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SPECTRAL INCLUSIONS OF EXPONENTIALLY BOUNDED C-SEMIGROUPS

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This paper is dedicated to my Professor Abdelaziz Tajmouati on the occasion of his retirement

Abstract. In 1989 Ki Sik Ha [6] proved that if A is a generator of an exponentially bounded C-semigroup $(S_t)_{t\geq 0}$ in a Banach space and $T_t = C^{-1}S_t$ for all $t \geq 0$, then the spectral mapping theorem, $e^{t\sigma(A)} \subset \sigma(T_t)$ and $e^{t\sigma_p(A)} \subset \sigma_p(T_t) \subset e^{t\sigma_p(A)} \cup \{0\}$ for all $t \geq 0$, holds. In the present paper, we extend the results of [6] to Saphar, essentially Saphar, Kato, and essentially Kato spectrum.

У 1989 році Кі Сік Ха [6] довів, що якщо A є генератором експоненціально обмеженої C-напівгрупи $(S_t)_{t\geq 0}$ у банаховому просторі та $T_t = C^{-1}S_t$ для всіх $t \geq 0$, то виконується теорема про спектральне відображення: $e^{t\sigma(A)} \subset \sigma(T_t)$ і $e^{t\sigma_p(A)} \subset \sigma_p(T_t) \subset e^{t\sigma_p(A)} \cup \{0\}$ для всіх $t \geq 0$. Ми поширюємо результати [6] на спектр Сапфара, суттєвий спектр Сапфара, спектр Като і суттєвий спектра Като.

1. Introduction and preliminaries

1.1. **C-semigroups.** Davies and Pang [2] introduced the notion of an exponentially bounded C-semigroup, and characterized it and its generator in a Banach space. Let X be a complex Banach space, and let $B(X)$ be the set of all bounded linear operators from X into itself.

Throughout the paper, let $C \in B(X)$ be injective, with dense range $R(C)$ in X. A family $\{S_t, t \geq 0\} \subset B(X)$ is called a C-semigroup, (or C-regularized semigroup) if

$$S_0 = C, \tag{1.1}$$

$$\forall\, t, s \geq 0 : \qquad S_{t+s}C = S_t S_s, \tag{1.2}$$

$$S\,.\,x : [0, +\infty[\to X : \quad t \mapsto S_t x \quad \text{ is continuous for } x \in X. \tag{1.3}$$

It is called exponentially bounded if there exists $M \geq 0$ and $a \in \mathbb{R}$ such that

$$\|S_t\| \leq Me^{at} \quad \text{ for every } t \geq 0. \tag{1.4}$$

Example 1.1. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup, then for any injective operator C that commutes with $T_t, t \geq 0$, the family $S_t = T_t C$ is a C-semigroup.

Remark 1.2.
(1) If $C = I$ (the identity operator), then $(S_t)_{t\geq 0}$ is a C_0-semigroup in the ordinary sense.
(2) Unlike the case of semigroups, there exists a C_0-semigroup which is not exponentially bounded, see for example [2].
(3) Let $(S_t)_{t\geq 0}$ be a C-semigroup. Letting $s \mapsto 0$ in (1.2), we obtain:
- $\forall t \geq 0$: $S_t C = CS_t$;
- $\forall t \geq 0$, $\forall x \in R(C)$: $S_t x \in R(C)$;
- $\forall x \in R(C)$, $t \geq 0$: $C^{-1}S_t x = S_t C^{-1}x$;
- for every $t \geq 0$, let T_t be the closed operator defined by $T_t = C^{-1}S_t$ for

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$x \in D = D(T_t) := \{x \in X : S_t x \in R(C)\}$, then the family $(T_t)_{t\geq 0}$ is a semigroup of unbounded operators on $R(C^2)$.

Let $(S_t)_{t\geq 0}$ be an exponentially bounded C-semigroup on X. The linear operator A defined in $D(A) = \{x \in X : \lim\limits_{t\mapsto 0} \frac{S_t x - Cx}{t} \in R(C)\}$, by $Ax = C^{-1}(\lim\limits_{t\mapsto 0} \frac{S_t x - Cx}{t})$ for $x \in D(A)$ is the generator of a C-semigroup $(S_t)_{t\geq 0}$.

It is shown in [4] that the generator A of an exponentially bounded C-semigroup $(S_t)_{t\geq 0}$ has the following properties:

$$\forall x \in D(A),\ t \geq 0: \qquad S_t x - Cx = \int_0^t S_u Ax\, du \iff S_t x = Cx + \int_0^t S_u x\, du, \tag{1.5}$$

$$\forall x \in D(A): \qquad S_t x \in D(A) \quad \text{and} \quad AS_t x = S_t Ax; t \geq 0, \tag{1.6}$$

$$\forall x \in D(A): \qquad \frac{d}{dt} S_t x = AS_t x = S_t Ax, \tag{1.7}$$

$$\forall x \in D(A): \qquad \frac{d}{dt} CS_t x = CAS_t x = CS_t Ax, \tag{1.8}$$

If we put $L_\lambda(x) := \int_0^{+\infty} e^{-\lambda t} S_t x\, dt$ for $x \in X$, $\lambda > a$ in (1.4), then

$$(\lambda I - A) L_\lambda x = Cx \qquad \forall x \in X, \tag{1.9}$$

$$L_\lambda (\lambda I - A)x = Cx \qquad \forall x \in D(A), \tag{1.10}$$

$$\forall x \in X: \qquad \int_0^s S_r x\, dr \in D(A), \quad \int_0^s AS_r x\, dr = A(\int_0^s S_r x\, dr) = S_s x - Cx. \tag{1.11}$$

It is well known from [2, 4, 12] that a C-semigroup is connected with an abstract Cauchy problem (ACP). Suppose that X is a Banach space and A is a linear operator on X with domain $D(A)$ and range $R(A)$. Then for given $x \in D(A)$, the abstract Cauchy problem for A, with an initial value x, consists of finding a solution $u(t)$ to the initial value problem:

$$(ACP): \qquad \begin{cases} \frac{d}{dt} u(t,x) = Cu(t,x) & \text{for } t \geq 0, \\ u(0,x) = x. \end{cases}$$

Conversely, if A is a closed linear operator which commutes with C and if the ACP for A has a unique solution $u(t,x)$ such that $\|u(t,x)\| \leq Me^{at}\|C^{-1}x\|$ for every initial value $x \in D(A)$, then A generates an exponentially bounded C-semigroup, under some additional conditions on A. One excellent source for studying C-semigroups is [7]. For other results concerning this theory we refer to [3, 4, 9, 10, 12, 14].

1.2. **Saphar, and essentially Saphar spectrums.** A closed operator S is called a generalized inverse of A, if $R(A) \subset D(S), R(S) \subset D(A), ASA = A$ on $D(A)$ and $SAS = S$ on $D(S)$, which is equivalent to $R(A) \subset D(S), R(S) \subset D(A)$, and $ASA = A$ on $D(A)$. A closed operator A is called a Saphar operator if A has a generalized inverse and $N(A) \subset R^\infty(A) := \cap_{n\geq 0} R(A^n)$, ($N(A)$ is the null space of A).

The set $\sigma_{Sap} := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not a Saphar operator}\}$ is called the Saphar spectrum of A. It is known that $\partial\sigma(A) \subset \sigma_{Sap}(A) \subset (A)$ ($\partial\sigma(A)$ is the topological of $\sigma(A)$). For more details about this point, see [11].

2. Main Results

Throughout this section, let $(S_t)_{t\geq 0}$ be an exponentially bounded C-semigroup with generator A. For each $t \geq 0$ the operator $T_t = C^{-1}S_t$ is defined on

$$D(T_t) = \{x \in X : S_t x \in R(C)\}.$$

Remark 2.1 (see [6])**.** We have $R(C) \subset D(T_t)$ for every $t \geq 0$, and for all $x \in R(C^2)$ and $t, s \geq 0$ we have the following:

(1) $T_0 x = x$;
(2) $T_{t+s} x = T_t T_s x$;
(3) $t \mapsto T_t x$ is continuous.

Lemma 2.2. *Let $(S_t)_{t\geq 0}$ be a C-semigroup, with generator A, and f a continuous function from $[0, +\infty[$ to X. Then*

$$\lim_{h\mapsto 0} \frac{1}{h} \int_t^{t+h} S_u f(u) du = S_t f(t) \qquad \forall t \geq 0.$$

In particular,

$$\lim_{h\mapsto 0} \frac{1}{h} \int_0^{h} S_u f(u) du = C(f(t)).$$

Proof. For $t > 0$, by continuity of f and $t \mapsto S_t$ on $[0, +\infty[$, we see that the function $g(t) = S_t f(t)$ is continuous on $[0, +\infty[$. Define $F(x) = \int_t^{t+x} g(u)\, du$. Then

$$F'(0) = \lim_{h\mapsto 0} \frac{F(h) - F(0)}{h} = \lim_{h\mapsto 0} \int_t^{t+h} S_u f(u)\, du.$$

On the other hand, we have $F'(x) = g(t + x)$, so that $F'(0) = g(t) = S_t f(t)$. Finally,

$$\lim_{h\mapsto 0} \frac{1}{h} \int_t^{t+h} S_u f(u)\, du = S_t f(t) \qquad \forall t \geq 0. \qquad \square$$

Lemma 2.3. *Let $\lambda \in \mathbb{C}$, define a bounded linear operator,*

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-u)} S_u x\, du.$$

Then we have:

$$\forall t \geq 0,\ \forall x \in X: \qquad B_\lambda(t)x \in D(A), \tag{2.12}$$

$$\forall x \in X,\ \forall t \geq 0: \qquad (\lambda I - A)B_\lambda(t)x = (e^{\lambda t}C - S_t)x = (e^{\lambda t} - T_t)Cx, \tag{2.13}$$

$$\forall x \in D(A)\ \forall t \geq 0: \qquad B_\lambda(t)(\lambda I - A)x = (e^{\lambda t}C - S_t)x = (e^{\lambda t} - T_t)Cx. \tag{2.14}$$

Proof. Let $h > 0$. We have

$$\begin{aligned}
\frac{1}{h}(S_h(B_\lambda(t)x) - C(B_\lambda(t)x) &= \frac{1}{h}\Big(\int_0^t e^{\lambda(t-u)} S_h S_u x\, du - C(\int_0^t e^{\lambda(t-u)} S_u\, du)\Big) \\
&= \frac{1}{h}\Big(\int_0^t e^{\lambda(t-u)} C S_{h+u} x\, du - C(\int_0^t e^{\lambda(t-u)} S_u\, du)\Big) \\
&= \frac{1}{h} C\Big(\int_0^t e^{\lambda(t-u)} S_{h+u} x\, du - \int_0^t e^{\lambda(t-u)} S_u\, du\Big) \\
&= \frac{1}{h} C\Big(\int_h^{h+t} e^{\lambda(t-u+h)} S_u x\, du - \int_0^t e^{\lambda(t-u)} S_u\, du\Big) \\
&= \frac{1}{h} C\Big(\int_h^{0} e^{\lambda(t-u+h)} S_u x\, du + \int_0^t e^{\lambda(t-u+h)} S_u x\, du \\
&\qquad + \int_t^{h+t} e^{\lambda(t-u+h)} S_u x\, du - \int_0^t e^{\lambda(t-u)} S_u\, du\Big) \\
&= C\Big(\frac{e^{\lambda h} - 1}{h} \int_0^t e^{\lambda(t-u)} S_u x\, du + \frac{e^{\lambda h}}{h} \int_t^{t+h} e^{\lambda(t-u)} S_u x\, du
\end{aligned}$$

$$- \frac{e^{\lambda h}}{h} \int_0^h e^{\lambda(t-u)} S_u x \, du \Big).$$

Hence, by Lemma 2.2, we have that the limit $\lim_{h \mapsto 0} \frac{1}{h}(S_h(B_\lambda(t)x) - C(B_\lambda(t)x$ exists, and

$$\lim_{h \mapsto 0} \frac{1}{h}(S_h(B_\lambda(t)x) - C(B_\lambda(t)x)) = \lambda B_\lambda(t)x + S_t x - e^{\lambda t} Cx.$$

Thus we obtain that $B_\lambda(t)x \in D(A)$ and

$$A(B_\lambda(t)x) = \lambda B_\lambda(t)x + S_t x - e^{\lambda t} Cx.$$

Then $B_\lambda(t)x \in D(A)$, and

$$(\lambda I - A)B_\lambda(t)x = (e^{\lambda t} C - S_t)x = (e^{\lambda t} - T_t)Cx \qquad \forall t \geq 0.$$

Using (1.7), we get $AB_\lambda(t)x = B_\lambda(t)x$ on $D(A)$. So, if $x \in D(A)$ and $t \geq 0$, then

$$B_\lambda(t)(\lambda I - A)x = (e^{\lambda t} C - S_t)x = (e^{\lambda t} - T_t)Cx.$$

□

Lemma 2.4. *For all $t > 0, \lambda \in \mathbb{C}$ and $x \in D(T_t)$ we define $\alpha_\lambda(t)x = C^{-1}(\frac{1}{t} B_\lambda(t)x)$ and $\beta_\lambda(t)x = C^{-1}(\frac{1}{t} F_\lambda(t)x)$, with $F_\lambda(t)x = \int_0^t e^{-\lambda s} B_\lambda(s)x ds$. Then we have:*

$$(\lambda - A)\alpha_\lambda(t)x = e^{\lambda t} x - T_t x, \tag{2.15}$$

$$(\lambda - A)\beta_\lambda(t)x = x - e^{-\lambda t} \alpha_\lambda(t)x. \tag{2.16}$$

In addition, the operators $A, \alpha_\lambda(t)$ and $\beta_\lambda(t)$ commute pairwise.

Proof. We have

$$\begin{aligned} A(\alpha_\lambda(t)x) &= A\big(C^{-1}(\frac{1}{t} B_\lambda(t)x)\big) \\ &= C^{-1}\big(\frac{1}{t}(\lambda B_\lambda(t)x) + S_t x - e^{\lambda t} Cx\big) \\ &= \lambda \alpha_\lambda(t)x + C^{-1} S_t x - e^{\lambda t} x \\ &= \lambda \alpha_\lambda(t)x + T_t x - e^{\lambda t} x, \end{aligned}$$

so that $(\lambda - A)\alpha_\lambda(t)x = e^{\lambda t} x - T_t x$ and

$$\begin{aligned} A(\beta_\lambda(t)x) &= A\big(C^{-1}(\frac{1}{t} F_\lambda(t)x)\big) \\ &= C^{-1}\big(\frac{1}{t}(\lambda F_\lambda(t)x) + S_t x + \frac{1}{t} e^{-\lambda t} B_\lambda(t)x - Cx\big) \\ &= \lambda \beta_\lambda(t)x + e^{-\lambda t} \alpha_\lambda(t)x - x \end{aligned}$$

Thus we get $(\lambda - A)\beta_\lambda(t)x = x - e^{-\lambda t} \alpha_\lambda(t)x$. □

Lemma 2.5. *Let $(S_t)_{t \geq 0}$ be a C-semigroup on X, with a generator A. Then for all $t \geq 0$ and $n \in \mathbb{N}$, for every $x \in D(T_t)$ we have*

$$(\lambda I - A)^n \alpha_\lambda(t)^n x = (e^{\lambda t} - T_t)^n x, \tag{2.17}$$

$$\alpha_\lambda^n(t)(\lambda I - A)^n x = (e^{\lambda t} - T_t)^n x. \tag{2.18}$$

Proof. According to (2.13) and (2.14) we have

$$\begin{aligned} (e^{\lambda t} - T_t)x &= (\lambda I - A)\alpha_\lambda(t)x \qquad \forall x \in D(T_t),\ t > 0 \\ &= \alpha_\lambda(t)(\lambda I - A)x \qquad \forall x \in D(T_t),\ t > 0. \end{aligned}$$

Proceeding by induction on n, we finish the proof. □

Corollary 2.6. *For every $\lambda \in \mathbb{C}$ and $t > 0$ we have*

$$N(\lambda - A) \subset N(e^{\lambda t} - T_t), \tag{2.19}$$

$$R(e^{\lambda t} - T_t) \subset R(\lambda - A), \tag{2.20}$$

$$N(\lambda - A)^n \subset N(e^{\lambda t} - T_t)^n, \tag{2.21}$$

$$R^{\infty}(e^{\lambda t} - T_t) \subset R^{\infty}(\lambda - A). \tag{2.22}$$

Proof. The proof is a direct consequence of (2.15), (2.17), and (2.18). □

Theorem 2.7. *For all $t \geq 0$ we have that $exp(t\sigma_{Sap}(A)) \subset \sigma_{Sap}(T_t)$.*

Proof. It is sufficient to prove that $\{\lambda \in \mathbb{C}/exp(t\lambda) \in \rho_{Sap}(T_t)\} \subset \rho_{Sap}(A)$.

Let $\lambda \in \mathbb{C}$ be such that $e^{t\lambda} - T_t$ is a Saphar operator. Then $e^{t\lambda} - T_t$ has a generalized inverse and $N(e^{t\lambda} - T_t) \subset R^{\infty}(e^{t\lambda} - T_t)$. We show that $(\lambda - A)$ is a Saphar operator.

First let us prove that $N(\lambda - A) \subset R^{\infty}(\lambda - A)$. For his we will use (2.19) and (2.22). We have $N(\lambda - A) \subset N(e^{t\lambda} - T_t) \subset R^{\infty}(e^{t\lambda} - T_t) \subset R^{\infty}(\lambda - A)$.

We justify that $\lambda - A$ has a generalized inverse.

Since $e^{t\lambda} - T_t$ has a generalized inverse, there exists a $S \in B(X)$ such that

$$(e^{t\lambda} - T_t)S(e^{t\lambda} - T_t) = (e^{t\lambda} - T_t).$$

Using (2.16), for every $t > 0$ we have

$$x = (\lambda - A)\beta_\lambda(t)x + e^{-t\lambda}\alpha_\lambda(t)x \quad \forall x \in D(T_t)$$

and

$$\begin{aligned}(\lambda - A) &= (\lambda - A)\beta_\lambda(t)(\lambda - A) + e^{-t\lambda}\alpha_\lambda(t)(\lambda - A)\\ &= (\lambda - A)\beta_\lambda(t)(\lambda - A) + e^{-t\lambda}(e^{t\lambda} - T_t)\\ &= (\lambda - A)\beta_\lambda(t)(\lambda - A) + e^{-t\lambda}(e^{t\lambda} - T_t)S(e^{t\lambda} - T_t)\\ &= (\lambda - A)\beta_\lambda(t)(\lambda - A) + e^{-t\lambda}(\lambda - A)\alpha_\lambda(t)S\alpha_\lambda(t)(\lambda - A)\\ &= (\lambda - A)[\beta_\lambda(t) + e^{-t\lambda}\alpha_\lambda(t)S\alpha_\lambda(t)](\lambda - A).\end{aligned}$$

Therefore, we obtain that $(\lambda - A)$ has a generalized inverse, that is $\beta_\lambda(t) + e^{-t\lambda}\alpha_\lambda(t)S\alpha_\lambda(t)$. □

For subspaces M, N of X, we write $M \subset_e N$ (M is essentially contained in N) if there exists a finite-dimensional subspace $F \subset X$ such that $M \subset N + F$.

Let A be a closed operator. Then A is called an essentially Saphar operator if A has a generalized inverse and $N(A) \subset_e R^{\infty}(A)$.

The set

$$\sigma_{e-Sap}(A) := \{\lambda \in \mathbb{C}/\lambda - A \text{ is not essentially Saphar }\}$$

is called the essentially Saphar spectrum of A.

Corollary 2.8. *For all $t \geq 0$ we have that*

$$exp(t\sigma_{e-Sap}(A)) \subset \sigma_{e-Sap}(T_t).$$

Proof. It is sufficient to prove that

$$\{\lambda \in \mathbb{C}/exp(t\lambda) \in \rho_{e-Sap}(T_t)\} \subset \rho_{e-Sap}(A).$$

Let $\lambda \in \mathbb{C}$ be such that $e^{t\lambda} - T_t$ is an essentially Saphar operator. Then $e^{t\lambda} - T_t$ has a generalized inverse and $N(e^{t\lambda} - T_t) \subset R^{\infty}(e^{t\lambda} - T_t) + F$ for some subspace F of X. Let us prove that $(\lambda - A)$ is essentially Saphar operator.

Let us show that $N(\lambda - A) \subset R^{\infty}(\lambda - A) + F$. From (2.19) we have that $N(\lambda - A) \subset N(e^{t\lambda} - T_t) \subset R^{\infty}(e^{t\lambda} - T_t) + F \subset R^{\infty}(\lambda - A) + F$ by (2.22). And we know that if $(e^{t\lambda} - T_t)$ has a generalized inverse, then $(\lambda - A)$ also has a generalized ineverse. □

Let A be a closed operator on X. Then A is called a Kato operator if $R(A)$ is closed and $N(A) \subset R^{\infty}(A)$. Recall that the Kato resolvent is

$$\rho_{\gamma}(A) = \{\lambda \in \mathbb{C}/\lambda - A \text{ is a Kato operator }\},$$

and we denote by $\sigma_{\gamma}(A) := \mathbb{C}\backslash\rho_{\gamma}(A)$ the Kato spectrum of A. It is well known that $\rho_{\gamma}(A)$ is an open subset of $\mathbb{C}$. If $R(A)$ is closed and $N(A) \subset_e R^{\infty}(A)$, we say that A is an essential Kato operator. We denote

$$\sigma_{e-\gamma}(A) = \{\lambda \in \mathbb{C}/\lambda - A \text{ is not an essential Kato operator }\}$$

the essential Kato spectrum of A, for more the detail see [5].

Corollary 2.9. *For all $t \geq 0$ we have*

$$exp(t\sigma_{\gamma}(A)) \subset \sigma_{\gamma}(T_t), \qquad exp(t\sigma_{e-\gamma}(A)) \subset \sigma_{e-\gamma}(T_t).$$

Proof. It suffices to show that if $R(e^{t\lambda} - T_t)$ is closed, then $R(\lambda - A)$ is closed.

Let $y \in \overline{R(\lambda - A)}$. Then there exists $y_n \in R(\lambda - A)$ such that $y_n \to y$. Let $x_n \in D(A)$ be such that $(\lambda - A)x_n = y_n$. From (2.16) we have

$$x_n = (\lambda - A)\beta_{\lambda}(t)(x_n) + e^{-t\lambda}\alpha_{\lambda}(t)(x_n)$$

and

$$y_n = (\lambda - A)\beta_{\lambda}(t)(y_n) + e^{-t\lambda}\alpha_{\lambda}(t)(y_n) = (\lambda - A)\beta_{\lambda}(t)(y_n) + e^{-t\lambda}\alpha_{\lambda}(t)(\lambda - A)(x_n),$$

and by (2.15) we obtain

$$\begin{aligned} y_n &= (\lambda - A)\beta_{\lambda}(t)(y_n) + e^{-t\lambda}(e^{t\lambda} - T_t)(x_n) \\ &\implies e^{-t\lambda}(e^{t\lambda} - T_t)(x_n) = y_n - (\lambda - A)\beta_{\lambda}(t)(y_n) \to y - (\lambda - A)\beta_{\lambda}(t)(y) \\ &\implies y - (\lambda - A)\beta_{\lambda}(t)(y) \in \overline{R(e^{t\lambda} - T_t)} = R(e^{t\lambda} - T_t). \end{aligned}$$

Therfore there exists $z \in D(T_t)$ such that

$$\begin{aligned} y - (\lambda - A)\beta_{\lambda}(t)(y) &= (e^{t\lambda} - T_t)(z) \\ &\implies y = (e^{t\lambda} - T_t)(z) + (\lambda - A)\beta_{\lambda}(t)(y) \\ &= (\lambda - A)\alpha_{\lambda}(t)(x) + (\lambda - A)\beta_{\lambda}(t)(z) \\ &= (\lambda - A)(\alpha_{\lambda}(t)(x) + \beta_{\lambda}(t)(z)) \in R(\lambda - A) \end{aligned}$$

Finally, we have that $R(\lambda - A)$ is closed. ☐

Let $A \in B(X)$. The ascent of A is defined by

$$asc(A) = \min\{n \in \mathbb{N}/N(A^n) = N(A^{n+1})\},$$

and if no such n exists we put $asc(A) = \infty$. The descent of A is defined by

$$dsc(A) = \min\{n \in \mathbb{N}/R(A^n) = R(A^{n+1})\},$$

if no such n exists we put $dsc(A) = \infty$. It is shown in [8], that if A is bounded and if both $asc(A)$ and $dsc(A)$ are finite, then $asc(A) = dsc(A) = p$, and we have $X = N(A^p) \oplus R(A^p)$. Clearly, $asc(A) = 0$ if and only if A is injective and $dsc(A) = 0$ if and only if A is surjective. The ascent and descent spectrum are respectively defined by

$$\sigma_{asc}(A) := \{\lambda \in \mathbb{C}/a(\lambda I - A) = \infty\} \quad \text{and} \quad \sigma_{desc} := \{\lambda \in \mathbb{C}/d(\lambda I - A) = \infty\}.$$

For more detailed information on ascent and descent of an operator, we refer to [8, 1, 13].

Problem 2.10. Do these results hold true for other spectra such as the ascent and descent spectrum?

3. Conclusions

The objective of this article is to study the relation between the Saphar spectrum, the essential Saphar spectrum, the Kato spectrum, and the essential Kato spectrum of the generator of a C-semigroup $(S_t)_{t\geq 0}$ and those of the family $T_t = C^{-1}S_t$. We showed that if $(S_t)_{t\geq 0}$ is a C-semigroup with generator A and if $T_t = C^{-1}S_t$ then

$$exp(t\sigma_*(A)) \subset \sigma_*(T_t)$$

for all $t \geq 0$ with $\sigma_* \in \{\sigma_{Sap}, \sigma_{e-Sap}, \sigma_\gamma, \sigma_{e-\gamma}\}$.

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