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MORSE DECOMPOSITION AND INTRINSIC SHAPE IN TOPOLOGICAL SPACES

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Abstract. In this paper for the first time the shape of the chain recurrent set in a topological space is investigated. Given a compact Hausdorff space X and a continuous flow φ_t evolving on X we use intrinsic shape theory tools which combine continuity up to a covering and the corresponding homotopies of first order to formulate a theorem about the shape of members of a Morse decomposition and the shape of the chain recurrent set in topological spaces.

У цій роботі вперше вивчено форма ланцюгової рекурентної множини в топологічному просторі. Для заданого компактного хаусдорфого простору X і неперервного потіка φ_t, що ружається на X, ми використовуємо інструменти теорії внутрішніх форм, які поєднують неперервність до покриття та відповідні гомотопії першого порядку для формулювання теореми про форму членів в розкладі Морса і форму ланцюгової рекурентної множини в топологічному просторі.

1. Introduction

When studying the behavior of dynamical systems, the tools range from traditional techniques of classical analysis to various branches of topology born in the twentieth century at least partially in response to some dynamical systems questions. Shape theory surely fits in this description. This point of view is becoming a valuable asset in the study of topological dynamics. Namely, this theory turns out to be an appropriate tool for studying spaces with local pathologies which appear for example in dynamical settings.

One particular goal when studying dynamical systems is to find and isolate periodic solutions and equilibria. They are subsets of a chain recurrent set. Some significant results concerning this set, e.g., chain recurrence and Morse decomposition in compact metric spaces were established by Conley in [2].

The purpose of this paper is to shed a different light on the chain recurrent set in the realm of topological spaces. The main aim is to study the local shape properties of a Morse decomposition of flows on topological spaces. Namely, we consider a very general situation of a continuous flow φ_t evolving on a compact Hausdorff space X which need not be a metric space. Let us mention that this general context is not vacuous, since semi-flows appear naturally in practice, while abstract topological spaces arise, for instance, in compactifications of dynamical systems. We shall investigate the dynamical concept of the Morse decomposition in a shape theoretical framework and apply this result to a study of local shape properties of the chain transitive components for flows on compact Haudorff spaces which extends some previous results on compact metric spaces. For our shape theoretical insight of the Morse decomposition in compact Hausdorff spaces we shall use the intrinsic approach to shape for paracompact spaces given in [16]. We shall use a concept of chain and chain recurrence of a semi-flow developed in [10] by allowing jumps within open sets of families of open coverings of X. This extends the usual concept of chains for flows in metric spaces as well as the original definition of Conley [2] that

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takes the family of all open coverings of X. This concept of chain recurrence yields a Morse decomposition of the semi-flow as stated in [9] so that it has dynamical significance. The connection between Morse decompositions and chain recurrence is an extension to semi-flows on compact Hausdorff spaces of a classical result in the Conley theory: the finest Morse decomposition is given by the chain transitive components of the chain recurrence set.

2. Preliminaries

Let X be a compact Hausdorff space. A flow on X is a continuous map $\varphi : X \times \mathbb{T} \to X$, where $\mathbb{T}$ stands for the set of integers $\mathbb{Z}$ or the set of real numbers $\mathbb{R}$, such that

i) $\varphi_0 = id_X$, and
ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $s, t \in \mathbb{T}$.

As usual we write φ_t for the map $\varphi_t : X \to X$ defined by $\varphi_t(x) = \varphi(x,t)$. If we substitute $\mathbb{T}$ with the set of positive integers $\mathbb{Z}^+$ or the set of positive real numbers $\mathbb{R}^+$ we obtain the corresponding notion of a semi-flow on X. We follow [9] in the sequel.

Given a subset $Y \subseteq X$ and $t \in \mathbb{T}$ we write $Y_t^+ = \bigcup_{s \geq t} \varphi_s(Y)$ and $Y_+^t = \bigcup_{0 \leq s \leq t} \varphi_s(Y)$. We also write $Y_t^- = \bigcup_{s \geq t} \varphi_s^{-1}(Y)$ and $Y_-^t = \bigcup_{0 \leq s \leq t} \varphi_s^{-1}(Y)$. In particular, the forward orbit of Y under the flow is Y_0^+ while Y_0^- is the backward orbit.

The α and ω-limit set of a subset $Y \subseteq X$ is defined in the usual way as

$$\alpha(Y) = \bigcap_{t \in T} \overline{Y_t^-}, \qquad \omega(Y) = \bigcap_{t \in T} \overline{Y_t^+}.$$

If $x \in X$ we just write $x_t^+ = \{x\}_t^+$, $x_t^- = \{x\}_t^-$, $x_+^t = \{x\}_+^t$ and $x_-^t = \{x\}_-^t$. Consequently the semi-trajectories of x are denoted by $\gamma^+(x) = x_0^+$ and $\gamma^-(x) = x_0^-$.

For a semi-flow φ, a subset $Y \subseteq X$ is (forward) invariant if $\varphi_t(Y) = Y$ for all $t \in \mathbb{T}^+$. A subset Y is backward invariant if $\varphi_t^{-1}(Y) = Y$ for all $t \in \mathbb{T}^+$. For a flow φ, a subset $Y \subseteq X$ is positively invariant if $Y_0^+ \subseteq Y$ and negatively invariant if $Y_0^- \subseteq Y$. A subset Y is invariant if it is positively and negatively invariant.

Proposition 2.1. *Let $Y \subseteq X$. Then $\omega(Y)$ and $\alpha(Y)$ are closed invariant sets.*

The concept of Morse decomposition for flows on compact Hausdorff spaces is analogous to that for flows in compact metric spaces. Recall that a collection $\{M_1, M_2, \ldots, M_n\}$ of non-void, pairwise disjoint and compact invariant subsets of X is a Morse decomposition if the following holds.

i) For all $x \in X$ one has that $\omega(x)$ and $\alpha(x)$ belong to $\bigcup_{i=1}^n M_i$.
ii) If $\omega(x)$ and $\alpha(x)$ belong to M_i, for some $x \in X$, then $x \in M_i$.
iii) The relation $\preccurlyeq$ is a partial order,

where the relation $\preccurlyeq$ is defined on $\{M_1, M_2, \ldots, M_n\}$ as follows: $M_i \preccurlyeq M_j$ if and only if there are a chain of sets $\{M_i = M_{m_1}, \ldots, M_{m_{l+1}} = M_j\}$ and points $\{x_1, \ldots, x_l\}$, such that for all $k \in \{1, \ldots, l\}$ we have $\alpha(x_k) \subseteq M_{m_k}$ and $\omega(x_k) \subseteq M_{m_{k+1}}$.

Each element of M_i is called a Morse set. We can order the Morse sets in such way that $M_i \preccurlyeq M_j$ implies that $i \leq j$. Note that $i < j$ does not imply $M_i \preccurlyeq M_j$ and that it does not imply the existence of $x \in X$ with $\alpha(x) \subseteq M_i$ and $\omega(x) \subseteq M_j$. A Morse decomposition $\{M_1, M_2, \ldots, M_n\}$ is called finer then a Morse decomposition $\{M_1^*, M_2^*, \ldots, M_m^*\}$ if for all $j \in \{1, 2, \ldots, m\}$ there is $i \in \{1, 2, \ldots, n\}$ with $M_i \subseteq M_j^*$. A Morse decomposition describe the flow via its movement from Morse sets with lower indices toward those with higher ones.

3. Intrinsic shape for paracompact spaces

The classical homotopy theory studies the equivalence relation of homotopy for maps. The equivalence relation of homotopy for maps leads to a useful and rich theory only when we restrict to spaces with nice local properties like polyhedra and absolute neighborhood retracts. The problem arise when the target space Y is such that there are not many maps from $X \times I$ into Y so that the properties of Y prevent identifying maps that ought to be identified. In other words the definition of homotopy is too rigid because the map $H : X \times I \to Y$ must be continuous and single-valued and because it must take values in the space Y.

Shape theory was introduced by Borsuk [1] in order to study geometric properties of compact metric spaces with not necessarily good local properties. Namely, homotopy theory turns out to be an inappropriate tool for studying spaces with local pathology which appear in the mathematical formulation of many natural phenomena, for example solenoids, attractors etc. Hence, it is natural to look for another adequate tool for handling these problems. Shape theory takes the role in this context because it manages to smooth out local pathologies while preserving global properties. Besides, shape theory does not modify homotopy theory in the good framework.

The modification of Borsuk relies on the idea to relinquish the insistence in the definition of homotopy that the map H goes precisely into the space Y. The obvious alternative method which was undertaken by Sanjurjo in [13] and [14] and further followed in the paper [19] is to give up with the requirement that the function H is continuous and (or) single-valued while retaining the desirable condition that it takes values in the space Y. The last one is known as the intrinsic approach to shape theory.

The first intrinsic approach to shape is given in the papers [4] and [12]. In the paper [15] using the notion of a proximate sequence over cofinal sequences of finite coverings intrinsic shape category is constructed for compact metric spaces. For paracompact spaces the notion of a proximate sequence is replaced with a proximate net indexed by locally finite coverings from the set of all coverings $CovX$.

We shall follow the construction given in [16] for paracompact spaces using the notion of $\mathcal{V}$-continuity.

By a covering we understand a covering consisting of open sets and the set of all coverings is denoted by $CovX$. For technical reasons a covering containing the empty set will be considered the same as the covering without the empty set.

Let us start with some basic definitions.

For collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, $\mathcal{U} \prec \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$, i.e., each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$.

Definition 3.1. Suppose $\mathcal{V}$ is a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$-continuous at a point $x \in X$, if there exists a neighborhood U_x of x and $V \in \mathcal{V}$, such that

$$f(U_x) \subseteq V.$$

A function $f : X \to Y$ is $\mathcal{V}$-continuous, if it is $\mathcal{V}$-continuous at every point $x \in X$. In this case, the family of all U_x forms a covering of X.

According to this $f : X \to Y$ is $\mathcal{V}$-continuous if there exists a covering $\mathcal{U}$ of X, such that for any $x \in X$, there exists a neighborhood U of x and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We indicate this briefly as there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.

If $f : X \to Y$ is $\mathcal{V}$-continuous, then $f : X \to Y$ is $\mathcal{W}$-continuous for any $\mathcal{W}$ such that $\mathcal{V} \prec \mathcal{W}$.

If $\mathcal{V}$ is a covering of Y and $V \in \mathcal{V}$, the open set $\mathrm{st}(V)$ (star of V) is the union of all $W \in \mathcal{V}$ such that $W \cap V \neq \emptyset$. We form a new covering of Y, $\mathrm{st}(\mathcal{V}) = \{\mathrm{st}(V) | V \in \mathcal{V}\}$.

Definition 3.2. Functions $f, g : X \to Y$ are $\mathcal{V}$-homotopic, if there exists a function $F : X \times I \to Y$ such that

i) $F : X \times I \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous,
ii) $F : X \times I \to Y$ is $\mathcal{V}$-continuous at all points of $X \times \partial I$,

iii) $F(x,0) = f(x),\ F(x,1) = g(x)$.

The relation of $\mathcal{V}$-homotopy is denoted by $f \overset{\mathcal{V}}{\simeq} g$. This is an equivalence relation.

Definition 3.3. A proximate net $(f_{\mathcal{V}}) : X \to Y$ is a family $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ of $\mathcal{V}$-continuous functions $f_{\mathcal{V}} : X \to Y$ such that if $\mathcal{V} \succ \mathcal{W}$ then $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$-homotopic.

Two proximate nets $(f_{\mathcal{V}}) : X \to Y$ and $(g_{\mathcal{V}}) : X \to Y$ are homotopic if $f_{\mathcal{V}}$ and $g_{\mathcal{V}}$ are $\mathcal{V}$-homotopic for all $\mathcal{V} \in CovY$ which we denote by $(f_{\mathcal{V}}) \sim (g_{\mathcal{V}})$. This is an equivalence relation.

If $(f_{\mathcal{V}}) : X \to Y$ and $(g_{\mathcal{W}}) : Y \to Z$ are proximate nets then for a covering $\mathcal{W} \in CovZ$, there exists a covering $\mathcal{V} \in CovY$ such that $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$. Then the composition of these two proximate nets is a proximate net $(h_{\mathcal{W}}) : X \to Z$ defined by $(h_{\mathcal{W}}) = (g_{\mathcal{W}} f_{\mathcal{V}}) : X \to Z$.

Paracompact spaces and homotopy classes of proximate nets $[(f_{\mathcal{V}})]$ form a category whose isomorphisms induce classifications which coincide with the standard shape classification, i.e., isomorphic spaces in this category have the same shape.

At the end of this section we give two lemmas that will be used in the sequel. The proof is given in [17] and [16].

Definition 3.4. Let $\mathcal{V}$ be a covering of Y. Two functions $f, g : X \to Y$ are $\mathcal{V}$-near if for any $x \in X$ there exists $V \in \mathcal{V}$ such that $f(x), g(x) \in V$.

Lemma 3.5. *If $\mathcal{V}$ is a covering of Y and $f, g : X \to Y$ are $\mathcal{V}$-near and $\mathcal{V}$-continuous then f and g are $\mathcal{V}$-homotopic.*

Lemma 3.6. *Suppose $\mathcal{V}$ is a finite covering of Y, $X = X_1 \cup X_2$, X_i is closed, $i = 1, 2$, and $f_i : X_i \to Y$ are $\mathcal{V}$-continuous functions, $i = 1, 2$, such that $f_1(x) = f_2(x)$ for all $x \in X_1 \cap X_2$. Define a function by*

$$f(x) = f_i(x)$$

for $x \in X_i$, $i = 1, 2$. Then we have the following.

1) *If $x \in IntX_1$ or $x \in IntX_2$, then $f : X \to Y$ is $\mathcal{V}$-continuous at x.*
2) *If $x \in \partial X_1$ or $x \in \partial X_2$, then $f : X \to Y$ is* $\mathrm{st}(\mathcal{V})$*-continuous at x.*

4. Shape of Morse sets in topological spaces

In this paper we apply the theory of intrinsic shape in paracompact spaces to deduce a result for Morse set properties compared with the properties of its neighborhood in terms of shape theory.

In order to prove the main theorem we need the following.

A covering $\mathcal{V}$ of M in X is called *regular* if it satisfies the following conditions.

1) If $V \in \mathcal{V}$ then $V \cap M \neq \emptyset$
2) If $U, V \in \mathcal{V}$ and $U \cap V \neq \emptyset$, then $U \cap V \in \mathcal{V}$.

For a covering $\mathcal{V}$ of M we introduce the notation $|\mathcal{V}| = \bigcup_{V \in \mathcal{V}} V$.

For a finite regular covering $\mathcal{V}$ we define a function $r_{\mathcal{V}} : |\mathcal{V}| \to M$ in the following way:

- for points $y \in M$, we put $r_{\mathcal{V}}(y) = y$;
- for points $y \in |\mathcal{V}| \backslash M$, by induction we can choose the smallest member $V \in \mathcal{V}$ such that $y \in V$, then choose a fixed point $y_V \in V \cap M$ and put $r_{\mathcal{V}}(y) = y_V$.

The function $r_{\mathcal{V}}$ is $\mathcal{V}$-continuous.

Lemma 4.1 ([17])**.** *If $\mathcal{V} \succ \mathcal{W}$ then $r_{\mathcal{W}} : |\mathcal{W}| \to M$ and $r_{\mathcal{V}} : |\mathcal{W}| \to M$ (the restriction of $r_{\mathcal{V}} : |\mathcal{V}| \to M$ to $|\mathcal{W}|$) are $\mathcal{V}$-near and so $\mathcal{V}$-homotopic via a homotopy $r_{\mathcal{V}\mathcal{W}} : |\mathcal{W}| \times I \to M$.*

Lemma 4.2 ([18])**.** *If $\mathcal{V}$ is a finite regular covering of M in X then $i \circ r_{\mathcal{V}} : |\mathcal{V}| \to |\mathcal{V}|$ and $1_{\mathcal{V}} : |\mathcal{V}| \to |\mathcal{V}|$ are $\mathcal{V}$-homotopic via a homotopy $R_{\mathcal{V}} : |\mathcal{V}| \times I \to |\mathcal{V}|$ such that $R_{\mathcal{V}}(x,t) = x$ for $x \in M$.*

Recall that a subset $A \subseteq X$ is called an attractor if there is a neighborhood W of A such that $\omega(W) = A$. Similarly, a set $R \subseteq X$ is called a repeller if $\alpha(V) = R$, for some neighborhood V of R. The dual repeller of A is: $R = \{x \in X \mid \omega(x) \cap A = \emptyset\}$. The pair (A, R) is called an attractor-repeller pair decomposition of X. Also note that if $W = X$ or $V = X$ (in compact spaces) then we obtain the corresponding notions of global attractor and global repeller respectively.

There are two fundamental theorems associated with attractor-repeller pair decompositions. The first indicates that the recurrent dynamics in X is contained entirely in $A \cup R$ and that outside of these sets the dynamics is gradient-like (for a proof see [11]).

Theorem 4.3. *Let (A, R) be an attractor-repeller pair decomposition of X. Then*

$$X = A \cup R \cup C(R, A),$$

where $C(R, A) = \{x \in X \mid \omega(x) \subseteq A, \alpha(x) \subseteq R\}$. Furthermore, there exists a continuous function $L : X \to [0, 1]$ such that

i) $R = L^{-1}(1)$,
ii) $A = L^{-1}(0)$.
iii) *If $x \in C(R, A)$ and $t > 0$ then $L(x) > L(\varphi(x, t))$.*

Remark 4.4. From the previous theorem 4.3 it follows that every continuous flow on a compact metric space admits a Lyapunov function which strictly decreases along the nonrecurrent trajectories.The theorem holds true for compact Hausdorff spaces as well (see [9]).

The second result concerns homotopy index theory in which the notion of continuation plays a central role (for more details see [8]).

The generalization of an attractor-repeller decomposition is the already mentioned Morse decomposition. As with attractor-repeller pairs, Morse decompositions admit Lyapunov functions. Let us recall the definition of a Lyapunov function for a given family of disjoint compact invariant subsets of X (see [5]).

Definition 4.5. Let $\mathcal{M} = \{M_j \mid j \in J\}$ be a family of disjoint compact invariant subsets of the phase space X. A Lyapunov function for $\mathcal{M}$ is a continuous function $L_\mathcal{M} : X \to \mathbb{R}$ such that:

i) $L_\mathcal{M}(\varphi(x, t)) < L_\mathcal{M}(x), \forall t > 0, \forall x \notin \bigcup_{j \in J} M_j$
ii) $L_\mathcal{M}(M_j) = c_j, \forall j \in J, (c_j \neq c_i$ for $i \neq j)$.

The real numbers c_j are called the critical values of $L_\mathcal{M}$.

Theorem 4.6. *Let $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$ be a Morse decomposition of a flow φ in a compact Hausdorff space X. Then there exists a Lyapunov function for $\mathcal{M}$.*

The proof is a simple extension of the argument used for attractor-repeller pairs (for a proof for compact metric spaces, see for example [18] or [2]); for compact Hausdorff spaces, see [9].

The existence of a Lyapunov function for a Morse decomposition collection on compact Hausdorff spaces turns out to have an important role in proving our claim 4.23. We shall be interested in a Lyapunov function for a Morse decomposition $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$ of a flow φ which satisfies the following condition.

- For each critical value c_i, the set $L_\mathcal{M}^{-1}(c_i)$ is a Morse component, i.e., $L_\mathcal{M}^{-1}(c_i) = M_i$.

Before employing topological techniques in proving our claim 4.23 let us make a few comments. Notice that, generally speaking, a trajectory can come arbitrary close to a given Morse component and then leave. The condition $L_\mathcal{M}^{-1}(c_i) = M_i$ ensures that this cannot be the case. The following example illustrates that our imposed condition $L_\mathcal{M}^{-1}(c_i) = M_i$ on the Lyapunov function may not be always satisfied.

Example 4.7. Consider a dynamical system defined in the cylinder $D\times I$, where D stands for the unit disk. The points in the Hawaiian earring $H=\bigcup_{n=1}^{\infty} S((1/2n,0,1/2),1/2n)$ are stationary points. All the points in $D\times\{0,1\}$ are also stationary. The trajectories of the rest of the points are vertical straight lines joining two stationary points. The Hawaiian earring is a Morse set which does not satisfy our condition. Also notice that there are trajectories that come arbitrary close to the Hawaiian earing and then leave.

Now let us discuss an arbitrary Morse decomposition $\mathcal{M}=\{M_1,M_2,\dots,M_n\}$ of a flow φ on a compact Hausdorff space X for which there exists a Lyapunov function $L_{\mathcal{M}}$ for $\mathcal{M}$ which satisfies our condition $L_{\mathcal{M}}^{-1}(c_i)=M_i$. For an arbitrary member M_i from $\mathcal{M}$ we shall introduce the following neighborhood $U=U^+\cup U^-$ of M_i where

$$U^+=L_{\mathcal{M}}^{-1}([c_i,c_i+\epsilon]) \text{ and } U^-=L_{\mathcal{M}}^{-1}([c_i-\epsilon,c_i]),$$

for sufficiently small ϵ such that $U=U^+\cup U^-$ is disjoint from the other members of $\mathcal{M}$. Let us note that U^+ and U^- are positively and negatively invariant, respectively.

Remark 4.8. The latter claim follows from the existence of a Lyapunov function which satisfies our imposed condition.

We shall prove that the semi-flow restricted to U^+ admits a global attractor which coincides with M_i. Similarly, the semi-flow restricted to U^- admits a global repeller which coincides with M_i.

Lemma 4.9. *The semi-flow φ restricted to U^+ admits a global attractor that coincides with M_i.*

Proof. We shall use Lemma 3.7 in [9] for $X_{\mathcal{M}}=L_{\mathcal{M}}^{-1}([c_i,+\infty))$, $K=U^+=L_{\mathcal{M}}^{-1}([c_i,c_i+\epsilon])$ and $A=M_i$. Note that $A\subset intK$ and that A is a maximal (forward) invariant set in K for the semi-flow φ restricted to $X_{\mathcal{M}}=L_{\mathcal{M}}^{-1}([c_i,+\infty))$. Also note that for all $x\in K\setminus A$ the backward orbit of x is not contained in K. On the contrary, from the fact that the backward orbit of the semi-flow on $X_{\mathcal{M}}$ coincides with the negative orbit $\gamma(x)^-=x_0^-$ we have that $x_0^-\subset K$ which yields that $x\in\omega(K)$. But $\omega(K)$ is an invariant set according to Proposition 2.1 which contains A. Hence from the maximality of A we conclude that $\omega(K)=A$. But this implies that $x\in A$ which is a contradiction. Hence, according to Lemma 3.7 in [9], $A=M_i$ is an attractor. This implies that $\omega(U^+)=A=M_i$ which yields that $A=M_i$ is a global attractor for the semi-flow on $(U^+,\varphi_{|U^+})$. □

Remark 4.10. Similarly, it follows that the semi-flow φ restricted to U^- admits a global repeller which coincides with M_i, i.e., $\alpha(U^-)=R=M_i$.

Definition 4.11. A set M is said to be positively admissible, if for any sequences $x_n\in M$ and $t_n\to\infty$ with $\varphi(x_n,[0,t_n])\subset M$ for all n, the sequence $\varphi(x_n,t_n)$ has a convergent subsequence. Similarly, a set M is said to be negatively admissible, if for any sequences $x_n\in M$ and $t_n\to-\infty$ with $\varphi(x_n,[t_n,0])\subset M$ for all n, the sequence $\varphi(x_n,t_n)$ has a convergent subsequence.

We proceed by showing how the semi-flow $(\varphi_t:U^+\to U^+)$ with a global attractor M_i and the semi-flow $(\varphi_t:U^-\to U^-)$ with a global repeller M_i, both defined on compact Hausdorff spaces U^+ and U^-, respectively, induce a shape morphism $\psi:U=U^+\cup U^-\to M_i$ in a natural way, assuming that the sets U^+ and U^- are positively and negatively admissible, respectively. We shall also assume that the points in the Morse member M_i are all stationary.

Remark 4.12. Let X be a normal Hausdorff topological space and $M\subset X$ a closed subset of X. We denote the set of all finite regular coverings of the compact X in X by Cov_rX. Similarly, we denote the set of all finite regular coverings of the compact M in X by Cov_rM. For $\mathcal{V}\in Cov_rX$ we consider the covering $\mathcal{V}_M=\{V\in\mathcal{V}|V\cap M\neq\emptyset\}$. Let

$Cov_r^M X = \{\mathcal{V} \in Cov_r X | \mathcal{V}_M \in Cov_r M\}$. Then $Cov_r^M X$ is cofinal in the set $Cov^f X$ of all finite coverings of X ([3, p. 262] and [3, p .249]).

Remark 4.13. Let us also note that every compact Hausdorff space is normal. Hence instead of working on the set $Cov^f X$ of all finite coverings of X we can work on the set $Cov_r^M X$.

Let $\mathcal{V} \in Cov_r^{M_i} U$ be an arbitrary open covering of the neighborhood $U = U^+ \cup U^-$.

Remark 4.14. The map $h : Cov_r^{M_i} U \to Cov_r M_i$ given by $h(\mathcal{V}) = \mathcal{V}_{M_i}$ is order preserving.

Definition 4.15. An open cover $\mathcal{U}$ of X is normal iff there exists a sequence of open covers $\mathcal{V}_n$ such that $\mathcal{V}_0 = \mathcal{U}$ and $\mathcal{V}_n$ is a star refinement of $\mathcal{V}_{n-1}$.

We will also need the following important theorem from [6] in the discussion that follows.

Theorem 4.16. *A topological space X is normal if and only if each finite open covering is normal.*

Remark 4.17. According to Remark 4.13, compact Hausdorff spaces are normal hence we are working with normal coverings.

Let $h(\mathcal{V}) = \mathcal{V}_{M_i} \prec \mathcal{V}$. We consider the neighborhood $|\mathcal{V}_{M_i}| = \bigcup_{V \in \mathcal{V}_{M_i}} V$ of M_i. Now using Lemma 4.9, Remark 4.10 and admissibility of U^+ and U^-, respectively, we see that there exists a net of positive reals $t_\mathcal{V}$ such that

$$\varphi(U^+, [t_\mathcal{V}, \infty)) \subseteq |\mathcal{V}_{M_i}| \qquad \text{and} \qquad \varphi(U^-, (-\infty, -t_\mathcal{V}]) \subseteq |\mathcal{V}_{M_i}|.$$

Hence, we have defined a net of positive real numbers $(t_\mathcal{V} | \mathcal{V} \in Cov_r^{M_i} U)$.

Construction of a proximate net. We choose an arbitrary covering $\mathcal{V} \in Cov_r^{M_i} U$. Let $h(\mathcal{V}) = \mathcal{V}_{M_i}$.

We define $f_\mathcal{V} : U^+ \cup U^- \to U^+ \cup U^-$ by

$$f_\mathcal{V}(x) = r_{\mathcal{V}_{M_i}} \varphi(x, t_\mathcal{V}) \quad \text{for } x \in U^+, \quad \text{and} \quad f_\mathcal{V}(x) = r_{\mathcal{V}_{M_i}} \varphi(x, -t_\mathcal{V}) \quad \text{for } x \in U^-.$$

Note that for any $x \in M_i$, $f_\mathcal{V}(x) = x$ in either case, so $f_\mathcal{V} : U^+ \cup U^- \to U^+ \cup U^-$ is well defined.

Also note that the function is $\mathcal{V}_{M_i} \cap M_i$-continuous and hence $\mathcal{V}$-continuous.

In this way we have defined a collection of $\mathcal{V}$-continuous functions $f_\mathcal{V} : U \to U$ for arbitrary $\mathcal{V} \in Cov_r^{M_i} U$ such that $f_\mathcal{V}(U) \subseteq M_i$.

Lemma 4.18. *The net of functions $\psi_U = (f_\mathcal{V} \mid \mathcal{V} \in Cov_r^{M_i} U) : U \to U$ is a proximate net.*

Proof. For two coverings $\mathcal{W} \prec \mathcal{V}$ we have the following.

If $t_\mathcal{V} < t_\mathcal{W}$ we define a $\mathcal{V}$-homotopy $R_{\mathcal{V}\mathcal{W}} : U^+ \times I \to U^+$ by

$$R_{\mathcal{V}\mathcal{W}}(x, t) = r_{\mathcal{V}_{M_i}} \varphi(x, (1-t)t_\mathcal{V} + t t_\mathcal{W})$$

that connects $f_\mathcal{V}(x)$ and $r_{\mathcal{V}_{M_i}} \varphi(x, t_\mathcal{W})$. Now $r_{\mathcal{V}_{M_i}} \varphi(x, t_\mathcal{W})$ and $r_{\mathcal{W}_{M_i}} \varphi(x, t_\mathcal{W}) = f_\mathcal{W}(x)$ are $\mathcal{V}$-near and hence $\mathcal{V}$-homotopic.

If $t_\mathcal{V} > t_\mathcal{W}$ we define a $\mathcal{W}$-homotopy $R_{\mathcal{V}\mathcal{W}} : U^+ \times I \to U^+$ by

$$R_{\mathcal{V}\mathcal{W}}(x, t) = r_{\mathcal{W}_{M_i}} \varphi(x, (1-t)t_\mathcal{W} + t t_\mathcal{V})$$

that connects $f_\mathcal{W}(x)$ and $r_{\mathcal{W}_{M_i}} \varphi(x, t_\mathcal{V})$. Since $\mathcal{W} \prec \mathcal{V}$ this is also a $\mathcal{V}$-homotopy. Now $r_{\mathcal{W}_{M_i}} \varphi(x, t_\mathcal{V})$ and $r_{\mathcal{V}_{M_i}} \varphi(x, t_\mathcal{V}) = f_\mathcal{V}(x)$ are $\mathcal{V}$-near and hence $\mathcal{V}$-homotopic.

This yields a $\mathcal{V}$-homotopy $r^+_{\mathcal{V}\mathcal{W}} : U^+ \times I \to U^+$ connecting $f_\mathcal{V}(x)$ and $f_\mathcal{W}(x)$ for $x \in U^+$.

In the same way we define a $\mathcal{V}$-homotopy $r^{-}_{\mathcal{V}\mathcal{W}} : U^{-} \times I \to U^{-}$ connecting $f_{\mathcal{V}}(x)$ and $f_{\mathcal{W}}(x)$ for $x \in U^{-}$. Since both homotopies coincide for $x \in M_i$ by Lemma 3.6 we conclude that there is a $\mathcal{V}$-homotopy $r_{\mathcal{V}\mathcal{W}} : (U^{+} \cup U^{-}) \times I \to U^{+} \cup U^{-}$ connecting $f_{\mathcal{V}} : U^{+} \cup U^{-} \to U^{+} \cup U^{-}$ and $f_{\mathcal{W}} : U^{+} \cup U^{-} \to U^{+} \cup U^{-}$. □

Remark 4.19. Note that the homotopies between the functions $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$, where $\mathcal{W} \prec \mathcal{V}$ are all in M_i, i.e., $r_{\mathcal{V}\mathcal{W}}(U \times I) \subseteq M_i$.

Remark 4.20. If $\mathcal{V}$ is a covering of Y and $M \subseteq Y$ then by $\mathcal{V} \cap M$ we denote the following covering of M:

$$\mathcal{V} \cap M = \{V \cap M | V \in \mathcal{V}\}.$$

If a proximate net $(f_{\mathcal{V}}) : X \to Y$ satisfies $f_{\mathcal{V}}(X) \subseteq M$ and if for $\mathcal{V} \succ \mathcal{W}$, $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$-homotopic in M, then we can define a proximate net $(f_{\mathcal{W}}) : X \to M$ as follows.

For a covering $\mathcal{W}$ of M we choose a covering $\mathcal{V}$ of Y such that $\mathcal{V} \cap M = \mathcal{W}$. Note that the existence of $\mathcal{V}$ easily follows from the relative topology of M.

Then the function $f_{\mathcal{W}} : X \to M$ is defined by

$$f_{\mathcal{W}}(x) = f_{\mathcal{V}}(x), \qquad x \in X.$$

Then $(f_{\mathcal{W}}) : X \to M$ is a proximate net.

We say that the proximate net $(f_{\mathcal{W}}) : X \to M$ is inherited from $(f_{\mathcal{V}}) : X \to Y$ and the inherited proximate net we denote by $(f_{\mathcal{V} \cap M} | \mathcal{V} \in CovY)$.

Remark 4.21. Let us note that the inclusion $i : M \to X$ induces a proximate net by setting $i_{\mathcal{V}}(x) = x$ for every $x \in M$. It is easy to prove that the collection $(i_{\mathcal{V}} \mid \mathcal{V} \in CovX)$ is a proximate net.

Lemma 4.22. *Let X be a compact Hausdorff space and M a compact subset of X. Suppose there is a proximate net $(f_{\mathcal{V}}) : X \to X$ such that $f_{\mathcal{V}}(X) \subseteq M$ for all coverings $\mathcal{V}$, and for $\mathcal{V} \succ \mathcal{W}$, the functions $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$-homotopic in M and there is a homotopy $H_{\mathcal{V}} : X \times I \to X$ such that*

1) $H_{\mathcal{V}}(x, 0) = x, H_{\mathcal{V}}(x, 1) = f_{\mathcal{V}}(x)$,
2) $H_{\mathcal{V}}(M \times I) \subseteq M$.

Then the inherited proximate net $(f_{\mathcal{V} \cap M}) : X \to M$ induces a shape equivalence with the inclusion $i : M \to X$ as a shape inverse.

Proof. We consider the proximate net $(i_{\mathcal{V}}) : M \to X$ induced by the inclusion $i : M \to X$. First $i_{\mathcal{V}} \circ f_{\mathcal{V} \cap M}(x) = f_{\mathcal{V}}(x)$ and by 1), $H_{\mathcal{V}}$ connects $i_{\mathcal{V}} \circ f_{\mathcal{V} \cap M}$ and the identity map 1_X. On the other hand, $f_{\mathcal{V} \cap M} \circ i_{\mathcal{V}}(x) = f_{\mathcal{V} \cap M}(x)$ and by 2), $H_{\mathcal{V}}|_M$ connects $f_{\mathcal{V} \cap M} \circ i_{\mathcal{V}}$ and the identity map 1_M. □

Theorem 4.23. *The shape morphism $[\psi] = [(f_{\mathcal{V} \cap M_i})] : U^{+} \cup U^{-} \to M_i$ is a shape equivalence. Consequently $Sh(M_i) = Sh(U)$.*

Proof. We choose an arbitrary covering $\mathcal{V}$ of $U = U^{+} \cup U^{-}$. Let $h(\mathcal{V}) = \mathcal{V}_{M_i} \prec \mathcal{V}$ be the finite regular covering adjoined to $\mathcal{V}$ by the map h. Notice that since $r_{\mathcal{V}_{M_i}}, R_{\mathcal{V}_{M_i}}$ are $\mathcal{V}_{M_i}$-continuous they will be $\mathcal{V}$-continuous as well. We will define a homotopy $H^{+}_{\mathcal{V}} : U^{+} \times I \to U^{+} \cup U^{-}$ as concatenation of three homotopies. The first is a continuous map $F : U^{+} \times I \to U^{+} \cup U^{-}$, defined by

$$F(x, s) = \varphi(x, st_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i}).$$

This map satisfies

$$F(x, 0) = x, \ F(x, 1) = \varphi(x, t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i}).$$

The third is a function $G : U^+ \times I \to U^+ \cup U^-$ defined by

$$G(x, s) = r_{\mathcal{V}_{M_i}} \varphi(x, (1 - s)t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i} + st_{\mathcal{V}}).$$

The composition is well defined since

$$(1 - s)t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i} + st_{\mathcal{V}} \geq t_{\mathcal{V}},$$

and it follows that $\varphi(x, (1 - s)t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i} + st_{\mathcal{V}}) \in |\mathcal{V}_{M_i}|$.

The composition $r_{\mathcal{V}_{M_i}} \varphi$ is $\mathcal{V}$-continuous and this map satisfies

$$G(x, 0) = r_{\mathcal{V}_{M_i}} \varphi(x, t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i}), \qquad G(x, 1) = r_{\mathcal{V}_{M_i}} \varphi(x, t_{\mathcal{V}}) = f_{\mathcal{V} \cap M_i}(x).$$

The middle homotopy $Q : U^+ \times I \to U^+ \cup U^-$ is defined by

$$Q(x, s) = R_{\mathcal{V}_{M_i}}(\varphi(x, t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i}), s),$$

where $R_{\mathcal{V}_{M_i}}$ is the homotopy in Lemma 4.2. This homotopy satisfies

$$Q(x, 0) = F(x, 1) = \varphi(x, t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i}), \qquad Q(x, 1) = G(x, 0) = r_{\mathcal{V}_{M_i}} \varphi(x, t_{\mathcal{V}} \frac{L_{\mathcal{M}}(x)}{c_i}).$$

Since

$$F(x, 1) = Q(x, 0), \qquad Q(x, 1) = G(x, 0),$$

we can define concatenation of the three defined homotopies and finally define the required $\mathcal{V}$-homotopy $H^+_{\mathcal{V}} : U^+ \times I \to U^+ \cup U^-$ by $H^+_{\mathcal{V}} = G * Q * F$ and

$$H^+_{\mathcal{V}}(x, 0) = x, \qquad H^+_{\mathcal{V}}(x, 1) = f_{\mathcal{V}}(x).$$

Since $F(x, s) = x$ for $x \in M_i$ and the same holds for Q and G we deduce that the same holds for the homotopy $H^+_{\mathcal{V}}$, i.e. $H^+_{\mathcal{V}}(x, s) = x$, for any $x \in M_i$.

In the same way starting with

$$F(x, s) = \varphi(x, -st_{\mathcal{V}} \frac{c_i}{L_{\mathcal{M}}(x)})$$

and making the corresponding changes for Q and G, we define a $\mathcal{V}$-homotopy $H^-_{\mathcal{V}}$: $U^- \times I \to U^+ \cup U^-$. For this homotopy we have that $H^-_{\mathcal{V}}(x, s) = x$ for any $x \in M_i$.

Since $H^+_{\mathcal{V}} : U^+ \times I \to U^+ \cup U^-$ and $H^-_{\mathcal{V}} : U^- \times I \to U^+ \cup U^-$ coincide on $M_i \times I$, we can define $H_{\mathcal{V}} : (U^+ \cup U^-) \times I \to U^+ \cup U^-$. This is a $\mathcal{V}$-homotopy by Lemma 3.6. This homotopy connects the identity map 1_U and $f_{\mathcal{V}}$ and $H_{\mathcal{V}}(x, s) = x$ for any $x \in M_i$. $\square$

5. Shape of the chain recurrent set in topological spaces

Now, in order to consider the shape of a chain recurrent set in topological spaces let us recall the definition of chains for semi-flows based on admissible open coverings of the state space X. We follow the abstract theory of chain transitivity and chain recurrence developed in [10] for compact Hausdorff spaces.

Let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of X. We write $\mathcal{V} \prec \frac{1}{2}\mathcal{U}$ if for every $V, V' \in \mathcal{V}$ with $V \cap V' \neq \emptyset$ there exists $U \in \mathcal{U}$ with $V \cup V' \subseteq U$. We define inductively the relation $\mathcal{V} \prec \frac{1}{2^n}\mathcal{U}$ if $\mathcal{V} \prec \mathcal{W}$ and $\mathcal{W} \prec \frac{1}{2^{n-1}}\mathcal{U}$. Note that $\mathcal{U} \prec \frac{1}{2}\mathrm{st}(\mathcal{U})$. Also if $\mathcal{V} \prec \frac{1}{2}\mathcal{U}$ then $\mathcal{V} \prec \mathcal{U} \prec \frac{1}{2}\mathrm{st}(\mathcal{U})$.

Given an open covering $\mathcal{U}$ of X and a compact subset $M \subseteq X$ we write, as in the previous section, that

$$\mathcal{U}_M = \{U \in \mathcal{U} \mid M \cap U \neq \emptyset\}.$$

If $N \subseteq X$ is open with $M \subseteq N$ we say that $\mathcal{U}$ is M-subordinated to N if for each $U' \in \mathcal{U}_M$ we have $U' \subseteq N$.

Now we can introduce the conditions on families of open coverings of X, which will be used in the concept of chains of a semi-flow.

Definition 5.1. Let $\mathcal{O}$ be a family of open coverings of X. We say that $\mathcal{O}$ is admissible if

1) for each $\mathcal{U} \in \mathcal{O}$ there exists $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \prec \frac{1}{2}\mathcal{U}$.
2) Let $N \subseteq X$ be an open set and $M \subseteq N$ be compact. Then there exists $\mathcal{U} \in \mathcal{O}$ which is M-subordinated to N.

Let us note that in paracompact Hausdorff spaces the family of all coverings $Cov(X)$ is admissible. Furthermore in compact Hausdorff spaces the family of all finite coverings $Cov_f(X)$ is admissible as well.

Now we can start looking at chains for semi-flows, based on admissible open coverings of the state space X.

Definition 5.2. Let φ be a semi-flow on X and $\mathcal{U}$ an open covering of X. Given $x, y \in X$ and $t \in \mathbb{T}$, a $(\mathcal{U}, t)$-chain from x to y means a sequence of points $\{x = x_1, \ldots, x_{n+1} = y\} \subset X$, a sequence of times $\{t_1, \ldots, t_n\} \subset \mathbb{T}$, and a sequence of open sets $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}$ such that $t_i \geq t$ and $\varphi_{t_i}(x_i), x_{i+1} \subset U_i$ for all $i = 1, \ldots, n$.

Given a subset $Y \subseteq X$ we write $\Omega(Y, \mathcal{U}, t)$ for the set of all x such that there is a $(\mathcal{U}, t)$-chain from a point $y \in Y$ to x.

Now let $\mathcal{O}$ be a family of open coverings of X. Then the $\Omega_{\mathcal{O}}$-limit set of a subset $Y \subset X$ is defined by

$$\Omega_{\mathcal{O}}(Y) = \bigcap \{\Omega(Y, \mathcal{U}, t) \mid \mathcal{U} \in \mathcal{O}, t \in \mathbb{T}\}.$$

For $x \in X$ we write $\Omega_{\mathcal{O}}(x) = \Omega_{\mathcal{O}}(\{x\})$ and define the relation $x \preccurlyeq_{\mathcal{O}} y$ if $y \in \Omega_{\mathcal{O}}(x)$.

The following fact is from [10] as well.

Proposition 5.3. *If a family $\mathcal{O}$ is admissible then the relation $\preccurlyeq_{\mathcal{O}}$ is transitive, closed, and invariant with respect to φ, i.e., we have that $\varphi_t(x) \preccurlyeq_{\mathcal{O}} \varphi_s(x)$ if $x \preccurlyeq_{\mathcal{O}} y$ for all $s, t \in \mathbb{T}$. Also, for every $Y \subset X$ the set $\Omega_{\mathcal{O}}(Y)$ is invariant.*

Define a relation $x \sim_{\mathcal{O}} y$ if $x \preccurlyeq_{\mathcal{O}} y$ and $y \preccurlyeq_{\mathcal{O}} x$. Then we say that $x \in X$ is $\mathcal{O}$-chain recurrent if it is self-related under $\sim_{\mathcal{O}}$, that is, $x \sim_{\mathcal{O}} x$. The set $\mathcal{R}_{\mathcal{O}}$ of all $\mathcal{O}$-chain recurrent points is called a $\mathcal{O}$-chain recurrent set. It is easy to see that the restriction of $\sim_{\mathcal{O}}$ to $\mathcal{R}_{\mathcal{O}}$ is an equivalence relation.

An equivalence class of $\sim_{\mathcal{O}}$ is called an $\mathcal{O}$-chain transitive component. A set $Y \subseteq X$ is called $\mathcal{O}$-chain recurrent if $Y \subseteq \mathcal{R}_{\mathcal{O}}$ and Y is called $\mathcal{O}$-chain transitive if any two points of Y are equivalent. Finally a semi-flow φ is called $\mathcal{O}$-chain recurrent if $X = \mathcal{R}_{\mathcal{O}}$, and φ is called $\mathcal{O}$-chain transitive if X is $\mathcal{O}$-chain transitive.

Let us note that if X is compact and Hausdorff, the set $\Omega_{\mathcal{O}}(Y)$ is independent of the particular admissible family $\mathcal{O}$. Hence we can drop the subscript $\mathcal{O}$ and write simply $\Omega(Y)$.

The following result from [10] relates connected chain recurrent sets to chain transitive sets.

Proposition 5.4. *If a set is connected and chain recurrent then it is chain transitive. In particular, each chain transitive component is a union of connected components of a chain recurrent set.*

In order to state our final claim regarding the shape of the chain recurrent set $\mathcal{R}$ in topological spaces we shall need the following theorem from [10] as well.

Theorem 5.5. *There exists the finest Morse decomposition if and only if the number of chain transitive components is finite. In this case, the chain transitive components make the finest Morse decomposition.*

Hence we have the following claim.

Corollary 5.6. *An arbitrary chain recurrent set $\mathcal{R}$ with finitely many connected components satisfying the conditions of Theorem 4.23 admits a compact neighborhood U with the same shape as $\mathcal{R}$,i.e., $Sh(U) = Sh(\mathcal{R})$.*

Proof. According to Proposition 5.4 the number of chain transitive components is finite. Hence according to Theorem 5.5 there exists the finest Morse decomposition $\mathcal{M} = \{M_1^{ct}, \dots, M_n^{ct}\}$ and the chain transitive components are its members. Now using Theorem 4.23 we conclude that each chain transitive component M_i^{ct} admits a compact neighborhood U_i with the same shape as M_i^{ct}, i.e., $Sh(M_i^{ct}) = Sh(U_i)$ and such that it is disjoint from the others (have in mind that the space is normal). Now, if we choose $U = \bigcup_{i=1}^n U_i$ we get that

$$Sh(U) = Sh(\bigcup_{i=1}^{n} U_i) = Sh(\bigcup_{i=1}^{n} M_i^{ct}) = Sh(\mathcal{R}).$$

□

Remark 5.7. Let us note that the condition of no movement on the Morse sets is not too restrictive. Actually, by slowing down the motion as we move towards the Morse sets, or repel from them, (the measure of nearness can be for example a Lyapunov function) we can obtain a new flow in which this is the case. For metric spaces one can use the Keesling reformulation of Beck's theorem [7].

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