if} \quad t < 1$$

and

$$[\overline{\phi}_n(f)](0) = \begin{bmatrix} \overline{\gamma}[f(0)] & 0 & 0 \\ 0 & \overline{\delta}[f(1/2)] & \\ 0 & & \end{bmatrix}$$

$$[\overline{\phi}_n(f)](1) = \begin{bmatrix} (\overline{\beta}_2 \circ \overline{\delta})[f(1/2)] & 0 & 0 \\ 0 & (\overline{\beta}_1 \circ \overline{\gamma})[f(0)] & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $B^c$ symbolically represented as $3 \times 3$ matrices:

$$x \sim \begin{bmatrix} qxq & qxr & qxs \\ rxq & rxr & rxs \\ sxq & sxr & sxs \end{bmatrix}$$

Here the entries (23), (32), (33) of the first and the second matrices are missing (i.e., not written), since they are not important and a little cumbersome. For example, the element (33) of the first matrix, for $x = [\overline{\phi}_n(f)](t)$, in fact, has the form

$$xs + (\mathbb{1} - s)x + (\overline{\theta}_t \circ \overline{\gamma})[f(t/2)] + (\overline{\pi}_t \circ \overline{\delta})[f((t+1)/2)].$$

According to the third matrix we have $\overline{\sigma}_{n+1}([\overline{\phi}_n(f)](0)) = [\overline{\phi}_n(f)](\mathbb{1})$. Since all automorphisms and isomorphisms are invariant under the corresponding involutive $*$-anti-automorphisms, their restrictions to the corresponding real part, denoted by symbols without a dash (without a wave, without a cap), we obtain the following:

$$\sigma_{n+1} : B \to (q+r)B(q+r), \ \phi_n : A_n \to A_{n+1} \text{ and } \sigma_{n+1}([\phi_n(f)](0)) = [\phi_n(f)](\mathbb{1}).$$

Further, following [4], we put $p_{n+1} = q + r$, $\lambda_{n+1} = \mu + \lambda_n(1 - \mu)$. Now let the algebra $A$ (respectively, $A^c$) be constructed as an inductive limit of real (respectively, complex) $C^*$-algebras,

$$A = \varinjlim\{A_n, \phi_n\} \quad \text{and} \quad A^c = \varinjlim\{A_n^c, \overline{\phi}_n\}.$$

By construction, we also have $A + iA = A^c$. By [4, Lemma 3.2], the $C^*$-algebra $A^c$ is simple, by [4, Proposition 3.3 and Corollary 3.4], $A^c$ is a projectionless $C^*$-algebra, and by [4, Proposition 3.5], $A^c$ is nuclear. Since simplicity of the algebra $A + iA = A^c$ implies simplicity of $A$, the real $C^*$-algebra $A$ is simple. Since $A^c$ is projectionless, $A$ is also projectionless. By [12, Proposition 2.], $A$ is also nuclear.

Thus, following the scheme of the paper [4], we have constructed a real $C^*$-algebra $A$ which is separable, simple, nuclear and contains no nonzero projections. Moreover, its enveloping $C^*$-algebra, $A + iA$, is also separable, simple, nuclear and contains no nonzero projections.

3.2. **A connection between a projectionless real $C^*$-algebra and an enveloping $C^*$-algebra.** Obviously, every projection of a real $C^*$-algebra is a projection of an enveloping $C^*$-algebra, but the converse is not true. The set of all projections of (complex) algebras is larger than the set of all projections of real subalgebras. Therefore, if a (complex) algebra contains no nonzero projections, then any its subalgebra also does not have nonzero projections. Hence and in connection with the previous example, a natural question arises: *if a real $C^*$-algebra $A$ is projectionless, is then the complexification $A+iA$ of $A$ also projectionless?*

Despite the fact that the set of all projections of $A + iA$ is larger than the set of all projections of $A$, the answer to the question is positive. Namely, the following result holds.

**Theorem 3.4.** *A real $C^*$-algebra $A$ is projectionless if and only if its enveloping $C^*$-algebra $A + iA = A^c$ (i.e. its complexification) is projectionless.*

*Proof.* Sufficiency is obvious. Let us show the necessity. Let $\alpha$ be an involutive $*$-anti-automorphism of $A^c$, generating $A$, i.e., $A = (A^c, \alpha)$ (see [11, Proposition 5.1.3.] and [1]). Let $e = a + ib$ be a nonzero projection of $A^c$, where $a, b \in A$. Since

$$e = \frac{1}{2}(e + \alpha(e)) + i \cdot \frac{e - \alpha(e)}{2i}$$

for $f = e + \alpha(e)$ and $q = \dfrac{e - \alpha(e)}{2i}$ we have $a = \frac{1}{2}f,\ \ b = q \in A$ and $f^* = f,\ q^* = -q$. Put $p = e \wedge \alpha(e)$. Since $f, f^2 \in A$, we have $p \in A$. If $p \neq 0$, then the real $C^*$-algebra $A$ contains a nonzero projection and if $p = 0$, then $e \perp \alpha(e)$, i.e., $e\alpha(e) = 0$. Then $f = e + \alpha(e)$ is a nonzero projection of $A$. □

**Remark 3.5.** A slightly more general notion of real $C^*$-algebras was given also by Berberian [3, p. 26, Exercise 14A]. We define a *real $C^*$-algebra* (in the sense of Berberian) as a Banach $^*$-algebra over the field of real numbers such that $\|x^*x\| = \|x\|^2$ for all $x \in A$, i.e., here the condition of invertibility of $\mathbb{1} + x^*x$ (for any $x \in A$) is not required. In this case, the theorem 3.4 is not true, i.e., the condition of invertibility of $\mathbb{1} + x^*x$ ($\forall x \in A$) is essential as the following example shows.

**Example 3.6.** Let $A = \mathbb{C}$ be the field of complex numbers. Then $A$ with the identical involution $z^* = z$ becomes a real Banach $^*$-algebra. Since $\|zz^*\|^2 = |z^2|^2 = (a^2+b^2)^2 = |z|^4$ (where $\forall z = a + ib \in \mathbb{C}$), we have $\|zz^*\| = \|z\|^2$. Moreover, for $z = i$ the element $\mathbb{1} + zz^* = 1 + i^2 = 0$ is not invertible. Thus $A$ is not a real C$^*$-algebra, because it is not a *symmetric* $^*$-algebra, which means that $\mathbb{1} + x^*x$ is invertible for any $x \in A$. But $A$ is a real $C^*$-algebra in the sense of Berberian. And also $A$ contains no nonzero projections and it is simple.

Now we consider the complexification $A + iA = \mathbb{C} + i\mathbb{C}$ of $A = \mathbb{C}$. In order to give an explicit form of the complexification, note that we can not put formally $M = A + iA = \{x+iy : x, y \in A\}$, because $A \cap iA = \mathbb{C} \neq \{0\}$. Therefore let us consider the representation of $A$ in the form

$$A \cong A_0 := \{(\lambda, \overline{\lambda}) \ : \ \lambda \in \mathbb{C}\},$$

where $\cong$ means a real isometric $^*$-isomorphism. Now it is clear that $A_0 \cap iA_0 = \{0\}$ and therefore

$$A + iA \cong A_0 + iA_0 = \{(\lambda, \overline{\lambda}) + i(\mu, \overline{\mu}) \ : \ \lambda, \mu \in A\} = \{(\lambda + i\mu, \overline{\lambda} + i\overline{\mu}) \ : \ \lambda, \mu \in A\}.$$

Let us show that the algebra $M \cong A_0 + iA_0$ has nontrivial projections. For this, we describe projections of $M$. Let $p = (\lambda + i\mu, \overline{\lambda} + i\overline{\mu})$ be a projection, i.e., $p^* = p = p^2$. From the last equalities we get

$$\lambda^2 - \mu^2 = \lambda \qquad \text{and} \qquad 2\lambda\mu = \mu.$$

If $\mu = 0$ we get $\lambda = 0$ or $\lambda = 1$ and if $\mu \neq 0$ we get $\lambda = 1/2$, $\mu = \pm i/2$. Then

$$p = \left(\frac{1}{2} + i\frac{\pm i}{2}, \frac{1}{2} - i\frac{\pm i}{2}\right) = \left(\frac{1}{2} \mp \frac{1}{2}, \frac{1}{2} \pm \frac{1}{2}\right) = (0,1) \text{ or } (1,0).$$

Since $p_1 = (0, 1)$ and $p_2 = (1, 0)$ are non-trivial projections of $M$, the (complex) $C^*$-algebra $M = A + iA$ has non-trivial projections, therefore, it is not a projectionless $C^*$-algebra. Moreover, the $C^*$-algebra

$$M = A + iA \cong A_0 + iA_0 = \{a + ib : \ | \ a = (\lambda, \overline{\lambda}), b = (\mu, \overline{\mu}) \in A_0\}$$

is not simple, because $M$ has nonzero proper ideals $\{(0,b) : b \in A_0\}$ and $\{(a,0) : a \in A_0\}$. Thus $A$ is a projectionless simple real $C^*$-algebra (in the sense of Berberian), for which the enveloping $C^*$-algebra $A + iA$ is not simple and has non-trivial projections.

**Remark 3.7.** Unlike the $C^*$-algebras, the real and complex $W^*$-algebras always have non-trivial projections. Moreover, the set of all projections of real and complex $W^*$-algebras is rich enough.

Indeed, let $M \subset B(H)$ be a $W^*$-algebra. In the finite-dimensional case, the $C^*$- and $W^*$- algebras coincide, and in this case, all projectors are described, i.e., the explicit form of the projectors is known. Therefore, suppose that $M$ is infinite-dimensional. We take an arbitrary nonzero vector $\xi \in H$ and consider the projection map $e_\xi : H \to \overline{M'\xi}$. Let us show that the mapping $e_\xi$ is a projection of $M$, i.e., $e_\xi \in M$. Let $\forall x, x' \in M'$ and $\forall \eta, \gamma \in H$. Let $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1 \in \overline{M'\xi}$ and $\gamma_2 \in \left(\overline{M'\xi}\right)^\perp$. Since $(x'(\xi), x^*(\gamma_2)) = (xx'(\xi), \gamma_2) = 0$, we have $x^*\gamma_2 \in \left(\overline{M'\xi}\right)^\perp$, therefore $e_\xi(x^*(\gamma_2)) = 0$. Then we will get

$$\begin{aligned}
(e_\xi x(\eta), \gamma) &= (e_\xi x(\eta), \gamma_1) + (e_\xi x(\eta), \gamma_2) = (x(\eta), e_\xi(\gamma_1)) + (x(\eta), e_\xi(\gamma_2)) \\
&= (x(\eta), e_\xi(\gamma_1)) = (x(\eta), \gamma_1) = (\eta, x^*(\gamma_1)) = (\eta, e_\xi(x^*(\gamma_1))) \\
&= (\eta, e_\xi(x^*(\gamma_1))) + (\eta, e_\xi(x^*(\gamma_2))) = (\eta, e_\xi(x^*(\gamma_1 + \gamma_2))) \\
&= (\eta, e_\xi(x^*(\gamma))) = (e_\xi(\eta), x^*(\gamma)) = (xe_\xi(\eta), \gamma),
\end{aligned}$$

hence $e_\xi x = xe_\xi$. Then $e_\xi \in M''$. As mentioned above (see: Preliminaries), by the bicommutant theorem we have $M'' = M$. Hence we get $e_\xi \in M$. Recall that the bicommutant theorem is also true for real $W^*$-algebras (see [11, Theorem 4.3.8]). □

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