Methods of Functional Analysis and Topology
Vol. 28 (2022), no. 2, pp. 176–188
https://doi.org/10.31392/MFAT-npu26_2.2022.10

ON SOME SUPERCRITICAL PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN OPERATOR

JINGUO ZHANG

Abstract. In this paper, a fractional Laplacian equation is investigated, which involve critical or supercritical Sobolev exponent as follows:

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-2}u + |u|^{r-2}u + \mu |u|^{q-2}u, & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplacian operator with $0 < s < 1$, $1 < p < 2 < r < 2_s^* \le q$, $2_s^* := \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, λ, $\mu \ge 0$ are parameters and $\Omega \subseteq \mathbb{R}^N (N > 2s)$ is a bounded domain with smooth boundary $\partial\Omega$. By using variational methods, truncation and Moser iteration techniques, we show that the problem has at least two nontrivial solutions.

У цій роботі досліджується наступне дробове рівняння Лапласа, яке включають критичний або надкритичний показник Соболєва:

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-2}u + |u|^{r-2}u + \mu |u|^{q-2}u, & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

де $(-\Delta)^s$ — дробовий оператор Лапласа з $0 < s < 1$, $1 < p < 2 < r < 2_s^* \le q$, $2_s^* := \frac{2N}{N-2s}$ це дробовий критичний показник Соболєва, λ, $\mu \ge 0$ є параметри та $\Omega \subseteq \mathbb{R}^N (N > 2s)$ — обмежена область з гладкою границею $\partial\Omega$. За допомогою варіаційних методів, скорочення та ітераційних методів Мозера показано, що задача має принаймні два нетривіальних розв'яків.

1. Introduction

In this article, we consider the following fractional Laplacian equation involving the critical or supercritical nonlinearities term

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-2}u + |u|^{r-2}u + \mu |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < s < 1$, $1 < p < 2 < r < 2_s^* \le q$ and $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, λ and μ are nonnegative parameters.

The powers $(-\Delta)^s$ of the Laplacian operator $-\Delta$, in a bounded domain Ω with zero Dirichlet boundary data, are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. Let (ρ_k, φ_k) be the eigenvalues and the corresponding eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary value data. Then (ρ_k^s, φ_k) are the eigenvalues and the corresponding eigenfunctions of $(-\Delta)^s$ with zero Dirichlet boundary value condition. So, the fractional Laplacian $(-\Delta)^s$ is well-defined in the space of the functions

$$X_0^s(\Omega) = \{u = \sum_{k=1}^{\infty} a_k \varphi_k \in L^2(\Omega) : \|u\|_{X_0^s} := \Big(\sum_{k=1}^{\infty} a_k^2 \rho_k^s\Big)^{\frac{1}{2}} < \infty\},$$

2020 *Mathematics Subject Classification.* 35J60,47J30.
Keywords. Fractional Laplacian; Variational methods; Moser iteration; Supercritical exponent.

and, as a consequence,

$$(-\Delta)^s u = \sum_{i=1}^{\infty} a_k \rho_k^s \varphi_k.$$

Motivated by the work of Caffarelli and Silvestre in [7], several authors have considered an equivalent definition of the operator $(-\Delta)^s$ in a bounded domain with zero Dirichlet boundary condition, see [4, 8, 18, 3, 1, 2, 7, 6, 10, 11, 16]. Associated to the bounded domain, let us consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty)$ and the lateral boundary of the cylinder $\partial_L \mathcal{C}_\Omega = \partial\Omega \times (0,\infty)$. For a functional $u \in X_0^s(\Omega)$, we define the s-harmonic extension $U = E_s(u)$ to the cylinder $\mathcal{C}_\Omega$ as a solution to the problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ U = u & \text{on } \Omega \times \{0\}. \end{cases}$$

The extension function U belongs to the space

$$H_{0,L}^1(\mathcal{C}_\Omega) = \{w \in L^2(\mathcal{C}_\Omega) : w = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \ \|w\|_{H_{0,L}^1} := \Big(k_s \int_{\mathcal{C}_\Omega} t^{1-2s} |\nabla w|^2 dxdt\Big)^{\frac{1}{2}} < \infty\}.$$

Moreover, the relevance of the extension function U is that it is related to the fractional Laplacian of the original function u through the formula

$$(-\Delta)^s u(x) = \frac{\partial U}{\partial \nu^s} := -k_s \lim_{t\to 0^+} t^{1-2s} \frac{\partial U}{\partial t}(x,t),$$

where $k_s = \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$ is a normalization constant. With this constant we have that the extension operator is an isometry between $X_0^s(\Omega)$ and $H_{0,L}^1(\mathcal{C}_\Omega)$. That is

$$\|E_s(u)\|_{H_{0,L}^1} = \|u\|_{X_0^s}, \qquad \forall u \in X_0^s(\Omega).$$

With this extension, we can reformulate our problem (1.1) as

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial U}{\partial \nu^s} = \lambda |u|^{p-2}u + |u|^{r-2}u + \mu |u|^{q-2}u & \text{in } \Omega, \end{cases} \tag{1.2}$$

where the function $u = U(\cdot,0)$ defined in the sense of traces, belongs to the space $X_0^s(\Omega)$. Obviously, the equation (1.2) is a local problem.

An energy solution to problem (1.2) is a function $U \in H_{0,L}^1(\mathcal{C}_\Omega)$ such that

$$k_s \int_{\mathcal{C}_\Omega} t^{1-2s} \nabla U \nabla V dxdt = \lambda \int_\Omega |u|^{p-2}uvdx + \int_\Omega |u|^{r-2}uvdx + \mu \int_\Omega |u|^{q-2}uvdx$$

for all $V \in H_{0,L}^1(\mathcal{C}_\Omega)$, where $v = V(x,0)$. For any energy solution $U \in H_{0,L}^1(\mathcal{C}_\Omega)$ to problem (1.2), the function $u = U(\cdot,0)$ belongs to the space $X_0^s(\Omega)$ and is an energy solution to problem (1.1). The converse is also true. Therefore, both formulations are equivalent.

The associated energy functional to problem (1.2) is

$$I(U) = \frac{k_s}{2} \int_{\mathcal{C}_\Omega} t^{1-2s} |\nabla U|^2 dxdt - \frac{\lambda}{p} \int_\Omega |u|^p dx - \frac{1}{r} \int_\Omega |u|^r dx - \frac{\mu}{q} \int_\Omega |u|^q dx.$$

Clearly, if U is a critical point of I in $H_{0,L}^1(\mathcal{C}_\Omega)$, the function $u = U(\cdot,0)$ is a weak solution of (1.1) in $X_0^s(\Omega)$. Therefore, in the sequel, and in view of the above equivalence, we will use both formulations of the problem, in Ω or in $\mathcal{C}_\Omega$, whenever we may take some advantage. In particular, we will use the extension version (1.2) when dealing with the fractional operator acting on products of functions, since it is not clear how to calculate this action.

In this paper, we study the existence and multiplicity of solutions for the problem with subcritical term and critical or supercritical term. In our problem, the first difficulty lie in that the fractional Laplacian operator $(-\Delta)^s$ is nonlocal, and this makes some calculation difficult. To overcome this difficulty, we do not work on the problem (1.1) directly, and we transform the nonlocal problem into a local problem by s-harmonic extension. The second difficulty lies in which problem (1.2) is supercritical, and we can not use directly the variational techniques because the corresponding energy functional is not well-defined on the space $H^1_{0,L}(\mathcal{C}_\Omega)$. To overcome this difficulty, one usually uses the truncation and the Moser iteration. This idea has been widely applied in the supercritical Laplacian problem in the past decades, see [9, 14, 12, 13] and references therein.

Now, we are ready to state the main results of this paper.

Theorem 1.1. *Assume $1 < p < 2 < r < 2^*_s \le q$, then (1.1) has at least two nontrivial solutions if λ and μ are sufficiently small.*

For the general problem

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-2}u + |u|^{r-1}u + \mu h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N > 2s)$, $1 < p < 2 < r < 2^*_s$ and the general perturbation $h(x,u)$ satisfies

(h) $|h(x,u)| \le C_0\left(1 + |u|^{q-1}\right)$, where $q \ge 2^*_s$ and $C_0 > 0$ is a constant.

We also have the following result similar to Theorem 1.1.

Theorem 1.2. *Let h satisfy (h) and $1 < p < 2 < r < 2^*_s$. Then the problem (1.3) has at least two nontrivial solutions for λ and μ small enough.*

This article is organized as follows. In Section 2, we consider a truncated problem (2.7) and obtain two nontrivial solutions by using variational methods. In Section 3, we finish the proof of Theorems 1.1 and 1.2 by demonstrating that solutions of (2.7) are actually solutions of the original problem (1.2), this reduces to an L^∞ estimate.

For convenience we fix some notations. The Lebesgue space $L^q(\Omega)$ $(1 < q < \infty)$ with the norm $\|u\|_{L^q} = \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}}$; C or $C_i (i = 1, 2, \cdots,)$ denote different positive constants; S is the best Sobolev embedding constant

$$S = \inf_{U \in H^1_{0,L}(\mathcal{C}_\Omega)\setminus\{0\}} \frac{k_s \int_\Omega t^{1-2s}|\nabla U|^2 dx dt}{\left(\int_\Omega |u|^{2^*_s} \mathrm{d}x\right)^{\frac{2}{2^*_s}}}, \quad \text{where } u(x) = U(x,0). \tag{1.4}$$

2. Truncated problem

One of the main difficulty to prove the existence solutions of problem (1.2) by using variational methods is that I does not satisfying the (PS) condition for large energy level for $q = 2^*_s$ and I is not well defined on $H^1_{0,L}(\mathcal{C}_\Omega)$ for $q > 2^*_s$.

Let $K > 0$ be a real number whose value will be fixed later. Following the idea in [15, 17, 20], define the following truncate function

$$g_K(u) = \begin{cases} |u|^{q-2}u, & \text{if } 0 \le |u| \le K, \\ K^{q-\theta}|u|^{\theta-2}u, & \text{if } |u| > K, \end{cases} \tag{2.5}$$

where $\theta \in (2, 2^*_s)$ and $\theta \ge r$. Thus, g_K satisfies

$$|g_K(u)| \le K^{q-\theta}|u|^{\theta-1}, \quad \forall\, u \in \mathbb{R}. \tag{2.6}$$

Now, we investigate the truncated problem

$$\begin{cases}(-\Delta)^s u = \lambda|u|^{p-2}u + |u|^{r-2}u + \mu g_K(u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.7}$$

where $u = U(x, 0)$.

Define in $X_0^s(\Omega)$ the corresponding energy functional of the problem (2.7) by

$$I_K(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{1}{r}\int_\Omega |u|^r dx - \mu\int_\Omega G_K(u)dx,$$

where

$$G_K(u) = \int_0^u g_K(t)dt = \begin{cases}\frac{1}{q}|u|^q, & \text{if } |u| \le K,\\ \frac{1}{\theta}K^{q-\theta}|u|^\theta, & \text{if } |u| \ge K.\end{cases} \tag{2.8}$$

Then G_K satisfies

$$G_K(u) \le \frac{1}{\theta}K^{q-\theta}|u|^\theta \tag{2.9}$$

and

$$g_K(u)u \ge rG_K(u), \quad \forall\, u \in \mathbb{R}. \tag{2.10}$$

From (2.6), we get g_K is a superlinear function with subcritical growth, then $I_K \in C^1(X_0^s(\Omega), \mathbb{R})$. Let $I_K'(u)$ denote the Frechet derivative of I_K at u. A function $u \in X_0^s(\Omega)$ is said to be a nontrivial solution of the problem (2.7) if

$$u \ne 0, \quad \langle I_K'(u), v\rangle = 0, \quad \forall v \in X_0^s(\Omega).$$

Remark 2.1. The original problem (1.1) is critical or supercritical, after truncation, it becomes subcritical and the functional $I_K(u) \in C^1$ is well-definedıo This fact allows us to use the usual minimax methods.

By using sub-super solution principle, we get one nontrivial solution for (2.7).

Theorem 2.2. *There exist two constants λ_0, $\mu_0 > 0$ such that for all λ, μ with $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$, problem (2.7) has one nontrivial solutions.*

Proof. Let e denote the nonnegative solution of

$$\begin{cases}(-\Delta)^s e = 1 & \text{in } \Omega,\\ e = 0 & \text{on } \partial\Omega.\end{cases}$$

Then $e \in C_0^\infty(\Omega)$ and $\|e\|_{L^\infty} \le C$ for some positive constant $C > 0$. Since $1 < p < 2 < r < 2_s^*$ and $2 < \theta < 2_s^*$, we can find $\lambda_1 > 0$ and $\mu_1 > 0$ such that for all $\lambda \in (0, \lambda_1)$ and $\mu \in (0, \mu_1)$ there exits $M = M(\lambda, \mu) > 0$ satisfying

$$M \ge \lambda M^{p-1}\|e\|_{L^\infty}^{p-1} + M^{r-1}\|e\|_{L^\infty}^{r-1} + \mu K^{q-\theta}M^{\theta-1}\|e\|_{L^\infty}^{\theta-1}.$$

Thus, Me satisfies

$$\begin{aligned}(-\Delta)^s(Me) &= [(-\Delta)^s e]M\\ &= M \ge \lambda M^{p-1}\|e\|_{L^\infty}^{p-1} + M^{r-1}\|e\|_{L^\infty}^{r-1} + \mu K^{q-\theta}M^{\theta-1}\|e\|_{L^\infty}^{\theta-1}\\ &\ge \lambda(Me)^{p-1} + (Me)^{r-1} + \mu g_K(Me),\end{aligned}$$

and $Me = 0$ on $\partial\Omega$, which implies that Me is a supersolution of equation (2.7). On the other hand, let (γ_1, φ_1) be the first eigenvalue and corresponding eigenfunction of the fractional Laplacian operator $(-\Delta)^s$ in Ω with zero Dirichlet boundary value on $\partial\Omega$. Then for all $\varepsilon > 0$ small enough and λ, $\mu > 0$ such that

$$(-\Delta)^s(\varepsilon\varphi_1) = \gamma_1(\varepsilon\varphi_1) \le \lambda(\varepsilon\varphi_1)^{p-1} + (\varepsilon\varphi_1)^{r-1} + \mu g_K(\varepsilon\varphi_1).$$

This implies that $\varepsilon\varphi_1$ is a subsolution of (2.7). Taking ε possibly smaller, we also have $\varepsilon\varphi_1 < Me$. It follows that (2.7) has a solution $\varepsilon\varphi_1 \le u_1 \le Me$ whenever $\lambda < \lambda_0$ and $\mu < \mu_0$. Moreover, from [4] we obtain that u_1 is a local minimum of I_K in the C^1-topology, hence a local minimum for I_K in the $X_0^s(\Omega)$-topology. □

Next, we look for a second solution of (2.7) by mountain mass theorem. From the s-harmonic extension, the problem (2.7) can be reformulated to the following local problem

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial U}{\partial \nu^s} = \lambda |u|^{p-2}u + |u|^{r-2}u + \mu g_K(u) & \text{in } \Omega, \end{cases} \tag{2.11}$$

where $u(x) = U(x,0)$ defined in the sense of traces. The associated functional in $H^1_{0,L}(\mathcal{C}_\Omega)$ is

$$\mathcal{I}_K(u) = \frac{k_s}{2}\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U(x,t)|^2 dxdt - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{1}{r}\int_\Omega |u|^r dx - \mu \int_\Omega G_K(u)dx.$$

Lemma 2.3. *The functional $\mathcal{I}_K$ satisfies $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof. Let $\{U_n\} \subset H^1_{0,L}(\mathcal{C}_\Omega)$ be a Palais-Smale sequence of $\mathcal{I}_K$ at level c, that is,

$$\mathcal{I}_K(U_n) \to c \quad \text{and} \quad \mathcal{I}_K'(U_n) \to 0 \quad \text{as} \quad n \to \infty.$$

By Sobolev embedding theorem and (2.10), we get

$$\begin{aligned} c + o_n(\|U_n\|_{H^1_{0,L}}) &= J_K(U_n) - \frac{1}{r}\langle J_K'(U_n), U_n\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{r}\Big)k_s \int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U_n|^2 dxdt - \Big(\frac{\lambda}{p} - \frac{\lambda}{r}\Big)\int_\Omega |u_n|^p dx \\ &\quad + \frac{\mu}{r}\int_\Omega \Big[g_K(u_n)u_n - rG_K(u_n)\Big]dx \\ &\ge \frac{r-2}{2r}k_s \int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U_n|^2 dxdt - \lambda\frac{r-p}{pr}\int_\Omega |u_n|^p dx \\ &\ge \frac{r-2}{2r}\|U_n\|^2_{H^1_{0,L}} - \lambda \frac{r-p}{pr} S^p \|U_n\|^p_{H^1_{0,L}}, \end{aligned} \tag{2.12}$$

where $u_n = U_n(x,0)$, $S > 0$ is the Sobolev constant and $1 < p < 2 < r$. Thus, $\{U_n\}$ is bounded in $H^1_{0,L}(\mathcal{C}_\Omega)$.

Up to a subsequence, there exists $U \in H^1_{0,L}(\mathcal{C}_\Omega)$ such that $U_n \rightharpoonup U$ weakly in $H^1_{0,L}(\mathcal{C}_\Omega)$, $u_n \to u$ strongly in $L^\alpha(\Omega)$ $(1 \le \alpha < 2_s^*)$, and $u_n(x) \to u(x)$ a.e in Ω, where $u_n = U_n(x,0)$ and $u = U(x,0)$. Then

$$\int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \to 0,$$
$$\int_\Omega (|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u)dx \to 0,$$

and

$$\int_\Omega \left[g_K(u_n) - g_K(u)\right](u_n - u)\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

So, we conclude by computing

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'_K(U_n) - \mathcal{I}'_K(U),\, U_n - U\rangle \\
&= k_s \int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla(U_n - U)|^2 dxdt - \lambda \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \\
&\quad - \int_\Omega (|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u)dx \\
&\quad - \mu \int_\Omega \left[g_K(u_n) - g_K(u)\right](u_n - u)dx \\
&= \|U_n - U\|^2_{H^1_{0,L}} + o(1),
\end{aligned}$$

which shows that $U_n \to U$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$. This proves Lemma 2.3. □

Lemma 2.4. *For every $U \in H^1_{0,L}(\mathcal{C}_\Omega) \setminus \{0\}$, we have*

$$\mathcal{I}_K(tU) \to -\infty \quad as \quad t \to +\infty.$$

Proof. Let $U \in H^1_{0,L}(\mathcal{C}_\Omega) \setminus \{0\}$ and $t > 0$ we have

$$\begin{aligned}
\mathcal{I}_K(tU) &= \frac{t^2}{2}\|U\|^2_{H^1_{0,L}} - \frac{\lambda t^p}{p}\int_\Omega |u|^p dx - \frac{t^r}{r}\int_\Omega |u|^r dx - \mu \int_\Omega G_K(tu)dx \\
&= \frac{t^2}{2}\|U\|^2_{H^1_{0,L}} - \frac{\lambda t^p}{p}\int_\Omega |u|^p dx - \frac{t^r}{r}\int_\Omega |u|^r dx \\
&\quad - \frac{\mu t^q}{q}\int_{\{|tu|\le K\}} |u|^q dx - \frac{\mu t^\theta K^{q-\theta}}{\theta}\int_{\{|tu|\ge K\}} |u|^\theta dx,
\end{aligned}$$

where $u = U(x,0)$. Since $\int_{\{|tu|\le K\}} |u|^q dx \to 0$ as $t \to +\infty$ and $1 < p < 2 < r \le \theta < 2^*_s$, it follows that $\mathcal{I}_K(tU) \to -\infty$ as $t \to +\infty$. The proof is completed. □

Theorem 2.5. *Let $1 < p < 2 < r < 2^*_s$, then the problem (2.7) has a nontrivial solution $U_2 \in H^1_{0,L}(\mathcal{C}_\Omega)$.*

Proof. From Lemmas 2.3 and 2.4, by the mountain pass theorem there exists a $U_2 \in H^1_{0,L}(\mathcal{C}_\Omega)\setminus\{0\}$ such that $\mathcal{I}_K(U_2) = c_m$, where

$$c_m = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_K(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], H^1_{0,L}(\mathcal{C}_\Omega)) : \ \gamma(0) = U_1,\ \mathcal{I}_K(\gamma(1)) < 0\}.$$

Thus, the problem (2.7) has a nontrivial solution U_2 and $U_2 \ne U_1$. □

Lemma 2.6. *The solutions for problem (2.7) obtained in Theorems 2.2 and 2.5 are bounded in $H^1_{0,L}(\mathcal{C}_\Omega)$, that is, there exists $C_0 > 0$ independent of μ such that*

$$\|U_i\|_{H^1_{0,L}} \le C_0, \quad \forall\, i = 1,\ 2.$$

Proof. Let c_m be the mountain pass level for $\mathcal{I}_K$ obtained in Theorem 2.5 and $u_i(x) = U_i(x,0)$ $(i = 1,\, 2)$. Then from (2.10) we have

$$\begin{aligned}
c_m &\ge \mathcal{I}_K(U_i) \\
&= \mathcal{I}_K(U_i) - \frac{1}{r}\langle \mathcal{I}'_K(U_i), U_i\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{r}\Big)\|U_i\|^2_{H^1_{0,L}} - \lambda\Big(\frac{1}{p} - \frac{1}{r}\Big)\int_\Omega |u_i|^p dx + \frac{\mu}{r}\int_\Omega \Big[g_K(u_i)u_i - rG_K(u_i)\Big]dx \\
&\ge \frac{r-2}{2r}\|U_i\|^2_{H^1_{0,L}} - \lambda\frac{r-p}{pr}\int_\Omega |u_i|^p dx
\end{aligned}$$

$$\geq \frac{r-2}{2r}\|U_i\|^2_{H^1_{0,L}} - \lambda\frac{r-p}{pr}S^p\,\|U_i\|^p_{H^1_{0,L}}.$$

Since $1 < p < 2 < r$, we infer that there exists $C_0 > 0$ which is independent of μ such that $\|U_i\|_{H^1_{0,L}} \leq C_0$ for all $i = 1,\, 2$. □

Remark 2.7. If these two nontrivial solution satisfy $|u_i| \leq K$ for all $i = 1,\, 2$, then in view of the definition of g_K, we have $g_K(u_i) = \mu|u_i|^{q-2}u_i$, and there U_i, $i = 1,\, 2$, are also solutions of the original problem (1.2). This implies that problem (1.1) has at least two solutions $u_i = U_i(x, 0)$ $(i = 1,\, 2)$. To show this, we only need to prove $\|u_i\|_{L^\infty} \leq K$ for all $i = 1, 2$.

3. Proof of the main result

To prove Theorem 1.1, we only need to show that solutions of (2.7) are actually bounded by some K. Our approach is a variant of Moser iteration technique inspired by [5, 15, 20, 19].

Proof of Theorem 1.1. For convenience, set $U := U_i$, $i = 1, 2$, $u(x) = U(x, 0)$, and $\varphi(x) = \Phi(x, 0)$. Since $U \in H^1_{0,L}(\mathcal{C}_\Omega)$ is a weak solution of (2.7), for any $\Phi \in H^1_{0,L}(\mathcal{C}_\Omega)$, we get

$$\begin{aligned} k_s\int_{\mathcal{C}_\Omega} t^{1-2s}\nabla U\nabla\Phi dxdt &= \lambda\int_\Omega |u|^{p-2}u\varphi dx \\ &\quad + \int_\Omega |u|^{r-2}u\varphi dx + \mu\int_\Omega g_K(u)\varphi dx. \end{aligned} \tag{3.13}$$

Set $U_+ = \max\{U, 0\}$, $U_- = -\min\{U, 0\}$. Then $|U| = U_+ + U_-$. We can argue with the positive and negative part of U separately.

We first deal with U_+. For each $L > 0$, let us define the following functions:

$$U_L(x,t) = \begin{cases} U_+, & \text{if} \quad U_+ \leq L, \\ L, & \text{if} \quad U_+ > L, \end{cases}$$

and

$$Z_L = U_L^{2(\beta-1)}U_+, \qquad W_L = U_L^{\beta-1}U_+,$$

where $\beta > 1$ will be fixed later.

Taking Z_L as a test function in (3.13), we obtain

$$\begin{aligned} k_s\int_{\mathcal{C}_\Omega} t^{1-2s}\nabla U\nabla Z_L dxdt &= \lambda\int_\Omega |u|^{p-2}u\, z_L dx \\ &\quad + \int_\Omega |u|^{r-2}u\, z_L dx + \mu\int_\Omega g_K(u)\, z_L dx, \end{aligned} \tag{3.14}$$

where $z_L(x) = Z_L(x, 0)$.

The left-hand side of the above equality (3.14) is

$$\begin{aligned} &\int_{\mathcal{C}_\Omega} t^{1-2s}\nabla U\nabla Z_L dxdt \\ &= \int_{\mathcal{C}_\Omega} t^{1-2s}\nabla(U_+ - U_-)\nabla(U_L^{2(\beta-1)}U_+)dxdt \\ &= \int_{\mathcal{C}_\Omega} t^{1-2s}(\nabla U_+ - \nabla U_-)(U_L^{2(\beta-1)}\nabla U_+ + 2(\beta-1)U_L^{2(\beta-1)-1}U_+\nabla U_L)dxdt \\ &= \int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U_+|^2U_L^{2(\beta-1)}dxdt \\ &\quad + 2(\beta-1)\int_{\mathcal{C}_\Omega} t^{1-2s}U_+U_L^{2(\beta-1)-1}\nabla U_+\nabla U_L dxdt. \end{aligned} \tag{3.15}$$

From the definition of U_L, we have

$$
\begin{aligned}
&2(\beta-1)\int_{\mathcal{C}_\Omega} t^{1-2s}U_+U_L^{2(\beta-1)-1}\nabla U_+\nabla U_L dxdt\\
&=2(\beta-1)\int_{\mathcal{C}_\Omega\cap\{0\le U\le L\}} t^{1-2s}U_L^{2(\beta-1)-1}U_+\nabla U_+\nabla U_L dxdt\\
&=2(\beta-1)\int_{\mathcal{C}_\Omega\cap\{0\le U\le L\}} t^{1-2s}|\nabla U_+|^2U_+^{2(\beta-1)}dxdt\\
&\ge 0.
\end{aligned}
\tag{3.16}
$$

Thus, it follows from (3.15) and (3.16) that

$$
\int_{\mathcal{C}_\Omega} t^{1-2s}\nabla U\nabla Z_L dxdt \ge \int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U_+|^2U_L^{2(\beta-1)}dxdt. \tag{3.17}
$$

On the other hand,

$$
\begin{aligned}
&\lambda\int_\Omega |u|^{p-2}u\, z_L dx+\int_\Omega |u|^{r-2}u\, z_L dx+\mu\int_\Omega g_K(u)\, z_L dx\\
&=\lambda\int_\Omega (u_++u_-)^{p-2}(u_+-u_-)u_L^{2(\beta-1)}u_+dx\\
&\quad+\int_\Omega (u_++u_-)^{r-2}(u_+-u_-)u_L^{2(\beta-1)}u_+dx+\mu\int_\Omega g_K(u)u_L^{2(\beta-1)}u_+dx\\
&\le\lambda\int_\Omega u_+^pu_L^{2(\beta-1)}\,dx+\int_\Omega u_+^ru_L^{2(\beta-1)}dx+\mu K^{q-\theta}\int_\Omega u_+^\theta u_L^{2(\beta-1)}dx,
\end{aligned}
\tag{3.18}
$$

where $u_\pm=U_\pm(x,0)$, $u_L=U_L(x,0)$ and $z_L=Z_L(x,0)$. Then, we deduce from (3.14), (3.17) and (3.18) for $\beta>1$ that

$$
\begin{aligned}
k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U_+|^2U_L^{2(\beta-1)}dxdt \le \lambda\int_\Omega u_+^pu_L^{2(\beta-1)}\,dx+\int_\Omega u_+^ru_L^{2(\beta-1)}dx&\\
+\mu K^{q-\theta}\int_\Omega u_+^\theta u_L^{2(\beta-1)}dx.&
\end{aligned}
\tag{3.19}
$$

Let $W_L=U_L^{\beta-1}U_+$. We have

$$
\nabla W_L=U_L^{\beta-1}\nabla U_++(\beta-1)U_L^{\beta-2}\nabla U_L\, U_+.
$$

By the Sobolev embedding theorem, we obtain

$$\begin{aligned}\Big(\int_\Omega |w_L|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}} &\le S^{-1}k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla W_L|^2dxdt\\ &= S^{-1}k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|(\beta-1)U_+U_L^{\beta-2}\nabla U_L+U_L^{\beta-1}\nabla U_+|^2dxdt\\ &\le 2S^{-1}\Big(k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|(\beta-1)U_+U_L^{\beta-2}\nabla U_L|^2dxdt\\ &\qquad + k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|U_L^{\beta-1}\nabla U_+|^2dxdt\Big)\\ &\le 2S^{-1}\Big((\beta-1)^2k_s\int_{\mathcal{C}_\Omega} t^{1-2s}U_L^{2(\beta-1)}|\nabla U_+|^2dxdt\\ &\qquad + k_s\int_{\mathcal{C}_\Omega} t^{1-2s}U_L^{2(\beta-1)}|\nabla U_+|^2dxdt\Big)\\ &\le 2S^{-1}\Big((\beta-1)^2+1\Big)k_s\int_{\mathcal{C}_\Omega} t^{1-2s}U_L^{2(\beta-1)}|\nabla U_+|^2dxdt\\ &= 2S^{-1}\beta^2\Big(\big(\frac{\beta-1}{\beta}\big)^2+\frac{1}{\beta^2}\Big)k_s\int_{\mathcal{C}_\Omega} t^{1-2s}U_L^{2(\beta-1)}|\nabla U_+|^2dxdt,\end{aligned} \tag{3.20}$$

where $S>0$ is the best fractional Sobolev embedding constant and $w_L = W_L(\cdot,0)$.

Since $\beta>1$, we have $\big(\frac{\beta-1}{\beta}\big)^2+\frac{1}{\beta^2}<2$. Thus, we can use (3.19) and (3.20) to obtain

$$\begin{aligned}\Big(\int_\Omega |w_L|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}} &\le 4S^{-1}\beta^2k_s\int_{\mathcal{C}_\Omega} t^{1-2s}U_L^{2(\beta-1)}|\nabla U_+|^2dxdt\\ &\le 4S^{-1}\beta^2\Big(\lambda\int_\Omega |u_+|^pu_L^{2(\beta-1)}dx+\int_\Omega |u_+|^ru_L^{2(\beta-1)}dx\\ &\qquad +\mu K^{q-\theta}\int_\Omega |u_+|^\theta u_L^{2(\beta-1)}dx\Big)\\ &\le 4S^{-1}\beta^2\Big[(\lambda\Big(\int_\Omega |u_+|^2u_L^{2(\beta-1)}dx\Big)^{\frac{p}{2}}\Big(\int_\Omega u_L^{2(\beta-1)}dx\Big)^{\frac{2-p}{2}}\\ &\qquad +\int_\Omega |u_+|^ru_L^{2(\beta-1)}dx+\mu K^{q-\theta}\int_\Omega |u_+|^\theta u_L^{2(\beta-1)}dx\Big]\\ &\le 4S^{-1}\beta^2\Big(\lambda\frac{2-p}{2}\int_\Omega u_L^{2(\beta-1)}dx+\lambda\frac{p}{2}\int_\Omega |u_+|^2u_L^{2(\beta-1)}dx\\ &\qquad +\int_\Omega |u_+|^ru_L^{2(\beta-1)}dx+\mu K^{q-\theta}\int_\Omega |u_+|^\theta u_L^{2(\beta-1)}dx\Big)\\ &\le 4S^{-1}\beta^2\Big(\lambda\int_\Omega u_L^{2(\beta-1)}dx+\lambda\int_\Omega |u_+|^2u_L^{2(\beta-1)}dx\\ &\qquad +\int_\Omega |u_+|^ru_L^{2(\beta-1)}dx+\mu K^{q-\theta}\int_\Omega |u_+|^\theta u_L^{2(\beta-1)}dx\Big).\end{aligned} \tag{3.21}$$

Considering the Sobolev embedding $H^1_{0,L}(\mathcal{C}_\Omega))\hookrightarrow L^{2^*}(\Omega)$ and $\|U_+\|_{H^1_{0,L}}\le C_0$ (see Lemma 2.6), we have

$$\Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{\frac{1}{2_s^*}} \le S^{-\frac12}\Big(k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla U_+|^2dxdt\Big)^{\frac12}\le S^{-\frac12}C_0. \tag{3.22}$$

Let $\alpha=\frac{2_s^*\cdot 2}{2_s^*-(\theta-2)}$. Since $u_+^\theta u_L^{2(\beta-1)}=u_+^{\theta-2}w_L^2$, $u_+^ru_L^{2(\beta-1)}=u_+^{r-2}w_L^2$ and $u_+^2u_L^{2(\beta-1)}=w_L^2$, we now use the Hölder inequality, (3.20), (3.21) and (3.22) to conclude that, whenever

$w_L \in L^\alpha(\Omega)$, we have that

$$\begin{aligned}
&\Big(\int_\Omega |w_L|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}\\
&\le 4S^{-1}\beta^2\Big[\lambda\int_\Omega u_L^{2(\beta-1)}dx+\lambda\int_\Omega |w_L|^2dx\\
&\qquad +\int_\Omega |u_+|^{r-2}|w_L|^2dx+\mu K^{q-\theta}\int_\Omega |u_+|^{\theta-2}|w_L|^2dx\Big]\\
&\le 4S^{-1}\beta^2\Big[\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}}\Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{\frac{2(\beta-1)}{2_s^*}}+\lambda|\Omega|^{\frac{\alpha-2}{\alpha}}\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\\
&\qquad +\Big(\int_\Omega |u_+|^{\frac{r-2}{\theta-2}2_s^*}dx\Big)^{1-\frac{2}{\alpha}}\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\\
&\qquad +\mu K^{q-\theta}\Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{1-\frac{2}{\alpha}}\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\Big]\\
&\le 4S^{-1}\beta^2\Big[\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}}S^{1-\beta}C_0^{2(\beta-1)}+\lambda|\Omega|^{\frac{\alpha-2}{\alpha}}\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\\
&\qquad +|\Omega|^{\frac{\theta-r}{2_s^*}}\Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{\frac{r-2}{2_s^*}}\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\\
&\qquad +\mu K^{q-\theta}\Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{1-\frac{2}{\alpha}}\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\Big]\\
&\le 4S^{-1}\beta^2\Big[\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}}S^{1-\beta}C_0^{2(\beta-1)}+\Big(\lambda|\Omega|^{\frac{\alpha-2}{\alpha}}+|\Omega|^{\frac{\theta-r}{2_s^*}}(C_0S^{-\frac12})^{r-2}\\
&\qquad +\mu K^{q-\theta}(C_0S^{-\frac12})^{\theta-2}\Big)\Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\Big]\\
&\le \beta^2 C_{\lambda,\mu,K}\cdot\max\Big\{1\ ,\ \Big(\int_\Omega |w_L|^\alpha dx\Big)^{\frac{2}{\alpha}}\Big\}, \qquad (3.23)
\end{aligned}$$

where

$$\begin{aligned}
C_{\lambda,\mu,K}=4S^{-1}\Big(&\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}}S^{1-\beta}C_0^{2(\beta-1)}+\lambda|\Omega|^{\frac{\alpha-2}{\alpha}}\\
&+|\Omega|^{\frac{\theta-r}{2_s^*}}(C_0S^{-\frac12})^{r-2}+\mu K^{q-\theta}(C_0S^{-\frac12})^{\theta-2}\Big)>0.
\end{aligned}$$

So, from (3.23) and the definition of W_L, we obtain

$$\Big(\int_\Omega u_L^{(\beta-1)2_s^*}u_+^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}\le \beta^2C_{\lambda,\mu,K}\max\Big\{1,\Big(\int_\Omega u_L^{(\beta-1)\alpha}u_+^\alpha dx\Big)^{\frac{2}{\alpha}}\Big\}.$$

By Fatou's Lemma on the variable L, we get

$$\Big(\int_\Omega u_+^{\beta 2_s^*}dx\Big)^{\frac{2}{2_s^*}}\le \beta^2C_{\lambda,\mu,K}\max\Big\{1,\Big(\int_\Omega u_+^{\beta\alpha}dx\Big)^{\frac{2}{\alpha}}\Big\},$$

i.e.,

$$\Big(\int_\Omega u_+^{\beta 2_s^*}dx\Big)^{\frac{1}{\beta 2_s^*}}\le \beta^{\frac1\beta}C_{\lambda,\mu,K}^{\frac{1}{2\beta}}\max\Big\{1,\Big(\int_\Omega u_+^{\beta\alpha}dx\Big)^{\frac{1}{\alpha\beta}}\Big\}. \qquad (3.24)$$

Set $\beta := \frac{2_s^*}{\alpha} = 1+\frac{2_s^*-\theta}{2} > 1$. Then, (3.24) implies that $u_+ \in L^{\beta\, 2_s^*}(\Omega)$. Since $\beta^2\alpha = \beta 2_s^*$, we have that the inequality (3.24) also holds with β replaced by β^2. Hence, we get

$$\begin{aligned}
&\Big(\int_\Omega |u_+|^{2_s^*\beta^2}dx\Big)^{\frac{1}{2_s^*\beta^2}} \\
&\le (\beta^2)^{\frac{1}{\beta^2}} C_{\lambda,\mu,K}^{\frac{1}{2\beta^2}} \max\Big\{1\ ,\ \Big(\int_\Omega |u_+|^{\alpha\beta^2}dx\Big)^{\frac{1}{\alpha\beta^2}}\Big\} \\
&= (\beta^2)^{\frac{1}{\beta^2}} C_{\lambda,\mu,K}^{\frac{1}{2\beta^2}} \max\Big\{1\ ,\ \Big(\int_\Omega |u_+|^{2_s^*\beta}dx\Big)^{\frac{1}{2_s^*\beta}}\Big\} \\
&\le \beta^{\frac{2}{\beta^2}+\frac{1}{\beta}} C_{\lambda,\mu,K}^{\frac{1}{2\beta^2}+\frac{1}{2\beta}} \max\Big\{1\ ,\ \Big(\int_\Omega |u_+|^{\alpha\beta}dx\Big)^{\frac{1}{\alpha\beta}}\Big\} \\
&= \beta^{\frac{2}{\beta^2}+\frac{1}{\beta}} C_{\lambda,\mu,K}^{\frac{1}{2\beta^2}+\frac{1}{2\beta}} \max\Big\{1\ ,\ \Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{\frac{1}{2_s^*}}\Big\}.
\end{aligned}$$

By iterating this process, we obtain

$$\Big(\int_\Omega |u_+|^{2_s^*\beta^m}dx\Big)^{\frac{1}{2_s^*\beta^m}} \le \beta^{\frac{m}{\beta^m}+\cdots+\frac{2}{\beta^2}+\frac{1}{\beta}} C_{\lambda,\mu,K}^{\frac{1}{2\beta^m}+\cdots+\frac{1}{2\beta^2}+\frac{1}{2\beta}} \max\Big\{1\,,\ \Big(\int_\Omega |u_+|^{2_s^*}dx\Big)^{\frac{1}{2_s^*}}\Big\}. \tag{3.25}$$

Taking the limit as $m \to +\infty$ in (3.25), we have

$$\|u_+\|_{L^\infty} \le \beta^{\tau_1} C_{\lambda,\mu,K}^{\tau_2} \max\Big\{1\,,\ S^{-\frac{1}{2}}C_0\Big\}$$

where

$$\tau_1 = \sum_{m=1}^{\infty}\frac{m}{\beta^m}, \qquad \tau_2 = \sum_{m=1}^{\infty}\frac{1}{2\beta^m}.$$

Now, to prove Theorem 1.1, we need to choose suitable values of λ, μ, K carefully so that

$$\beta^{\tau_1} C_{\lambda,\mu,K}^{\tau_2} \max\Big\{1\ ,\ S^{-\frac{1}{2}}C_0\Big\} \le \frac{K}{2}, \tag{3.26}$$

this is equivalent to

$$C_{\lambda,\mu,K} \le \left(\frac{K}{2\beta^{\tau_1}\max\Big\{1\,,\ S^{-\frac{1}{2}}C_0\Big\}}\right)^{\frac{1}{\tau_2}}.$$

That is,

$$\begin{aligned}
\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}} S^{1-\beta} C_0^{2(\beta-1)} + \lambda|\Omega|^{\frac{\alpha-2}{\alpha}} + |\Omega|^{\frac{\theta-r}{2_s^*}}(C_0S^{-\frac{1}{2}})^{r-2} \\
+\mu K^{q-\theta}(C_0S^{-\frac{1}{2}})^{\theta-2} \le \frac{S}{4}\left(\frac{K}{2\,\beta^{\tau_1}\max\Big\{1\,,\ S^{-\frac{1}{2}}C_0\Big\}}\right)^{\frac{1}{\tau_2}}.
\end{aligned}$$

Choose $K > 0$ to satisfy the inequality (note that $\lambda \le \lambda_0$)

$$\begin{aligned}
\frac{S}{4}\left(\frac{K}{2\,\beta^{\tau_1}\max\{1\,,\ S^{-\frac{1}{2}}C_0\}}\right)^{\frac{1}{\tau_2}} - \Big(\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}} S^{1-\beta} C_0^{2(\beta-1)} \\
+\lambda|\Omega|^{\frac{\alpha-2}{\alpha}} + |\Omega|^{\frac{\theta-r}{2_s^*}}(C_0S^{-\frac{1}{2}})^{r-2}\Big) > 0
\end{aligned} \tag{3.27}$$

and then fix μ_K such that

$$\mu_K := \frac{1}{K^{q-\theta}(C_0 S^{-\frac{1}{2}})^{\theta-2}} \left[\frac{S}{4} \Big(\frac{K}{2\beta^{\tau_1}\{1\,,\, S^{-\frac{1}{2}}C_0\}} \Big)^{\frac{1}{\tau_2}} \right.$$
$$\left. -\lambda|\Omega|^{\frac{2_s^*-2(\beta-1)}{2_s^*}} S^{1-\beta} C_0^{2(\beta-1)} - \lambda|\Omega|^{\frac{\alpha-2}{\alpha}} - |\Omega|^{\frac{\theta-r}{2_s^*}} (C_0 S^{-\frac{1}{2}})^{r-2} \right].$$

Let $\mu^* := \min\{\mu_0, \mu_K\}$, we obtain (3.26) for $\mu \in [0, \mu^*]$ and some K satisfying (3.27), that is

$$\|u_+\|_{L^\infty} \le \frac{K}{2}, \quad \forall\, \mu \in [0, \mu^*], \;\; \lambda \in [0, \lambda_0]. \tag{3.28}$$

Similarly, we can also have the estimate for u_- as follows:

$$\|u_-\|_{L^\infty} \le \frac{K}{2}, \quad \forall\, \mu \in [0, \mu^*], \;\; \lambda \in [0, \lambda_0]. \tag{3.29}$$

Then, from (3.28) and (3.30), we have

$$\|u\|_{L^\infty} \le K, \quad \forall\, \mu \in [0, \mu^*], \;\; \lambda \in [0, \lambda_0]. \tag{3.30}$$

This completes the proof. □

Proof of Theorem 1.2 In fact, the truncation of $h(x,u)$ can be given by

$$h_K(x,u) = \begin{cases} h(x,u), & \text{if } |u| \le K, \\ \min\{h(x,u)\ ,\ C_0(1 + K^{q-\theta}|u|^{\theta-2}u)\}, & \text{if } |u| > K, \end{cases} \tag{3.31}$$

where $\theta \in (2, 2_s^*)$ and $\theta \ge r$. Then h_K satisfies

$$|h_K(x,u)| \le C_0(1 + K^{q-\theta}|u|^{\theta-1}), \quad \forall u \in \mathbb{R}. \tag{3.32}$$

The truncated problem associated to problem (1.3) is the following

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathcal{C}_\Omega, \\ U = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial U}{\partial \nu^s} = \lambda |u|^{p-2}u + |u|^{r-2}u + \mu h_K(x,u), & \text{in } \Omega, \end{cases} \tag{3.33}$$

where $u(x) = U(x,0)$. By (3.31), (3.32) and a technique similar to the one in Theorem 1.1, we can prove that the truncated problem (3.33) has two nontrivial solutions U_1, U_2, one is a local minimum, the other is of Mountain Pass type, and satisfying $\|u_i\|_{L^\infty} \le K$, $i = 1, 2$. In view of the definition of h_K, we know that U_1 and U_2 are also solutions of the problem (3.33). This implies that original problem (1.3) has two solutions $u_i = U_i(\cdot, 0)$ $(i = 1, 2)$. This completes the proof of Theorem 1.2.

References

[1] V. Ambrosio, *Periodic solutions for critical fractional equations*, Calc. Var. Partial Differ. Equ **57(2)** (2018), https://doi.org/10.1007/s00526-018-1317-y.

[2] V. Ambrosio, *Concentration phenomena for critical fractional Schrödinger systems*, Commun. Pure Appl. Anal. **17(5)** (2018), 2085–2123.

[3] V. Ambrosio and H. Hajaiej, *Multiple solutions for a class of nonhomogeneous fractional Schrödinger equations in* $\mathbb{R}^N$, J. Dynam. Differ. Equ. **30(3)** (2018), 1119–1143.

[4] B.Barrios, E.Colorado, A.de Pablo, U.Sánchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations, **252** (2012), 6133–6162.

[5] F. J. S. A. Corrêa and G. M. Figueiredo, *On an elliptic equation of p-Kirchhoff type via variational methods*, Bull. Austral. Math. Soc. ,**74** (2006), 263–277.

[6] M. Caponi and P. Pucci, *Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations*, Ann. Mat. Pura Appl. **195(6)** (2016), 2099–2129.

[7] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, **32** (2007), 1245–1260.

[8] X. Cabré and J. Tan, *Positive solutions for nonlinear problems involving the square root of the Laplacian*, Adv. Math., **224** (2010), 2052–2093.

[9] J. Cohabrowski and J. Yang, *Existence theorems for elliptic equations involving supercritical Sobolev exponent*, Adv. Differential Equations, **2** (1997), 231–256.

[10] S. Dipierro, M. Medina and E. Valdinoci, *Fractional elliptic problems with critical growth in the whole of* $\mathbb{R}^n$, Appunti. In: Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], 15, pp. 8–152. Edizioni della Normale, 2017.

[11] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136(5)** (2012), 521–573.

[12] J. Francisco, S. A. Correa and M. Figueiredo Giovany, *On an elliptic equation of p-Kirchhoff type via variational methods*, Bull. Australian Math. Soc., **74** (2006), 263–277.

[13] G. Figueiredo and M. Furtado, *Positive solutions for some quasilinear equations with critical and supercritical growth*, Nonlinear Anal. TMA, **66** (2007), 1600–1616.

[14] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math., **13** (1960), 457–468.

[15] S. Miyajima, D. Motreanu and M. Tanaka, *Multiple existence results of solutions for the Neumann problems via super- and sub-solutions*, J. Funct. Anal., **262** (2012), 1921–1953.

[16] G. Molica Bisci, V. Radulescu and Servadei, *Variational methods for nonlocal fractional problems.* In: With a foreword by Jean Mawhin. Encyclopedia of Mathematics and its Applications, vol. 162, pp. 97–383. Cambridge University Press, 2016.

[17] P. H. Rabinowitz, *Variational methods for nonlinear elliptic eigenvalue problems*, Indiana Univ. Math. J. **23** (1974), 729–754.

[18] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var., **42** (2011), 21–41.

[19] J. Zhang, X. Liu, *Three solutions for a fractional elliptic problems with critical and supercritical growth*, Acta Math. Sci., **36B(6)** (2016), 1819–1831.

[20] L. Zhao and P. Zhao, *The existence of threee solutions for p-Laplacian problems with critical and supercritical growth*, Rocky Mountain J. Math., **44** (2014), 1383–1397.

J. Zhang: jgzhang@jxnu.edu.cn
School of Mathematics and Statistics, Jiangxi Normal University, Nanchang 330022, China

Received 26/05/2022