PROJECTIONLESS REAL $C^\ast$-ALGEBRAS

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ABSTRACT. In this paper the projectionless real $C^\ast$-algebras are investigated. Following construction of [4] a real $C^\ast$-algebra is constructed, which is separable, simple, nuclear, nonunital, and contains no nonzero projections. It is proved that a real $C^\ast$-algebra is projectionless if and only if the enveloping $C^\ast$-algebra is projectionless. An example of a projectionless real Banach $*$-algebra with the $C^\ast$-property is constructed, the complexification of which contains a non-trivial projection.

A real or complex $C^\ast$-algebra is said to be projectionless if it contains no projections other than $I$ (if present) and 0. It has long been an open question whether there exists a projectionless simple $C^\ast$-algebra. The problem of whether simple infinite-dimensional $C^\ast$-algebras with this property exist was posed in 1958 by Irving Kaplansky and the first example of one was published in 1981 by Bruce Blackadar (see [4, 6, 8]). In the paper [4] a sufficient condition for a unital $C^\ast$-algebra to have no nontrivial projections is given.

In this paper the projectionless real $C^\ast$-algebras are investigated. It is proved that a real $C^\ast$-algebra is projectionless if and only if the enveloping $C^\ast$-algebra is projectionless. Following construction of [4] a real $C^\ast$-algebra is constructed, which is separable, simple, nuclear, nonunital, and contains no nonzero projections. An example of a projectionless real Banach $*$-algebra with the $C^\ast$-property is constructed, the complexification of which contains a non-trivial projection.

1. INTRODUCTION.

Let $A$ be a Banach $*$-algebra over the field $\mathbb{C}$. The algebra $A$ is called a $C^\ast$-algebra if $\|aa^\ast\| = \|a\|^2$ for any $a \in A$. A $C^\ast$-algebra $M$ is called a $W^\ast$-algebra if there exists a Banach space $M_*$, so-called a predual of $M$ such that $(M_*)^\ast = M$. Let $B(H)$ be the algebra of all bounded linear operators acting on a complex Hilbert space $H$ and let $M \subset B(H)$ be a $*$-subalgebra. The subset $M' = \{a \in B(H) : ba = ab, \forall b \in M\}$ is called the commutant of $M$. It is easy to see that $M \subset M'' = M''' = M^V = \ldots$ and $M' = M''$ = $M''' = \ldots$, where $M'' = (M')'$. If $M = M''$, then it is called a von Neumann algebra. By the bicommutant theorem any von Neumann algebra is $W^\ast$-algebra, i.e. $M$ is weakly closed with $I \in M$. The converse is also true. Therefore, $W^\ast$-algebras are also called von Neumann algebras.

Now, a real $*$-subalgebra $R \subset B(H)$ with $I$ is called a real $W^\ast$-algebra if it is weakly closed and $R \cap iR = \{0\}$. The smallest (complex) $W^\ast$-algebra $M$ containing $R$ coincides...
with its complexification $R + iR$, i.e. $M = R + iR$. It is known that $R$ generates a natural involutive (i.e. of order 2) *-antiautomorphism $\alpha_R$ of $M$, namely $\alpha_R(x + iy) = x^* + iy^*$, where $x + iy \in M$, $x, y \in R$. In this case $R = \{x \in M : \alpha_R(x) = x^*\}$. Conversely, given a $W^*$-algebra $M$ and any involutive *-antiautomorphism $\alpha$ on $M$, the set $(M, \alpha) = \{a \in M : \alpha(a) = a^*\}$ is a real $W^*$-algebra (see [2]).

A real Banach *-algebra $A$ is called a real $C^*$-algebra if $\|aa^*\| = \|a\|^2$ and the element $1 + a a^*$ is invertible for any $a \in A$. It is known that $A$ is a real $C^*$-algebra if and only if the norm on $A$ can be extended on the complexification $A^c = A + iA$ of $A$ so that $A^c$ is a $C^*$-algebra (see [11, 5.1.1]).

Let $A$ be a real or complex algebra. A subspace $I$ of an algebra $A$ is called an left ideal (resp. right ideal) if $xy \in I$ for all $x \in A$ and $y \in I$ (resp. for all $x \in I$ and $y \in A$). A left and right ideal is called a two-sided ideal or ideal. An algebra $A$ is said to be simple if it contains no non-trivial two-sided ideals and the multiplication operation is not zero (that is, there is some a and some b such that $ab \neq 0$). The second condition in the definition precludes the following situation; consider the algebra with the usual matrix operations:

$$\left\{ \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

This is a one-dimensional algebra in which the product of any two elements is zero. This condition ensures that the algebra has a minimal nonzero left ideal, which simplifies certain arguments.

It is easy to see that if the complexification $A + iA$ of a real algebra $A$ is simple, then a real algebra $A$ is also simple. But the converse is not true. We will show this below (see Example 3.6).

Now recall that [12] a real (resp. complex) $C^*$-algebra $A$ is nuclear if for all real (resp. complex) $C^*$-algebras $B$ the algebraic tensor product $A \otimes_B \mathbb{B}$ (resp. $A \otimes_{\mathbb{B}} \mathbb{B}$) has a unique $C^*$-norm. It is known that (see [12, Proposition 2.1]) a real $C^*$-algebra $A$ is nuclear if and only if the $C^*$-algebra $A + iA$ is nuclear.

3. Main results.

3.1. A simple real $C^*$-algebra with no nontrivial projections. Following the construction of the paper [4], we construct a projectionless real $C^*$-algebras as follows and herewith the similar moments we will skip, or briefly summarize.

The algebra $B$ will be the (unique) simple unital AF algebra whose ordered group $K_0(B)$ is isomorphic to the additive group of real algebraic numbers [5, 7]. $B$ has the following properties:

1. $B$ has a unique normalized trace $\tau$, which is faithful.
2. If $p$ and $q$ are projections in $B$, then $p \sim q$ if and only if $\tau(p) = \tau(q)$.
3. If $\lambda$ is a number with $0 < \lambda < 1$, then there is a projection $p \in B$ with $\tau(p) = \lambda$.
4. If $p$ is any nonzero projection of $B$, then $pBp \cong B$.

The fact that $B$ satisfies (l)-(4) follows easily from the results of [6] and [7, III.2.11, III.2.12, III.3.4]

Let us define an algebra $A_1 = A(\sigma_1)$ as the real $C^*$-algebra of continuous real functions $f : [0,1] \rightarrow B$ such that $f(1) = \sigma_1(f(0))$. Since $B$ is a real $C^*$-algebra (even is simple AF $C^*$-algebra), and the norm and the involution on $A_1$ are defined by $\|f\| = \sup_{0 \leq t \leq 1} \|f(t)\|$ and $f^*(t) = f(t)^*$, respectively. We have

$$\|ff^*\| = \sup_{0 \leq t \leq 1} \|(ff^*)(t)\| = \sup_{0 \leq t \leq 1} \|f(t)f(t)^*\| = \sup_{0 \leq t \leq 1} \|f(t)\|^2 = \|f\|^2,$$

$$(1_A + ff^*)(t) = 1_A(t) + (ff^*)(t) = 1_A(t) + f(t)f^*(t),$$

where $1_A(t)$ is the characteristic function of $A$.
therefore, the invertibility of $1 + f(t)f^*(t)$ implies the invertibility of $1 + f$, $f$. Hence $A_1$ is a real $C^*$-algebra. Then the complexification $A_1^c = A_1 + iA_1$ of $A_1$ is a (complex) $C^*$-algebra. Similarly to [4, Proposition 1.1.] we have

**Proposition 3.1.** $A(\sigma_1)$ is projectionless if and only if $\sigma_1(1) \neq 1$, where $1 = p_1$ is the unit of $p_1Bp_1$.

The proof of Proposition 3.1 is carried out similarly to the proof of [4, Proposition 1.1]. Only here, it is necessary to clarify the following point. A trace $\tau$ is extended to the enveloping (complex) $C^*$-algebra $B^c = B + iB$ as $\tau(a + ib) = \tau(a)$, since $a + ib \geq 0$ implies $a \geq 0$ (see [2, Corollary 1.1.4]). Since any projector of a real $C^*$-algebra $B$ is automatically a projector in the enveloping $C^*$-algebra $B^c$, according to [9, Lemma 1.8] for projectors $e, h \in B$, the condition $\|e - h\| < 1$ implies $e \sim h$, i.e., $\tau(e) = \tau(h)$.

An isomorphism $\sigma_1$ can be extended by linearity to an isomorphism $\sigma_1$ between $B^c$ and $B^c = p_1Bp_1 + ip_1Bp_1$ as $\sigma_1(x + iy) = \sigma_1(x) + i\sigma_1(y)$, $x, y \in p_1Bp_1$. Then it is directly shown that $A_1^c = A(\sigma_1)$ and $A_1^c$ is the $C^*$-algebra of continuous functions $F : [0,1] \to B^c$ with $F(1) = \sigma_1(F(0))$. Moreover $A_1 = (A_1^c, \alpha) = \{ x \in A_1^c : \alpha(x) = x^* \}$, where $\alpha$ is an involutive $*$-antiautomorphism of $A_1^c$, defined by $\alpha(a + ib) = a^* + ib^*$ $(a, b \in A_1)$. And by [4, Proposition 1.1] we also have the following.

**Proposition 3.2.** $A(\sigma_1)$ is projectionless if and only if $\sigma_1(1) \neq 1$.

It is easily shown that $\sigma_1(1) \neq 1 \iff \sigma_1(1) \neq 1, \text{ therefore we have}$

**Proposition 3.3.** $A(\sigma_1)$ is projectionless if and only if $A(\sigma_1)$ is projectionless.

Further, sequences of real $C^*$-algebras $A_2, A_3, \ldots$ and of $C^*$-algebras $A_2, A_3, \ldots$ are constructed similarly to the scheme for constructing the same sequence from [4]: suppose $A_2, A_3, \ldots, A_n$ have been defined, with $A_n = A(\sigma_n)$, where $\sigma_n$ is an isomorphism of $B$ onto $p_nBp_n$, $p_n \in B$ is a projection with $0 < \lambda_n = \tau(p_n) < 1$. Let $\mu = \lambda_n^{1/2}/(1 + \lambda_n^{1/2})$ and let $q, r \in B$ be orthogonal projections with $\tau(q) = \mu, \tau(r) = \lambda_n(1 - \mu)$. Put $s = 1 - q - r$. Choose a fixed isomorphism $\sigma_{n+1}$ of $B$ onto $(q + r)B(q + r)$ such that $\sigma_{n+1}(q) = r$. This is possible because

$$\tau(\sigma_{n+1}(q)) = [\mu + \lambda_n(1 - \mu)] = \lambda_n(1 - \mu) = \tau(r).$$

Analogously, this isomorphism can be extended by linearity to an isomorphism $\sigma_{n+1}$ between $B^c$ and $(q + r)B^c(q + r)$. Set $A_{n+1} = A(\sigma_{n+1})$, $\sigma_{n+1}$ induces isomorphisms $\beta_1 : qBq \to rBr$ and $\beta_2 : (1 - q)B(1 - q) \to qBq$ by restriction. Let $\gamma : B \to qBq$ and $\delta : B \to (1 - q)B(1 - q)$ be arbitrary isomorphisms with $\delta(p) = r$. It is also possible since $\tau(\delta(p)) = \lambda_n(1 - \mu) = \tau(r)$. The linear extensions of these isomorphisms to the complexification of the corresponding algebras are denoted by $\overline{\beta}_1, \overline{\beta}_2, \overline{\gamma}, \overline{\delta}$, respectively.

By [4, Theorem 2.3] there is a pointwise-continuous path of automorphisms $\overline{\beta}_t \in [0 \leq t \leq 1]$ of $qB^*q$ with $\overline{\beta}_0 = id$ and $\overline{\beta}_1 = \overline{\beta}_2 \circ \overline{\delta} \circ \overline{\gamma}^{-1}$.

In the proof of the theorem, passing from a real $C^*$-algebra to its complexification, for automorphisms $\overline{\beta}_t$ one can obtain the $\dot{\alpha}$-invariance: $\overline{\beta}_t \circ \dot{\alpha} = \dot{\alpha} \circ \overline{\beta}_t$, where $\dot{\alpha}$ is the involutive $*$-antiautomorphism of $qB^*q$, generating $qBq$, i.e., $qBq = (qB^*q, \dot{\alpha})$. Let $w_t \in rB^*r$ ($0 \leq t < 1$) be a continuous path of unitaries with $w_0 = r$ such that the family of inner $*$-automorphisms $Ad(w_t) = w_t$, $w_t$ converges pointwise to $\overline{\beta}_1 \circ \overline{\gamma} \circ \overline{\beta}_n^{-1} \circ \overline{\delta}^{-1} |_{rB^*r}$ as $t \to 1$. Similarly, here we can also choose as: $Ad(w_t) \circ \dot{\alpha} = \dot{\alpha} \circ Ad(w_t)$, where $\dot{\alpha}$ is the involutive $*$-antiautomorphism of $rB^*r$, generating $rBr$, i.e., $rBr = (rB^*r, \dot{\alpha})$. Put $u_t = w_t + s$.

Then $u_t$ is unitary in $(1 - q)B^*(1 - q)$, for which the inner $*$-automorphisms $\pi_t = Ad(u_t)$ ($0 \leq t < 1$) are invariant under the corresponding involutive $*$-antiautomorphism. Now define $\overline{\phi}_n : A_{n}^c \to A_{n+1}^c$ as follows:
\[
(\bar{\sigma}_n(f))(t) = \begin{bmatrix}
(\bar{\pi}_t \circ \bar{\tau})[f(t/2)] & 0 & 0 \\
0 & (\pi_t \circ \bar{\delta})[f((t+1)/2)] & 0 \\
0 & 0 & \tau[f(t/2)]
\end{bmatrix}, \quad \text{if } t < 1
\]

and
\[
(\bar{\sigma}_n(f))(0) = \begin{bmatrix}
\tau[f(0)] & 0 & 0 \\
0 & \bar{\delta}[f(1/2)] & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
(\bar{\sigma}_n(f))(1) = \begin{bmatrix}
(\bar{\pi}_2 \circ \bar{\delta})[f(1/2)] & 0 & 0 \\
0 & (\bar{\pi}_1 \circ \tau)[f(0)] & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
with \(B^c\) symbolically represented as 3 \(\times\) 3 matrices:
\[
x \sim \begin{bmatrix}
qxq & qx\tau & qx\bar{s} \\
rxq & rx\tau & rx\bar{s} \\
sxq & sx\tau & sx\bar{s}
\end{bmatrix}
\]

Here the entries (23), (32), (33) of the first and the second matrices are missing (i.e., not written), since they are not important and a little cumbersome. For example, the element (33) of the first matrix, for \(x = (\bar{\sigma}_n(f))(t)\), in fact, has the form
\[
xr + (1-s)x + (\bar{\pi}_t \circ \bar{\tau})[f(t/2)] + (\pi_t \circ \bar{\delta})[f((t+1)/2)].
\]

According to the third matrix we have \(\bar{\sigma}_{n+1}(|\phi_n(f)|)(0)) = |\bar{\sigma}_n(f)|(\mathbb{I})\). Since all automorphisms and isomorphisms are invariant under the corresponding involutive *-anti-automorphisms, their restrictions to the corresponding real part, denoted by symbols without a dash (without a wave, without a cap), we obtain the following:
\[
\sigma_{n+1} : B \rightarrow (q+r)B(q+r), \ \phi_n : A_n \rightarrow A_{n+1} \text{ and } \sigma_{n+1}(|\phi_n(f)|)(0)) = |\phi_n(f)|(\mathbb{I}).
\]

Further, following [4], we put \(p_{n+1} = q+r, \ \lambda_{n+1} = \mu + \lambda_n(1 - \mu)\). Now let the algebra \(A\) (respectively, \(A^c\)) be constructed as an inductive limit of real (respectively, complex) \(C^*\)-algebras,
\[
A = \lim\{A_n, \phi_n\} \quad \text{and} \quad A^c = \lim\{A_{nc}, \bar{\sigma}_n\}.
\]

By construction, we also have \(A + iA = A^c\). By [4, Lemma 3.2], the \(C^*\)-algebra \(A^c\) is simple, by [4, Proposition 3.3 and Corollary 3.4], \(A^c\) is a projectionless \(C^*\)-algebra, and by [4, Proposition 3.5], \(A^c\) is nuclear. Since simplicity of the algebra \(A + iA = A^c\) implies simplicity of \(A\), the real \(C^*\)-algebra \(A\) is simple. Since \(A^c\) is projectionless, \(A\) is also projectionless. By [12, Proposition 2.1], \(A\) is also nuclear.

Thus, following the scheme of the paper [4], we have constructed a real \(C^*\)-algebra \(A\) which is separable, simple, nuclear and contains no nonzero projections. Moreover, its enveloping \(C^*\)-algebra, \(A + iA\), is also separable, simple, nuclear and contains no nonzero projections.

3.2. A connection between a projectionless real \(C^*\)-algebra and an enveloping \(C^*\)-algebra. Obviously, every projection of a real \(C^*\)-algebra is a projection of an enveloping \(C^*\)-algebra, but the converse is not true. The set of all projections of (complex) algebras is larger than the set of all projections of real subalgebras. Therefore, if a (complex) algebra contains no nonzero projections, then any its subalgebra also does not have nonzero projections. Hence and in connection with the previous example, a natural question arises: if a real \(C^*\)-algebra \(A\) is projectionless, is then the complexification \(A + iA\) of \(A\) also projectionless?

Despite the fact that the set of all projections of \(A + iA\) is larger than the set of all projections of \(A\), the answer to the question is positive. Namely, the following result holds.
Theorem 3.4. A real $C^*$-algebra $A$ is projectionless if and only if its enveloping $C^*$-algebra $A + iA = A^c$ (i.e. its complexification) is projectionless.

Proof. Sufficiency is obvious. Let us show the necessity. Let $\alpha$ be an involutive *-antiautomorphism of $A^c$, generating $A$, i.e., $A = (A^c, \alpha)$ (see [11, Proposition 5.1.3.] and [1]). Let $e = a + ib$ be a nonzero projection of $A^c$, where $a, b \in A$. Since

$$e = \frac{1}{2}(e + \alpha(e)) + i \cdot \frac{e - \alpha(e)}{2i}$$

for $f = e + \alpha(e)$ and $q = \frac{e - \alpha(e)}{2i}$ we have $a = \frac{1}{2}f$, $b = q \in A$ and $f^* = f$, $q^* = -q$. Put $p = e \wedge \alpha(e)$. Since $f, f^2 \in A$, we have $p \in A$. If $p \neq 0$, then the real $C^*$-algebra $A$ contains a nonzero projection and if $p = 0$, then $e \perp \alpha(e)$, i.e., $eo(e) = 0$. Then $f = e + \alpha(e)$ is a nonzero projection of $A$. \hfill \Box

Remark 3.5. A slightly more general notion of real $C^*$-algebras was given also by Berberian [3, p. 26, Exercise 14A]. We define a real $C^*$-algebra (in the sense of Berberian) as a Banach *-algebra over the field of real numbers such that $\|x^*x\| = \|x\|^2$ for all $x \in A$, i.e., the condition of invertibility of $1 + x^*x$ (for any $x \in A$) is not required. In this case, the theorem 3.4 is not true, i.e., the condition of invertibility of $1 + x^*x$ ($\forall x \in A$) is essential as the following example shows.

Example 3.6. Let $A = \mathbb{C}$ be the field of complex numbers. Then $A$ with the identical involution $z^* = z$ becomes a real Banach *-algebra. Since $\|zz^*\|^2 = |z|^2 = (a^2 + b^2)^2 = |z|^4$ (where $\forall z = a + ib \in \mathbb{C}$), we have $\|zz^*\| = |z|^2$. Moreover, for $z = i$ the element $1 + zz^* = 1 + i^2 = 0$ is not invertible. Thus $A$ is not a real $C^*$-algebra, because it is not a symmetric *-algebra, which means that $1 + x^*x$ is invertible for any $x \in A$. But $A$ is a real $C^*$-algebra in the sense of Berberian. And also $A$ contains no nonzero projections and it is simple.

Now we consider the complexification $A + iA = \mathbb{C} + i\mathbb{C}$ of $A = \mathbb{C}$. In order to give an explicit form of the complexification, note that we can not put formally $M = A + iA = \{x + iy : x, y \in A\}$, because $A \cap iA = \mathbb{C} \neq \{0\}$. Therefore let us consider the representation of $A$ in the form

$$A \cong A_0 := \{ (\lambda, \bar{\lambda}) : \lambda \in \mathbb{C} \},$$

where $\cong$ means a real isometric *-isomorphism. Now it is clear that $A_0 \cap iA_0 = \{0\}$ and therefore

$$A + iA \cong A_0 + iA_0 = \{(\lambda, \bar{\lambda}) + i(\mu, \bar{\mu}) : \lambda, \mu \in A\} = \{ (\lambda + i\mu, \bar{\lambda} + i\bar{\mu}) : \lambda, \mu \in A \}.$$

Let us show that the algebra $M \cong A_0 + iA_0$ has nontrivial projections. For this, we describe projections of $M$. Let $p = (\lambda + i\mu, \bar{\lambda} + i\bar{\mu})$ be a projection, i.e., $p^* = p = p^2$. From the last equalities we get

$$\lambda^2 - \mu^2 = \lambda \quad \text{and} \quad 2\lambda\mu = \mu.$$

If $\mu = 0$ we get $\lambda = 0$ or $\lambda = 1$ and if $\mu \neq 0$ we get $\lambda = 1/2, \mu = \pm i/2$. Then

$$p = \left( \frac{1}{2} + i \frac{\pm i}{2}, 2 - i \frac{\pm 1}{2} \right) = \left( \frac{1}{2} \pm \frac{1}{2}, 2 \pm \frac{1}{2} \right) = (0, 1) \text{ or } (1, 0).$$

Since $p_1 = (0, 1)$ and $p_2 = (1, 0)$ are non-trivial projections of $M$, the (complex) $C^*$-algebra $M = A + iA$ has non-trivial projections, therefore, it is not a projectionless $C^*$-algebra. Moreover, the $C^*$-algebra

$$M = A + iA \cong A_0 + iA_0 = \{ a + ib : | a = (\lambda, \bar{\lambda}), b = (\mu, \bar{\mu}) \in A_0 \}.$$
is not simple, because $M$ has nonzero proper ideals $\{(0, b): b \in A_0\}$ and $\{(a, 0): a \in A_0\}$. Thus $A$ is a projectionless simple real $C^*$-algebra (in the sense of Berberian), for which the enveloping $C^*$-algebra $A + iA$ is not simple and has non-trivial projections.

**Remark 3.7.** Unlike the $C^*$-algebras, the real and complex $W^*$-algebras always have non-trivial projections. Moreover, the set of all projections of real and complex $W^*$-algebras is rich enough.

Indeed, let $M \subset B(H)$ be a $W^*$-algebra. In the finite-dimensional case, the $C^*$- and $W^*$-algebras coincide, and in this case, all projectors are described, i.e., the explicit form of the projectors is known. Therefore, suppose that $M$ is infinite-dimensional. We take an arbitrary nonzero vector $\xi \in H$ and consider the projection map $e_\xi : H \rightarrow M\xi$. Let us show that the mapping $e_\xi$ is a projection of $M$, i.e., $e_\xi \in M$. Let $x, x' \in M'$ and $\forall \eta, \gamma \in H$. Let $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1 \in M\xi$ and $\gamma_2 \in (M\xi)^\perp$. Since $(x'(\xi), x^*(\gamma_2)) = (x\overline{x}'(\xi), 2\gamma) = 0$, we have $x'\gamma_2 \in (M\xi)^\perp$, therefore $e_\xi(x^*\gamma_2) = 0$. Then we will get

\[
(e_\xi x(\eta), \gamma) = (e_\xi x(\eta), \gamma_1) + (e_\xi x(\eta), \gamma_2) = (x(\eta), e_\xi(\gamma_1)) + (x(\eta), e_\xi(\gamma_2)) = (x(\eta), e_\xi(\gamma_1)) = (\eta, e_\xi(x^*\gamma_1)) = (\eta, e_\xi(x^*\gamma_1)) = (\eta, e_\xi(x^*\gamma_2)) = (\gamma, e_\xi(x^*\gamma_1)) = (x\overline{x}'(\xi), 2\gamma) = 0,
\]

hence $e_\xi x = xe_\xi$. Then $e_\xi \in M''$. As mentioned above (see: Preliminaries), by the bicommutant theorem we have $M'' = M$. Hence we get $e_\xi \in M$. Recall that the bicommutant theorem is also true for real $W^*$-algebras (see [11, Theorem 4.3.8]).

**References**


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Received 26/10/2020; Revised 29/06/2022