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THE EXISTENCE OF EIGENVALUES OF SCHRÖDINGER OPERATOR ON THREE DIMENSIONAL LATTICE

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Abstract. We consider a three-particle discrete Schrödinger operator $H_{\mu,\gamma}(\mathbf{K})$, $\mathbf{K} \in \mathbb{T}^3$, associated to a system of three particles (two fermions and one another particle) interacting through zero range pairwise potential $\mu > 0$ on the three-dimensional lattice $\mathbb{Z}^3$. It is proved that the operator $H_{\mu,\gamma}(\mathbf{K})$, $\|\mathbf{K}\| < \delta$, for $0 < \gamma < \gamma_0$ ($\gamma_0 \approx 4,7655$) has no eigenvalues and for $\gamma > \gamma_0$ has exactly three eigenvalues lying below the essential spectrum for sufficiently large μ and small δ.

Ми розглядаємо тричастинковий дискретний оператор Шрӧдінгера $H_{\mu,\gamma}(\mathbf{K})$, $\mathbf{K} \in \mathbb{T}^3$, який асоціюється з системою з трьох частинок (двох ферміонів і одна інша частинка), які попарно взаємодіють через потенціал нульового радіусу $\mu > 0$ на тривимірній решітці $\mathbb{Z}^3$. Доведено, що оператор $H_{\mu,\gamma}(\mathbf{K})$, $\|\mathbf{K}\| < \delta$, для $0 < \gamma < \gamma_0$ ($\gamma_0 \approx 4,7655$) не має власних значень, а для $\gamma > \gamma_0$ має рівно три власні значення, що лежать нижче суттєвого спектру для достатньо великих μ і малих δ.

1. Introduction

In models of solid state physics [1, 2] and also in lattice quantum field theory [3], discrete operators are considered as lattice analogs of the three-particle Schrödinger operator in the continuum.

Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. Periodicity of the potential gives rise to a band structure for the dynamics of the atoms. The dynamics of the ultracold atoms loaded in the lower or upper band is well described by the Bose-Hubbard Hamiltonian [4]; in Section 3, we give a corresponding Schrödinger operator.

In the continuum case, due to rotational invariance, the Hamiltonian separates into a free Hamiltonian for the center of mass and a Hamiltonian H_{rel} for the relative motion. Bound states are eigenstates of H_{rel}.

Kinematics of quantum particles on a lattice is rather exotic [5]. The discrete Laplacian *is not rotationally invariant* and, therefore, one cannot separate the motion of the center of mass.

In this work we consider a Hamiltonian $\mathrm{H}_{\mu,\gamma}$ for a system of three quantum particles, two fermions with mass 1 and another particle with mass $m = 1/\gamma > 0$ with zero range pair potentials $\mu > 0$ on the three dimensional lattice $\mathbb{Z}^3$. In the momentum representation, the total three-body Hamiltonian appears to be decomposable (see, e.g. [6])

$$\mathrm{H}_{\mu,\gamma} = \int_{\mathbb{T}^3} \oplus H_{\mu,\gamma}(\mathbf{K}) d\mathbf{K}.$$

The fiber Hamiltonian $H_{\mu,\gamma}(\mathbf{K}) = H_{0,\gamma}(\mathbf{K}) - \mu V$ depends parametrically on the *total quasimomentum* $\mathbf{K} \in \mathbb{T}^3 \equiv \mathbb{R}^3/(2\pi\mathbb{Z}^3)$. It is the sum of a free part $H_{0,\gamma}(\mathbf{K})$ and an interaction term $-\mu V$, both bounded and the dependence on $\mathbf{K}$ of the free part is continuous. Eigenfunctions of $H_{\mu,\gamma}(\mathbf{K})$ are interpreted as the bound states of the Hamiltonian $\mathrm{H}_{\mu,\gamma}$

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and the corresponding eigenvalues, as the bound state energy. The main results of this paper are given for sufficiently large values $\mu > 0$ of the energy of the interaction of two particles (fermion and another particle), that is, for the case where two-particle subsystems have bound states with negative energy there is a threshold value of the particle mass ratio γ_0 such that, if $\gamma > \gamma_0$, then the fiber Hamiltonian $H_{\mu,\gamma}(\mathbf{K})$ with a fixed total quasimomentum $\mathbf{K}$ has three eigenvalues below the essential spectrum (for sufficiently large values of the interaction energy μ and sufficiently small modulus of values of the total quasimomentum $\mathbf{K}$), and if $\gamma < \gamma_0$ then it has no eigenvalues lying below the essential spectrum (with the same assumption on the interaction energy and total quasimomentum).

The existence of the bound states of the three particle systems has been studied in many works, see e.g. [7]–[23]. Efimov [7] discovered the existence of an infinite number of eigenvalues (Efimov's effect). Since then this problem has been studied in many physics journals and books [8, 9, 10]. A rigorous mathematical proof of the existence of Efimov's effect was originally carried out in [11] by Yafaev and then this theory has been rapidly developed see, e.g. [12]–[15]. Gridnev [16] proved the existence of the so-called super Efimov effect for the system of 3 nonrelativistic spinless fermions in two dimensions, which interact through spherically-symmetric pair interactions. In [17], the Hamiltonian H for the system of three quantum particles (two fermions with mass 1 and another particle with mass $m > 0$) with point-like interaction in Euclid space $\mathbb{R}^3$ is considered. For $0 < m < m_1$ any selfadjoint extension of $T_{l=1}$ of thr auxiliary operator $T = \oplus_{l=0}^{\infty} T_l$ involving the construction of the resolvent for the operator H has a sequence of eigenvalues $\{\lambda_n < 0, n > n_0\}$ diverging to $-\infty$ generate negative eigenvalues (Efimov's effect) of the corresponding extension of H_ε that are equal to $-\left(\varepsilon/\lambda_n\right)^2$. Three particle quantum system in dimension three composed of two identical fermions (of mass one and another particle of mass m) with two-body short range potentials was considered in [18]. Under some assumptions, the Hamiltonian of all two-particle subsystems do not have bound states with negative energy and the two subsystems made of a fermion and another particle have a zero-energy resonance, for $m < m^* = (13.607)^{-1}$ there were proved the existence of the Efimov effect and $m > m^*$ the number of negative eigenvalues of H is finite. In [19] the Hamiltonian H of the system consisting of three point-like particles, two identical fermions and another particle of mass ratio m with respect two fermions was considered. It was shown that a self-adjoint extension H_α of H has some critical values m^* and M^* of m such that for mass $m \in (m^*, M^*)$ it admits eigenvalues and for $m > M^*$ has no eigenvalues below the essential spectrum of the operator H_α.

Now we give some results concerning the three-particle systems on the lattice. The existence of at least one eigenvalues of the three particle discrete Schrödinger operator $H_\mu(\mathbf{K}) = H_0(\mathbf{K}) - \mu V$ (μ is arbitrary) for dimensions $d = 1, 2$ is shown in [6] and [20]. The proofs are based on unboundedness of the norm of the Faddeev type operator $T(\mathbf{K}, z)$ for the spectral parameter z close to the lower bound of the essential spectrum. If $d \geq 3$ then the operator $T(\mathbf{K}, z)$ is bounded at the bottom of the essential spectrum, i.e., in this case the methods for $d = 1, 2$ are not applicable.

In [21] the authors studied a model operator H_γ associated with the three-particle discrete Schrödinger operator on the three-dimensional cubical lattice, with zero-range pair potentials, where the role of the two-particle discrete Schrödinger operators is taken by a family of the Friedreich models with parameters $h_\alpha(\mathbf{k}), \alpha = 1, 2, \mathbf{k} \in \mathbb{T}^3$. It was proved that there is a critical value γ^* of the parameter such that only for $\gamma < \gamma^*$ the Efimov effect is absent for the Hilbert space of antisymmetric functions with respect to the two identical particles. In addition, if the two-particle subsystems have a zero-energy resonance without having negative energy bound states, then it is shown that H_γ has an infinite number of eigenvalues (Efimov's effect). This result is also preserved for the operator $H_{\mu,\gamma}(\mathbf{0}) := H_{\gamma}^{as}$ we are considering for every $\gamma > \gamma^*$ and fixed $\mu_0 = (1+\gamma)\left(\int_{\mathbb{T}^3} \frac{d\mathbf{p}}{\varepsilon(\mathbf{p})}\right)^{-1}$.

In this paper, we first prove that, for the upper and the lower values of γ of the critical value $\gamma_0 \approx 4,7655$, the operator $H_{\mu,\gamma}(\mathbf{0})$ has exactly three eigenvalues and has no eigenvalues lying below the bottom of the essential spectrum for large $\mu > 0$, respectively. Then applying the perturbation theory we show that the obtained results are preserved for small values of $\mathbf{K}$. The problem of finding the number of eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{K})$ less than z ($z < \tau_{\min,\gamma}(\mu,\mathbf{K})$) reduces to the problem of finding the number of eigenvalues of a Faddeev-type operator $A_{\mu,\gamma}(\mathbf{K},z)$ greater than 1 (see Section 5). The sensitivity of the kernel of the integral operator $A_{\mu,\gamma}(\mathbf{K},z)$ with respect to a change in $\mathbf{K}$ leads to a change in the number of eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{K})$. For example, it can be shown that for γ sufficiently close to γ_0, the operator $H_{\mu,\gamma}(\boldsymbol{\pi})$, ($\boldsymbol{\pi} = (\pi,\pi,\pi)$) has no eigenvalues lying below the bottom of the essential spectrum for large $\mu > 0$. The "two-particle branch" (see Section 4) of the essential spectrum of the operator $H_{\mu,\gamma}(\mathbf{K})$ moves to $-\infty$ with the order μ as $\mu \to +\infty$. However, for $\gamma > \gamma_0$ and sufficiently large values of μ, there are three eigenvalues of $H_{\mu,\gamma}(\mathbf{K})$ that are not absorbed by the essential spectrum.

The paper is organized as follows. Section 1 is an introduction. In Section 2 we describe the Hamiltonians of the two-body and three-body case in the Schrödinger representation and give main results. In Section 3 we study spectral properties of a two-particle operator. The essential spectrum of the three-particle Shrödinger operator is described in Section 4. In Section 5 we prove the absence of eigenvalues of the three-particle operator on the right of the essential spectrum and obtained the Faddeev type compact equation. In the last section we prove a result on existence or absence of eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{0})$ lying below the essential spectrum for sufficiently large μ. At the end of the section we give proofs of the main results.

2. Representations of Hamiltonians and main results

Let $\ell^2[(\mathbb{Z}^3)^d]$, $(d = 2,3)$ be the Hilbert space of square-summable functions defined on the Cartesian product $(\mathbb{Z}^3)^d$, and $\ell^{2,as}[(\mathbb{Z}^3)^d] \subset \ell^2[(\mathbb{Z}^3)^d]$ be a subspace of antisymmetric functions with respect to the first two coordinates.

We consider the Hamiltonian of the system of three quantum particles (two fermions with mass 1 and another particle with mass $m = 1/\gamma$) interacting via zero-range attractive pair potentials on $\mathbb{Z}^3$. Since the Hamiltonian corresponding to a system of fermions acts in $\ell^{2,as}[(\mathbb{Z}^3)^2]$, the first two particles in our system have no zero-range two-particle interaction (see, [6, 22]).

In coordinate representation, the Hamiltonian of a system of two free arbitrary particles (fermion and another particle) on $\mathbb{Z}^3$ is associated with a bounded self-adjoint operator $\hat{\mathrm{h}}_{0,\gamma}$ in $\ell^2[(\mathbb{Z}^3)^2]$,

$$\hat{\mathrm{h}}_{0,\gamma} = -\frac{1}{2}\Delta \otimes I - \frac{\gamma}{2} I \otimes \Delta,$$

where Δ is the lattice Laplacian, I is the identity operator in $\ell^2[\mathbb{Z}^3]$ and $\gamma = \frac{1}{m}$.

The total Hamiltonian $\hat{\mathrm{h}}_{\mu,\gamma}$ of a system of two arbitrary particles with zero-range interaction acts in $\ell^2[(\mathbb{Z}^3)^2]$ and is a bounded perturbation of the free Hamiltonian $\hat{\mathrm{h}}_{0,\gamma}$,

$$\hat{\mathrm{h}}_{\mu,\gamma} = \hat{\mathrm{h}}_{0,\gamma} - \mu\hat{v}.$$

Here $\mu > 0$ is the energy of interaction of two particles (a fermion and another particle), and the operator $\hat{v}$ describes their zero-range interaction:

$$(\hat{v}\hat{\psi})(\mathbf{x},\mathbf{y}) = \delta_{\mathbf{x}\mathbf{y}}\hat{\psi}(\mathbf{x},\mathbf{y}),$$

and $\delta_{\mathbf{xy}}$ is the Kronecker delta.

Similarly, the free Hamiltonian $\hat{\mathrm{H}}_{0,\gamma}$ of the system of the three particles on $\mathbb{Z}^3$ is defined on $\ell^{2,as}[(\mathbb{Z}^3)^3]$ by

$$\hat{\mathrm{H}}_{0,\gamma} = -\frac{1}{2}\Delta \otimes I \otimes I - \frac{1}{2} I \otimes \Delta \otimes I - \frac{\gamma}{2} I \otimes I \otimes \Delta.$$

The total Hamiltonian $\hat{\mathrm{H}}_{\mu,\gamma}$ of the system of the three-particles with the pairwise zero-range interaction is a bounded perturbation of the free Hamiltonian $\hat{\mathrm{H}}_{0,\gamma}$,

$$\hat{\mathrm{H}}_{\mu,\gamma} = \hat{\mathrm{H}}_{0,\gamma} - \mu(\hat{\mathrm{V}}_1 + \hat{\mathrm{V}}_2),$$

where

$$(\hat{\mathrm{V}}_1 \hat{\psi})(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \delta_{\mathbf{x}_2\mathbf{x}_3} \hat{\psi}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

and

$$(\hat{\mathrm{V}}_2 \hat{\psi})(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \delta_{\mathbf{x}_3\mathbf{x}_1} \hat{\psi}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

Let $\mathbb{T}^3$ be a three-dimensional torus, $L_2^{as}[(\mathbb{T}^3)^3] \subset L_2[(\mathbb{T}^3)^3]$ be the Hilbert space of square-integrable functions, defined on $(\mathbb{T}^3)^3$ and antisymmetric with respect to the first two coordinates. In the momentum representation, the two-and three-particle Hamiltonians act accordingly on the Hilbert spaces $L_2[(\mathbb{T}^3)^2]$ and $L_2^{as}[(\mathbb{T}^3)^3]$. Assume that $d\mathbf{p}$ is a unit measure on the torus $\mathbb{T}^3$, i.e., $\int_{\mathbb{T}^3} d\mathbf{p} = 1$.

The study of spectra of the operators $h_{\mu,\gamma}$ and $H_{\mu,\gamma}$ is reduced to the study of the spectra of families of the operators $h_{\mu,\gamma}(\mathbf{k}), \mathbf{k} \in \mathbb{T}^3$, and $H_{\mu,\gamma}(\mathbf{K}), \mathbf{K} \in \mathbb{T}^3$, respectively (see [6, 23, 24]).

The two-particle discrete Schrödinger operator

$$h_{\mu,\gamma}(\mathbf{k}) = h_{0,\gamma}(\mathbf{k}) - \mu v \tag{2.1}$$

acts on $L_2(\mathbb{T}^3)$, where

$$\begin{aligned} (h_{0,\gamma}(\mathbf{k})f)(\mathbf{p}) &= \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{p}) f(\mathbf{p}), \quad \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{p}) = \varepsilon(\mathbf{p}) + \gamma\varepsilon(\mathbf{k} - \mathbf{p}), \quad \varepsilon(\mathbf{p}) = \sum_{i=1}^{3}(1 - cosp_i) \\ (vf)(\mathbf{p}) &= \int_{\mathbb{T}^3} f(\mathbf{s}) d\mathbf{s}. \end{aligned}$$

The corresponding three-particle Schrödinger operator

$$H_{\mu,\gamma}(\mathbf{K}) = H_{0,\gamma}(\mathbf{K}) - \mu(V_1 + V_2)$$

acts on $L_2^{as}[(\mathbb{T}^3)^2]$, where

$$(H_{0,\gamma}(\mathbf{K})f)(\mathbf{p}, \mathbf{q}) = E_{\mathbf{K},\gamma}(\mathbf{p}, \mathbf{q}) f(\mathbf{p}, \mathbf{q}), \quad E_{\mathbf{K},\gamma}(\mathbf{p}, \mathbf{q}) = \varepsilon(\mathbf{p}) + \varepsilon(\mathbf{q}) + \gamma\varepsilon(\mathbf{K} - \mathbf{p} - \mathbf{q}),$$

$$(V_1 f)(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{T}^3} f(\mathbf{p}, \mathbf{s}) d\mathbf{s}, \quad (V_2 f)(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{T}^3} f(\mathbf{s}, \mathbf{q}) d\mathbf{s}.$$

Let

$$\gamma_0 = \left(\int_{\mathbb{T}^3} \frac{\sin^2 s_1}{\varepsilon(\mathbf{s})} d\mathbf{s} \right)^{-1} \approx 4,7655.$$

The main results of the paper are the following theorems:

Theorem 2.1. *Let $\gamma > \gamma_0$. Then there exist $\mu_\gamma > 0$ and $\delta > 0$ such that, for any $\mu > \mu_\gamma$ and $\mathbf{K}$ satisfying $\|\mathbf{K}\| < \delta$, the operator $H_{\mu,\gamma}(\mathbf{K})$ has exactly three eigenvalues, counted with multiplicities, that lay below the essential spectrum.*

Theorem 2.2. *Let $0 < \gamma < \gamma_0$. Then there exist $\mu_\gamma > 0$ and $\delta > 0$ such that, for any $\mu > \mu_\gamma$ and $\mathbf{K}$ satisfying $\|\mathbf{K}\| < \delta$, the operator $H_{\mu,\gamma}(\mathbf{K})$ has no eigenvalues below the essential spectrum.*

3. The spectrum of a two-particle operator $h_{\mu,\gamma}(\mathbf{k})$

By the Weyl theorem [25], the essential spectrum $\sigma_{ess}(h_{\mu,\gamma}(\mathbf{k}))$ of the operator $h_{\mu,\gamma}(\mathbf{k})$ coincides with the spectrum $\sigma(h_{0,\gamma}(\mathbf{k}))$ of the unperturbed operator $h_{0,\gamma}(\mathbf{k})$, i.e.,

$$\sigma_{ess}(h_{\mu,\gamma}(\mathbf{k})) = [\mathcal{E}_{min,\gamma}(\mathbf{k}), \mathcal{E}_{max,\gamma}(\mathbf{k})],$$

where

$$\mathcal{E}_{min,\gamma}(\mathbf{k}) = \min_{\mathbf{q}\in\mathbb{T}^3} \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{q}) = 3(1+\gamma) - \sum_{i=1}^{3} \sqrt{1+2\gamma\cos k_i + \gamma^2}$$

and

$$\mathcal{E}_{max,\gamma}(\mathbf{k}) = \max_{\mathbf{q}\in\mathbb{T}^3} \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{q}) = 3(1+\gamma) + \sum_{i=1}^{3} \sqrt{1+2\gamma\cos k_i + \gamma^2}.$$

Note that the functions $\mathcal{E}_{min,\gamma}(\mathbf{k})$ and $\mathcal{E}_{max,\gamma}(\mathbf{k})$ are symmetric with respect to any two variables k_i and k_j, are even in each $k_i \in [-\pi,\pi], i = 1,2,3$. The function $\mathcal{E}_{min,\gamma}(\mathbf{k})$ is increasing, and $\mathcal{E}_{max,\gamma}(\mathbf{k})$ is decreasing in each $k_i \in [0,\pi]$. Therefore,

$$\min_{\mathbf{k}\in\mathbb{T}^3} \mathcal{E}_{min,\gamma}(\mathbf{k}) = \mathcal{E}_{min,\gamma}(\mathbf{0}) = 0, \quad \max_{\mathbf{k}\in\mathbb{T}^3} \mathcal{E}_{max,\gamma}(\mathbf{k}) = \mathcal{E}_{max,\gamma}(\mathbf{0}) = 6(1+\gamma).$$

Let $z \in \mathbb{C} \setminus [\mathcal{E}_{min,\gamma}(\mathbf{k}), \mathcal{E}_{max,\gamma}(\mathbf{k})]$ and $\Delta_{\mu,\gamma}(\mathbf{k}, z)$ be the Fredholm determinant of the operator $I - \mu v r_{0,\gamma}(\mathbf{k}; z)$, where $r_{0,\gamma}(\mathbf{k}; z)$ is the resolvent of $h_{0,\gamma}(\mathbf{k})$, v is an integral operator with the kernel $v(\mathbf{q}, \mathbf{q}') = 1$. Then $\Delta_{\mu,\gamma}(\mathbf{k}; z)$ has the form

$$\Delta_{\mu,\gamma}(\mathbf{k}, z) = 1 - \mu D_\gamma(\mathbf{k}, z), \quad D_\gamma(\mathbf{k}, z) = \int_{\mathbb{T}^3} \frac{d\mathbf{q}}{\mathcal{E}_{\mathbf{k},\gamma}(\mathbf{q}) - z}. \tag{3.2}$$

Lemma 3.1. *A number $z \in \mathbb{C} \setminus [\mathcal{E}_{min,\gamma}(\mathbf{k}), \mathcal{E}_{max,\gamma}(\mathbf{k})]$ is an eigenvalue of the operator $h_{\mu,\gamma}(\mathbf{k})$ if and only if $\Delta_{\mu,\gamma}(\mathbf{k}, z) = 0$.*

Lemma 3.1 is proved analogously to Lemma 2.1 in [15].

The function $D_\gamma(\mathbf{k}, \mathcal{E}_{min,\gamma}(\mathbf{k}))$ is symmetric with respect to any two variables k_i and k_j, is even in each $k_i \in [-\pi,\pi]$ and increases in $k_i \in [0,\pi], i = 1,2,3$. For any $z \le 0$, the function $D_\gamma(\cdot, z)$ is decreasing in each $k_i \in [0,\pi]$, while the other coordinates of $\mathbf{k}$ are fixed. Moreover, the following relations hold:

$$\min_{\mathbf{k}\in\mathbb{T}^3} D_\gamma(\mathbf{k}, \mathcal{E}_{min,\gamma}(\mathbf{k})) = \max_{\mathbf{k}\in\mathbb{T}^3} D_\gamma(\mathbf{k}, 0) = D_\gamma(\mathbf{0}, 0) = \frac{1}{\mu_0},$$

where

$$\mu_0 = \left(\int_{\mathbb{T}^3} \frac{d\mathbf{q}}{(1+\gamma)\varepsilon(\mathbf{q})} \right)^{-1}$$

is the harmonic mean value of the kinetic energy of the fermion and the other particle.

The proofs of these statements follow from the cosine function properties and monotonicity of the Lebesgue integral.

Theorem 3.2. *Let $\mu > \mu_0$. Then for any $\mathbf{k} \in \mathbb{T}^3$ the operator $h_{\mu,\gamma}(\mathbf{k})$ has a unique simple eigenvalue $z_{\mu,\gamma}(\mathbf{k})$ below the essential spectrum.*

Proof. The function $D_\gamma(\mathbf{k}, \cdot)$ is continuous and increasing in each interval $(-\infty, \mathcal{E}_{min,\gamma}(\mathbf{k}))$ and $(\mathcal{E}_{max,\gamma}(\mathbf{k}), \infty)$. The continuity of the function $D_\gamma(\mathbf{k}, \cdot)$ follows from the continuity of the integrand, and the monotonicity follows from $\dfrac{\partial D_\gamma(\mathbf{k}, z)}{\partial z} > 0$. The monotonicity of $D_\gamma(\mathbf{k}, \cdot)$ in $(-\infty, \mathcal{E}_{min,\gamma}(\mathbf{k}))$ implies the existence of the limit

$$\lim_{z\mapsto\mathcal{E}_{min,\gamma}(\mathbf{k})} D_\gamma(\mathbf{k}, z) = D_\gamma(\mathbf{k}, \mathcal{E}_{min,\gamma}(\mathbf{k})).$$

By definition of μ_0 we have $1 < \mu D_\gamma(\mathbf{k}, \mathcal{E}_{min,\gamma}(\mathbf{k}))$. Thus, we conclude that the function $\Delta_{\mu,\gamma}(\mathbf{k}, z)$ has a unique zero in the interval $(-\infty, \mathcal{E}_{min,\gamma}(\mathbf{k}))$. From Lemma 3.1 it follows that the operator $h_{\mu,\gamma}(\mathbf{k})$ has a unique simple eigenvalue $z_{\mu,\gamma}(\mathbf{k}) < \mathcal{E}_{min,\gamma}(\mathbf{k})$.

The theorem is proved. □

Lemma 3.3. *The eigenvalue $z_{\mu,\gamma}(\mathbf{k}) = z_{\mu,\gamma}(k_1, k_2, k_3)$ is symmetric and even in each $k_i \in [-\pi, \pi]$, and increases in each $k_i \in [0, \pi], i = 1, 2, 3$. Moreover, $z_{\mu_0,\gamma}(\mathbf{0}) = 0$.*

Proof. By virtue of Lemma 3.1 an eigenvalue $z_{\mu,\gamma}(k_1, k_2, k_3)$ of the operator $h_{\mu,\gamma}(\mathbf{k})$ is a solution of the equation $\mu D_\gamma(\mathbf{k}, z) = 1$. Since the function $D_\gamma(\cdot, z)$ is symmetric and even, the eigenvalue $z_{\mu,\gamma}(\mathbf{k})$ is also symmetric and even in each $k_i \in [-\pi, \pi], i = 1, 2, 3$.

Now we prove that the function $z_{\mu,\gamma}(k_1, k_2, k_3)$ is increasing in k_1 that ranges in $[0, \pi]$. Let $k_1' < k_1'' \in [0, \pi]$ be arbitrary and

$$\mu D_\gamma(\mathbf{k}', z') = \mu D_\gamma(\mathbf{k}'', z'') = 1,$$

where $\mathbf{k}' = (k_1', k_2, k_3), \mathbf{k}'' = (k_1'', k_2, k_3)$. For any $z \le 0$ the function $D_\gamma(\cdot, z)$ is decreasing in $k_1 \in [0, \pi]$, therefore $D_\gamma(\mathbf{k}'', z') < D_\gamma(\mathbf{k}', z') = D_\gamma(\mathbf{k}'', z'')$. Since the function $D_\gamma(\mathbf{k}, \cdot)$ is increasing we have $z' < z''$, i.e., $z_{\mu,\gamma}(\mathbf{k}') < z_{\mu,\gamma}(\mathbf{k}'')$. From this and the definition of μ_0 we have $\min\limits_{\mathbf{k}\in\mathbb{T}^3} z_{\mu_0,\gamma}(\mathbf{k}) = z_{\mu_0,\gamma}(\mathbf{0}) = 0$. □

Lemma 3.4. *For any $\gamma > 0$ and $\mu > 3(1+\gamma)$ the following estimates hold true:*

$$-\mu + 3(1+\gamma) - \frac{9(1+\gamma)^2}{\mu} < z_{\mu,\gamma}(\mathbf{0}) < -\mu + 3(1+\gamma).$$

Proof. If we show the existence of $z', z'' < 0$ such that

$$\Delta_{\mu,\gamma}(\mathbf{0}, z') < 0 = \Delta_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0})) < \Delta_{\mu,\gamma}(\mathbf{0}, z'')$$

then? since the function $\Delta_{\mu,\gamma}(\mathbf{0}, \cdot)$ is monotone on $(-\infty, 0)$, we obtain that

$$z'' < z_{\mu,\gamma}(\mathbf{0}) < z'.$$

Denoting by $\xi(s) = \sum_{i=1}^3 \cos s_i$ we have

$$\begin{aligned}\Delta_{\mu,\gamma}(\mathbf{0}, z) &= 1 - \frac{\mu}{3(1+\gamma) - z}\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{1 - \frac{(1+\gamma)\xi(\mathbf{s})}{3(1+\gamma)-z}} \\ &= 1 - \frac{\mu}{3(1+\gamma) - z}\int_{\mathbb{T}^3}\left[1 + \frac{(1+\gamma)\xi(\mathbf{s})}{3(1+\gamma)-z} + \left(\frac{(1+\gamma)\xi(\mathbf{s})}{3(1+\gamma)-z}\right)^2 + \dots\right] d\mathbf{s} \\ &= 1 - \frac{\mu}{3(1+\gamma) - z}\int_{\mathbb{T}^3}\left[1 + \left(\frac{(1+\gamma)\xi(\mathbf{s})}{3(1+\gamma)-z}\right)^2 + \left(\frac{(1+\gamma)\xi(\mathbf{s})}{3(1+\gamma)-z}\right)^4 + \dots\right] d\mathbf{s}\end{aligned} \tag{3.3}$$

for $z < 0$. Here we used the identity

$$\int_{\mathbb{T}^3} \xi^{2n-1}(\mathbf{s}) d\mathbf{s} = 0, \qquad n = 1, 2, \dots.$$

Let $z' = -\mu + 3(1+\gamma)$. Then from (3.3) we get

$$\Delta_{\mu,\gamma}(\mathbf{0}, z') < 1 - \frac{\mu}{3(1+\gamma) - z'}\int_{\mathbb{T}^3} d\mathbf{s} = 1 - \frac{\mu}{3(1+\gamma) - z'} = 0.$$

Taking into account that $|\xi(\mathbf{s})| \leq 3$ in (3.3) we have

$$\Delta_{\mu,\gamma}(\mathbf{0},z) > 1 - \frac{\mu}{3(1+\gamma)-z}\int_{\mathbb{T}^3}\left[1+\left(\frac{3(1+\gamma)}{3(1+\gamma)-z}\right)^2+\left(\frac{3(1+\gamma)}{3(1+\gamma)-z}\right)^4+\dots\right]d\mathbf{s}$$
$$= 1 - \frac{\mu}{3(1+\gamma)-z}\frac{1}{1-\left(\frac{3(1+\gamma)}{3(1+\gamma)-z}\right)^2}.$$

Now choosing $z'' = -\mu + 3(1+\gamma) - \dfrac{9(1+\gamma)^2}{\mu}$ we obtain

$$\Delta_{\mu,\gamma}(\mathbf{0},z'') \quad > \quad \frac{\left(\frac{9(1+\gamma)^2}{\mu}\right)^2}{\mu^2+9(1+\gamma)^2+\left(\frac{9(1+\gamma)^2}{\mu}\right)^2} > 0.$$

The proof is complete. □

Lemma 3.5. *For any $\gamma > 0$ and $\mu > 3(1+\gamma)$, we have*

$$-\mu + 3(1+\gamma) - \frac{9(1-\gamma)^2}{\mu} < z_{\mu,\gamma}(\boldsymbol{\pi}) < -\mu + 3(1+\gamma),$$

where $\boldsymbol{\pi} = (\pi,\pi,\pi)$.

Proof. The lemma is proved analogously to Lemma 3.4. □

Lemmas 3.3 — 3.5 immediately give the following.

Lemma 3.6. *For any $\gamma > 0$ and $\mu > 3(1+\gamma)$,*

$$0 \leq z_{\mu,\gamma}(\boldsymbol{\pi}) - z_{\mu,\gamma}(\mathbf{0}) \leq \frac{9(1+\gamma)^2}{\mu}.$$

4. Essential spectrum of the three-particle operator

In this section, we introduce so-called "channel operators" the spectrum of which describes the essential spectrum of the Schrödinger operator $H_{\mu,\gamma}(\mathbf{K})$.

Since in the three-particle system we are considering, two particles are same (i.e., the operators V_1 and V_2 are unitarily equivalent), there is only one channel operator $H^{ch}_{\mu,\gamma}(\mathbf{K})$, and, in the momentum representation, it is defined as a self-adjoint operator acting on the Hilbert space $L_2[(\mathbb{T}^3)^2]$ according to the formula

$$H^{ch}_{\mu,\gamma}(\mathbf{K}) = H_{0,\gamma}(\mathbf{K}) - \mu V_1.$$

The operator $H^{ch}_{\mu,\gamma}(\mathbf{K})$ commutes with the group $\{U_\mathbf{s},\ \mathbf{s}\in\mathbb{Z}^3\}$ of the unitary operators

$$(U_s f)(\mathbf{p},\mathbf{q}) = exp\{-i(\mathbf{s},\mathbf{p})\}\, f(\mathbf{p},\mathbf{q}), \quad f \in L_2[(\mathbb{T}^3)^2].$$

The operator $H^{ch}_{\mu,\gamma}(\mathbf{K})$ decomposes into a direct operator integral ([25], Theorem XIII.84)

$$H^{ch}_{\mu,\gamma}(\mathbf{K}) = \int\limits_{\mathbb{T}^3} \oplus H^{ch}_{\mu,\gamma}(\mathbf{K},\mathbf{p})d\mathbf{p}. \tag{4.4}$$

The space $L_2[(\mathbb{T}^3)^2]$ also expands into the corresponding direct integral

$$L_2[(\mathbb{T}^3)^2] = \int\limits_{\mathbb{T}^3} \oplus L_2(\mathbb{T}^3)d\mathbf{p}.$$

From uniqueness of the decomposition (4.4) it follows that the fiber operator $H^{ch}_{\mu,\gamma}(\mathbf{K},\mathbf{p})$ has the form

$$H^{ch}_{\mu,\gamma}(\mathbf{K},\mathbf{p}) = h_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})I,$$

where I is the identity operator and $h_{\mu,\gamma}(\mathbf{k})$ is the operator defined by (2.1) (see [25], Theorem XIII.85).

From the definition of $H^{ch}_{\mu,\gamma}(\mathbf{K},\mathbf{p})$ it follows that

$$\sigma(H^{ch}_{\mu,\gamma}(\mathbf{K},\mathbf{p})) = \{z_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})\} \bigcup \big[\mathcal{E}_{\min,\gamma}(\mathbf{K}-\mathbf{p}), \mathcal{E}_{\max,\gamma}(\mathbf{K}-\mathbf{p})\big], \tag{4.5}$$

where $z_{\mu,\gamma}(\mathbf{k})$ is the eigenvalue of the operator $h_{\mu,\gamma}(\mathbf{k})$ (see Theorem 3.2). For any $\mathbf{K} \in \mathbb{T}^3$ we put

$$E_{\min,\gamma}(\mathbf{K}) = \min_{\mathbf{p},\mathbf{q}\in\mathbb{T}^3} E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}), \quad E_{\max,\gamma}(\mathbf{K}) = \max_{\mathbf{p},\mathbf{q}\in\mathbb{T}^3} E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}).$$

By the theorem on the spectrum of decomposable operators (see, e.g. [25]) and the structure of the spectrum (4.5) of operator $H^{ch}_{\mu,\gamma}(\mathbf{K},\mathbf{p})$, we have

$$\sigma(H^{ch}_{\mu,\gamma}(\mathbf{K})) = \bigcup_{\mathbf{p}\in\mathbb{T}^3} \{z_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})\} \bigcup \big[E_{\min,\gamma}(\mathbf{K}), E_{\max,\gamma}(\mathbf{K})\big].$$

Let

$$\mu_{\min,\gamma}(\mathbf{K}) = \min_{\mathbf{p}\in\mathbb{T}^3} \Big(\int\limits_{\mathbb{T}^3} \frac{d\mathbf{q}}{E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}) - E_{\min,\gamma}(\mathbf{K})} \Big)^{-1},$$

$$\mu_{\max,\gamma}(\mathbf{K}) = \max_{\mathbf{p}\in\mathbb{T}^3} \Big(\int\limits_{\mathbb{T}^3} \frac{d\mathbf{q}}{E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}) - E_{\min,\gamma}(\mathbf{K})} \Big)^{-1},$$

and

$$\tau_{\min,\gamma}(\mu,\mathbf{K}) = \min_{\mathbf{p}\in\mathbb{T}^3}\{z_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})\}, \quad \tau_{\max,\gamma}(\mu,\mathbf{K}) = \max_{\mathbf{p}\in\mathbb{T}^3}\{z_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})\}.$$

Lemma 4.1. *The spectrum $\sigma(H^{ch}_{\mu,\gamma}(\mathbf{K}))$ satisfies*

$$\sigma(H^{ch}_{\mu,\gamma}(\mathbf{K})) = [\tau_{min,\gamma}(\mu,\mathbf{K}), \tau_{max,\gamma}(\mu,\mathbf{K})] \cup [E_{min,\gamma}(\mathbf{K}), E_{max,\gamma}(\mathbf{K})].$$

The following theorem describes the structure and location of the essential spectrum $\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K}))$ of the operator $H_{\mu,\gamma}(\mathbf{K})$.

Theorem 4.2. *The essential spectrum $\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K}))$ of the operator $H_{\mu,\gamma}(\mathbf{K})$ coincides with the spectrum of the channel operator, i.e., $\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K})) = \sigma(H^{ch}_{\mu,\gamma}(\mathbf{K}))$. More precisely,*

a) if $\mu \le \mu_{\min,\gamma}(\mathbf{K})$, then

$$\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K})) = [E_{\min,\gamma}(\mathbf{K}), E_{max,\gamma}(\mathbf{K})];$$

b) if $\mu_{\min,\gamma}(\mathbf{K}) < \mu \le \mu_{\max,\gamma}(\mathbf{K})$, then

$$\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K})) = [\tau_{\min,\gamma}(\mu,\mathbf{K}), E_{max,\gamma}(\mathbf{K})]$$

and $\tau_{\min,\gamma}(\mu,\mathbf{K}) < E_{min}(\mathbf{K})$;

c) if $\mu > \mu_{\max,\gamma}(\mathbf{K})$, then

$$\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K})) = [\tau_{\min,\gamma}(\mu,\mathbf{K}), \tau_{\max,\gamma}(\mu,\mathbf{K})] \cup [E_{\min,\gamma}(\mathbf{K}), E_{max,\gamma}(\mathbf{K})]$$

and $\tau_{\max,\gamma}(\mu,\mathbf{K}) < E_{min,\gamma}(\mathbf{K})$.

Proof. We omit the proof, since it can be proved analogously to Theorem 1 in [22]. □

The intervals $[\tau_{min,\gamma}(\mu,\mathbf{K}), \tau_{max,\gamma}(\mu,\mathbf{K})]$ and $[E_{min,\gamma}(\mathbf{K}), E_{max,\gamma}(\mathbf{K})]$ are called "two particle branch" and "three particle branch" of the essential spectrum of the operator $H_{\mu,\gamma}(\mathbf{K})$, respectively.

Corollary 4.3. *Let* $\mathbf{K} = \mathbf{0}$*. Then*

$$\mu_{\min,\gamma}(\mathbf{0}) = \mu_0; \quad \mu_{\max,\gamma}(\mathbf{0}) = \left(\int_{\mathbb{T}^3} \frac{d\mathbf{q}}{9 + 3\gamma + (\gamma - 1)\xi(\mathbf{q})} \right)^{-1}$$

and

$$\sigma_{ess}(H_{\mu,\gamma}(\mathbf{0})) = [z_{\mu,\gamma}(\mathbf{0}),\ z_{\mu,\gamma}(\boldsymbol{\pi}) + 6] \bigcup \left[0,\ 6 + \frac{15}{2}\gamma\right]$$

for $\mu > \mu_{\max,\gamma}(\mathbf{0})$.

5. Discrete spectrum of the three-particle operator

First we show that the operator $H_{\mu,\gamma}(\mathbf{K})$ does not have eigenvalues on the right of $E_{max,\gamma}(\mathbf{K})$.

Lemma 5.1. *Operator* $V = V_1 + V_2$ *is a projection in the space* $L_2^{as}((\mathbb{T}^3)^2)$.

Proof. Since the operators V_1 and V_2 are orthogonal projections and $V_1V_2 = V_2V_1 = 0$, $V_1 + V_2$ is also a projection. □

Theorem 5.2. *For any* $\mathbf{K} \in \mathbb{T}^3$, $\mu > 0, \gamma > 0$, *the operator* $H_{\mu,\gamma}(\mathbf{K})$ *does not have eigenvalues lying above* $E_{max,\gamma}(\mathbf{K})$.

Proof. Positivity of the operator $V = V_1 + V_2$ and the minimax principle imply that

$$\begin{aligned} \sup_{\|f\|\le 1} (H_{\mu,\gamma}(\mathbf{K})f, f) &= \sup_{\|f\|\le 1} [(H_{0,\gamma}(\mathbf{K})f, f) - \mu(Vf, f)] \\ &\le \sup_{\|f\|\le 1} (H_{0,\gamma}(\mathbf{K})f, f) = E_{max,\gamma}(\mathbf{K}) \end{aligned}$$

and then $\sigma(H_{\mu,\gamma}(\mathbf{K})) \cap (E_{max,\gamma}(\mathbf{K}), \infty) = \emptyset$. □

Now we find an equivalent equation for the eigenfunctions of the three-particle operator $H_{\mu,\gamma}(\mathbf{K})$. Let $z < \tau_{\min,\gamma}(\mu, \mathbf{K})$ and define a self-adjoint operator $A_{\mu,\gamma}(\mathbf{K}, z)$ as

$$(A_{\mu,\gamma}(\mathbf{K}, z)g)(\mathbf{p}) = \frac{-\mu}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{K} - \mathbf{p}, z - \varepsilon(\mathbf{p}))}} \int_{\mathbb{T}^3} \frac{(E_{\mathbf{K},\gamma}(\mathbf{p}, \mathbf{s}) - z)^{-1} g(\mathbf{s}) d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{K} - \mathbf{s}, z - \varepsilon(\mathbf{s}))}}, \tag{5.6}$$

which is defined on the subspace

$$D(A_{\mu,\gamma}(\mathbf{K}, z)) = \left\{ g \in L_2(\mathbb{T}^3) : \quad \int_{\mathbb{T}^3} \frac{g(\mathbf{s}) d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{K} - \mathbf{s}, z - \varepsilon(\mathbf{s}))}} = 0 \right\}.$$

Note that, for $z < \tau_{\min,\gamma}(\mu, \mathbf{K})$, the function $\Delta_{\mu,\gamma}(\mathbf{K} - \mathbf{s}, z - \varepsilon(\mathbf{s}))$ is positive for all $\mathbf{K}, \mathbf{s} \in \mathbb{T}^3$.

Lemma 5.3. *A number* $z < \tau_{\min,\gamma}(\mu, \mathbf{K})$ *is an eigenvalue of the operator* $H_{\mu,\gamma}(\mathbf{K})$ *if and only if* 1 *is an eigenvalue of* $A_{\mu,\gamma}(\mathbf{K}, z)$.

*Proof. (**Sufficiency**.)* Let $z < \tau_{\min,\gamma}(\mu, \mathbf{K})$ be an eigenvalue of the operator $H_{\mu,\gamma}(\mathbf{K})$ and $f \in L_2^{as}[(\mathbb{T}^3)^2]$ be the corresponding eigenfunction, i.e.,

$$H_{0,\gamma}(\mathbf{K})f - \mu \sum_{\alpha=1}^{2} V_\alpha f = zf. \tag{5.7}$$

Introducing the notation

$$\varphi(\mathbf{p}) = (V_1 f)(\mathbf{p}) = \int_{\mathbb{T}^3} f(\mathbf{p}, \mathbf{s}) d\mathbf{s}, \tag{5.8}$$

from (5.7) we obtain

$$f(\mathbf{p}, \mathbf{q}) = \mu \frac{\varphi(\mathbf{p}) - \varphi(\mathbf{q})}{E_{\mathbf{K},\gamma}(\mathbf{p}, \mathbf{q}) - z}. \tag{5.9}$$

Since the function f is antisymmetric, we see that the function φ belongs to the space $L_2(\mathbb{T}^3)$ and

$$\int_{\mathbb{T}^3} \varphi(\mathbf{p})\, d\mathbf{p} = 0.$$

Substituting the expression (5.9) into (5.8) we get that the equation

$$\varphi(\mathbf{p})\Big(1 - \mu \int_{\mathbb{T}^3} \frac{d\mathbf{s}}{E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{s}) - z}\Big) = -\mu \int_{\mathbb{T}^3} \frac{\varphi(\mathbf{s})d\mathbf{s}}{E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{s}) - z}$$

has a nonzero solution $\varphi \in L_2(\mathbb{T}^3)$. Using the notation (3.2) we have that

$$\varphi(\mathbf{p}) = \frac{-\mu}{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{p}, z - \varepsilon(\mathbf{p}))} \int_{\mathbb{T}^3} \frac{\varphi(\mathbf{s})d\mathbf{s}}{E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{s}) - z}.$$

If we denote $g(\mathbf{p}) = \sqrt{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{p}, z - \varepsilon(\mathbf{p}))}\,\varphi(\mathbf{p})$, we have

$$g(\mathbf{p}) = \frac{-\mu}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{p}, z - \varepsilon(\mathbf{p}))}} \int_{\mathbb{T}^3} \frac{g(\mathbf{s})d\mathbf{s}}{(E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{s}) - z)\sqrt{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{s}, z - \varepsilon(\mathbf{s}))}},$$

i.e. 1 is an eigenvalue of the operator $A_{\mu,\gamma}(\mathbf{K}, z)$ and

$$\int_{\mathbb{T}^3} \frac{g(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{s}, z - \varepsilon(\mathbf{s}))}} = 0.$$

(Necessity.) Let for some $z < \tau_{\min,\gamma}(\mu, \mathbf{K})$ the number 1 be an eigenvalue of the operator $A_{\mu,\gamma}(\mathbf{K}, z)$ corresponding to an eigenfunction $g \in D(A_{\mu,\gamma}(\mathbf{K}, z))$. Then the function f defined by (5.9) belongs to $L_2^{as}[(\mathbb{T}^3)^2]$ and satisfies the equation (5.7). □

Lemma 5.4. *Let $\gamma > 0$ and $\mu > \mu_{min,\gamma}(\mathbf{K})$. Then*

i) the operator $\mathrm{A}_{\mu,\gamma}(\mathbf{K}, z)$ is continuous in $z \in (-\infty, \tau_{\min,\gamma}(\mu, \mathbf{K})]$;

ii) there exists $\delta > 0$ such that the operator $\mathrm{A}_{\mu,\gamma}(\mathbf{K}, \tau_{\min,\gamma}(\mu, \mathbf{K}))$ is continuous in $\mathbf{K} \in U_\delta(\mathbf{0})$.

Proof. i) Continuity of the operator $\mathrm{A}_{\mu,\gamma}(\mathbf{K}, z)$ as an operator-valued function for $z < \tau_{\min,\gamma}(\mu, \mathbf{K})$ follows from (5.6).

Now we show that the operator $A_{\mu,\gamma}(\mathbf{K}, z)$ converges uniformly to the operator $A_{\mu,\gamma}(\mathbf{K}, \tau_{\min,\gamma}(\mu, \mathbf{K}))$ in the uniform operator topology. Using the following notation

$$A_{\mu,\gamma}(\mathbf{K}, z; \mathbf{p}, \mathbf{q}) = \frac{(E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}) - z)^{-1}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{p}, z - \varepsilon(\mathbf{p}))}\sqrt{\Delta_{\mu,\gamma}(\mathbf{K}-\mathbf{q}, z - \varepsilon(\mathbf{q}))}}$$

we have

$$\begin{aligned}
&\|A_{\mu,\gamma}(\mathbf{K}, z) - A_{\mu,\gamma}(\mathbf{K}, \tau_{\min,\gamma}(\mu, \mathbf{K}))\|^2 \\
&\qquad \le \int_{(\mathbb{T}^3)^2} \left(A_{\mu,\gamma}(\mathbf{K}, z; \mathbf{p}, \mathbf{q}) - A_{\mu,\gamma}(\mathbf{K}, \tau_{\min,\gamma}(\mu, \mathbf{K}); \mathbf{p}, \mathbf{q})\right)^2 d\mathbf{p}d\mathbf{q} \\
&\qquad = \int_{(\mathbb{T}^3 \setminus V_\varepsilon(\mathbf{0}))^2} (\cdot)^2\, d\mathbf{p}d\mathbf{q} + \int_{(\mathbb{T}^3)^2 \setminus (\mathbb{T}^3 \setminus V_\varepsilon(\mathbf{0}))^2} (\cdot)^2\, d\mathbf{p}d\mathbf{q},
\end{aligned}$$

where $V_\varepsilon(\mathbf{0})$ is an ε neighbourhood of the origin. For any $\varepsilon > 0$, the convergence of the first integral of the right-hand side of the last equality to zero follows from the uniform convergence of the integrand for $z \to \tau_{\min,\gamma}(\mu, \mathbf{K})$.

The convergence of the second integral to zero follows directly from a modification of Lemma 6.8, arbitrariness of $\varepsilon > 0$, and absolute continuity of the Lebesgue integral.

Statement ii) is proved similarly to the proof of statement i). □

6. Eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{0})$ for large μ

Now we give Birman-Schwinger's principle for the three-particle operator $H_{\mu,\gamma}(\mathbf{0})$.

For a bounded self-adjoint operator A given on a Hilbert space $\mathcal{H}$ and for some $\lambda \in \mathbb{R}$, we define numbers $n_+[\lambda, A]$ and $n_-[\lambda, A]$ as

$$n_+[\lambda, A] := \max\{\dim \mathcal{H}_A^+(\lambda) : \mathcal{H}_A^+(\lambda) \subset \mathcal{H};\ (A\varphi,\varphi) > \lambda, \varphi \in \mathcal{H}_A^+(\lambda),\ ||\varphi|| = 1\},$$

$$n_-[\lambda, A] := \max\{\dim \mathcal{H}_A^-(\lambda) : \mathcal{H}_A^-(\lambda) \subset \mathcal{H};\ (A\varphi,\varphi) < \lambda,\ \varphi \in \mathcal{H}_A^-(\lambda),\ ||\varphi|| = 1\}.$$

If some point of the essential spectrum of A is greater (resp., less) than λ, then $n_+[\lambda, A]$ (resp., $n_-[\lambda, A]$) is equal to infinity, and if $n_+[\lambda, A]$ (resp., $n_-[\lambda, A]$) is finite, then it is equal to the number of eigenvalues of the operator A, which is greater (resp., less) than λ (see., for example, Glazman's lemma [26]).

The following lemma follows from the well-known Birman–Schwinger principle for the operator $H_{\mu,\gamma}(\mathbf{0})$ (see [23]).

Lemma 6.1. *Let $\mu > \mu_0$. Then, for any $z \leq z_{\mu,\gamma}(\mathbf{0})$,*

$$n_-[z, H_{\mu,\gamma}(\mathbf{0})] = n_+[1, \mathrm{A}_{\mu,\gamma}(\mathbf{0}, z)].$$

In this section we discuss eigenvalues $z \in (-\infty, z_{\mu,\gamma}(\mathbf{0}))$ of the operator $H_{\mu,\gamma}(\mathbf{0})$ for sufficiently large μ. Without loss of generality we may suppose that $z \in (z_{\mu,\gamma}(\mathbf{0}) - 3(1 + \gamma), z_{\mu,\gamma}(\mathbf{0})]$.

Theorem 6.2. *Let $\gamma > \gamma_0$. Then there exists $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ the operator $H_{\mu,\gamma}(\mathbf{0})$ has exactly three eigenvalues, counted with multiplicities, lying below the essential spectrum $z_{\mu,\gamma}(\mathbf{0})$.*

Theorem 6.3. *Let $0 < \gamma < \gamma_0$. Then there exist $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ the operator $H_{\mu,\gamma}(\mathbf{0})$ has no eigenvalues below the essential spectrum $z_{\mu,\gamma}(\mathbf{0})$.*

To make further statements more clear, we assume that the operator $A_{\mu,\gamma}(\mathbf{0}, z)$ is defined on $L_2(\mathbb{T}^3)$. If necessary, use the domain of $A_{\mu,\gamma}(\mathbf{0}, z)$.

It is known that the Hilbert space $L_2(\mathbb{T}^3)$ is represented as

$$L_2(\mathbb{T}^3) = L_2^e(\mathbb{T}^3) \oplus L_2^o(\mathbb{T}^3),$$

where

$$L_2^e(\mathbb{T}^3) = \{f \in L_2(\mathbb{T}^3) : f(-\mathbf{p}) = f(\mathbf{p})\}, \quad L_2^o(\mathbb{T}^3) = \{f \in L_2(\mathbb{T}^3) : f(-\mathbf{p}) = -f(\mathbf{p})\}.$$

Lemma 6.4. *The subspaces $L_2^e(\mathbb{T}^3)$ and $L_2^o(\mathbb{T}^3)$ are invariant with respect to the operator $A_{\mu,\gamma}(\mathbf{0}, z)$.*

Proof. From the definition of $\Delta_{\mu,\gamma}(-\mathbf{p}, z - \varepsilon(\mathbf{p}))$ (see (3.2)) it follows that

$$\Delta_{\mu,\gamma}(-\mathbf{p}, z - \varepsilon(\mathbf{p})) = \Delta_{\mu,\gamma}(\mathbf{p}, z - \varepsilon(-\mathbf{p})) = \Delta_{\mu,\gamma}(\mathbf{p}, z - \varepsilon(\mathbf{p})).$$

If $g \in L_2^e(\mathbb{T}^3)$ then replacing the integrand $g(\mathbf{s})$ with $g(-\mathbf{s})$ and making the change of the variables $-\mathbf{s} = \mathbf{q}$ and also taking into account the equality $E_{\mathbf{0},\gamma}(-\mathbf{p}, -\mathbf{q}) = E_{\mathbf{0},\gamma}(\mathbf{p}, \mathbf{q})$, we get

$$\begin{aligned}\varphi(-\mathbf{p}) &= (A_{\mu,\gamma}(\mathbf{0}, z)g)(-\mathbf{p}) \\ &= \frac{-\mu}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p}, z - \varepsilon(-\mathbf{p}))}} \int_{\mathbb{T}^3} \frac{\mathbf{g}(\mathbf{s})d\mathbf{s}}{(E_{\mathbf{0},\gamma}(-\mathbf{p}, \mathbf{s}) - z)\sqrt{\Delta_{\mu,\gamma}(-\mathbf{s}, z - \varepsilon(\mathbf{s}))}} \\ &= \frac{-\mu}{\sqrt{\Delta_{\mu,\gamma}(-\mathbf{p}, z - \varepsilon(\mathbf{p}))}} \int_{\mathbb{T}^3} \frac{g(\mathbf{q})d\mathbf{q}}{(E_{\mathbf{0},\gamma}(\mathbf{p}, \mathbf{q}) - z)\sqrt{\Delta_{\mu,\gamma}(-\mathbf{q}, z - \varepsilon(\mathbf{q}))}} = \varphi(\mathbf{p}).\end{aligned}$$

Hence, the space $L_2^e(\mathbb{T}^3)$ is invariant under the operator $A_{\mu,\gamma}(\mathbf{0}, z)$.

Since the operator $A_{\mu,\gamma}(\mathbf{0}, z)$ is self-adjoint, the orthogonal complement $L_2^o(\mathbb{T}^3)$ of the subspace $L_2^e(\mathbb{T}^3)$ is also invariant under the operator $A_{\mu,\gamma}(\mathbf{0}, z)$. □

Using the equality $\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x}, \quad (x \neq 1)$ we have

$$\frac{1}{E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{q}) - z} = \frac{1}{6+3\gamma - z}\left(1 + \frac{\xi(\mathbf{p}) + \xi(\mathbf{q}) + \gamma\xi(\mathbf{p}+\mathbf{q})}{6+3\gamma-z} + \frac{\zeta(\gamma;\mathbf{p},\mathbf{q})}{6+3\gamma-z}\right),$$

where

$$\zeta(\gamma;\mathbf{p},\mathbf{q}) = \frac{(\xi(\mathbf{p}) + \xi(\mathbf{q}) + \gamma\xi(\mathbf{p}+\mathbf{q}))^2}{E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{q}) - z}.$$

From this we get

$$(A_{\mu,\gamma}(\mathbf{0},z)g)(\mathbf{p}) = (A^{(0)}_{\mu,\gamma}(\mathbf{0},z)g)(\mathbf{p}) + (A^{(1)}_{\mu,\gamma}(\mathbf{0},z)g)(\mathbf{p}),$$

where the self-adjoint operators $A^{(0)}_{\mu,\gamma}(\mathbf{0},z)$ and $A^{(1)}_{\mu,\gamma}(\mathbf{0},z)$ are given as

$$(A^{(0)}_{\mu,\gamma}(\mathbf{0},z)g)(\mathbf{p}) = -\frac{\mu(6+3\gamma-z)^{-2}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\int_{\mathbb{T}^3}\frac{(6+3\gamma-z+\xi(\mathbf{p})+\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s}))g(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}$$

and

$$(A^{(1)}_{\mu,\gamma}(\mathbf{0},z)g)(\mathbf{p}) = -\frac{\mu(6+3\gamma-z)^{-2}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\int_{\mathbb{T}^3}\frac{\zeta(\gamma;\mathbf{p},\mathbf{s})g(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}.$$

Let us note that, for $g \in D(A_{\mu,\gamma}(\mathbf{0},z))$, the operator $A^{(0)}_{\mu,\gamma}(\mathbf{0},z)$ takes the form

$$(A^{(0)}_{\mu,\gamma}(\mathbf{0},z)g)(\mathbf{p}) = -\frac{\mu(6+3\gamma-z)^{-2}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\int_{\mathbb{T}^3}\frac{(\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s}))g(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}.$$

Lemma 6.5. *The subspaces $L_2^e(\mathbb{T}^3)$ and $L_2^o(\mathbb{T}^3)$ are invariant with respect to the operators $A^{(0)}_{\mu,\gamma}(\mathbf{0},z)$ and $A^{(1)}_{\mu,\gamma}(\mathbf{0},z)$, respectively.*

Proof. The lemma is proved analogously to Lemma 6.4. □

Denote by $A^{(0+)}_{\mu,\gamma}(\mathbf{0},z)$ and $A^{(0-)}_{\mu,\gamma}(\mathbf{0},z)$ restrictions of the operator $A^{(0)}_{\mu,\gamma}(\mathbf{0},z)$ to the subspaces $L_2^o(\mathbb{T}^3)$ and $L_2^e(\mathbb{T}^3)$, respectively, i.e.,

$$A^{(0+)}_{\mu,\gamma}(\mathbf{0},z)\psi(\mathbf{p}) = \frac{\mu\gamma(6+3\gamma-z)^{-2}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\int_{\mathbb{T}^3}\frac{\sum_{i=1}^3 \sin p_i \sin s_i \psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}},$$

and

$$A^{(0-)}_{\mu,\gamma}(\mathbf{0},z)\psi(\mathbf{p}) = -\frac{\mu(6+3\gamma-z)^{-2}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\int_{\mathbb{T}^3}\frac{\sum_{i=1}^3(\cos s_i + \gamma\cos p_i\cos s_i)\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}.$$

Lemma 6.6. *The operator $A^{(0+)}_{\mu,\gamma}(\mathbf{0},z)$ is positive on $L_2^o(\mathbb{T}^3)$ and*

$$\lambda_{\mu,\gamma}(z) = \frac{\mu\gamma}{(6+3\gamma-z)^2}\int_{\mathbb{T}^3}\frac{\sin^2 s_1}{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}d\mathbf{s}$$

is the only nonzero eigenvalue, which has multiplicity three, with the corresponding eigenfunctions

$$g_i(\mathbf{p}) = \frac{\sin p_i}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}, \quad i=1,2,3.$$

Proof. Note, that $D(A_{\mu,\gamma}(\mathbf{0},z)) \cap L_2^o(\mathbb{T}^3) = L_2^o(\mathbb{T}^3)$ and g_1, g_2, g_3 are orthogonal elements of $L_2^o(\mathbb{T}^3)$. In terms of g_i, $i=1,2,3$, the operator $A^{(0+)}_{\mu,\gamma}(\mathbf{0},z)$ is written in the form

$$(A^{(0+)}_{\mu,\gamma}(\mathbf{0},z)\psi)(\mathbf{p}) = \frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{i=1}^3(\psi,g_i)g_i(\mathbf{p}),$$

where $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{T}^3)$. Then for any $\psi \in L_2^o(\mathbb{T}^3)$ we get

$$(A_{\mu,\gamma}^{(0+)}(\mathbf{0},z)\psi,\,\psi) = \frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{i=1}^{3}(\psi,g_i)(g_i,\psi) = \frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{i=1}^{3}|(\psi,g_i)|^2 \geq 0,$$

which proves positivity of $A_{\mu,\gamma}^{(0+)}(\mathbf{0},z)$.

The number $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $A_{\mu,\gamma}^{(0+)}(\mathbf{0},z)$ if and only if the equation

$$\frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{i=1}^{3}(\psi,g_i)g_i(\mathbf{p}) = \lambda\psi(\mathbf{p})$$

has a nonzero solution $\psi \in L_2^o(\mathbb{T}^3)$. If we denote $C_i = (\psi,g_i)$, $i=1,2,3$, in the last equation, we get

$$\lambda\psi(\mathbf{p}) \quad = \quad \frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{j=1}^{3}C_j g_j(\mathbf{p}).$$

Multiplying both sides of the equality by the function g_i and integrating over $\mathbb{T}^3$ we obtain

$$\lambda C_i \quad = \quad \frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{j=1}^{3}C_j(g_j,g_i), \;\; i=1,2,3.$$

By orthogonality of g_1,g_2,g_3 we have that the system of homogeneous equations has a nonzero solution if and only if

$$\prod_{i=1}^{3}\left(\lambda - \frac{\mu\gamma(g_i,g_i)}{(6+3\gamma-z)^2}\right) = 0.$$

Since $\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))$ is symmetric with respect to permutation of the variables p_i and p_j $(i,j=1,2,3)$ we get $(g_1,g_1)=(g_2,g_2)=(g_3,g_3)$. Therefore,

$$\lambda = \frac{\mu\gamma(g_1,g_1)}{(6+3\gamma-z)^2}$$

is a multiplicity three eigenvalue of $A_{\mu,\gamma}^{(0+)}(\mathbf{0},z)$ corresponding to the eigenfunctions $\psi(\mathbf{p}) = \sum_{j=1}^{3}\tilde{C}_j g_j(\mathbf{p})$, where $\tilde{C}_j$ are arbitrary constants. □

The following lemma will be used in the proof of Theorem 2.1.

Lemma 6.7. *For any $z \leq z_{\mu,\gamma}(\mathbf{0})$, the operator $A_{\mu,\gamma}^{(0-)}(\mathbf{0},z)$ is negative, i.e.,*

$$(A_{\mu,\gamma}^{(0-)}(\mathbf{0},z)\psi,\,\psi) \leq 0 \quad \textit{for all} \quad \psi \in D(A_{\mu,\gamma}(\mathbf{0},z)) \cap L_2^e(\mathbb{T}^3).$$

Proof. Using direct calculations and that $\psi \in D(A_{\mu,\gamma}(\mathbf{0},z)) \cap L_2^e(\mathbb{T}^3)$ we have

$$\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\frac{\sum_{i=1}^{3}\cos s_i\psi(\mathbf{s})\overline{\psi(\mathbf{p})}d\mathbf{s}d\mathbf{p}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}$$
$$= \left(\sum_{i=1}^{3}\int_{\mathbb{T}^3}\frac{\cos s_i\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}\right)\overline{\left(\int_{\mathbb{T}^3}\frac{\psi(\mathbf{p})d\mathbf{p}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\right)} = 0,$$

which implies that

$$\begin{aligned}(A^{(0-)}_{\mu,\gamma}(\mathbf{0},z)\psi,\,\psi) &= -\frac{\mu\gamma}{(6+3\gamma-z)^2}\int_{(\mathbb{T}^3)^2}\frac{\left(\sum_{i=1}^3\cos p_i\cos s_i\right)\psi(\mathbf{s})\overline{\psi(\mathbf{p})}d\mathbf{s}d\mathbf{p}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}}\\ &= -\frac{\mu\gamma}{(6+3\gamma-z)^2}\sum_{i=1}^3\left|\int_{\mathbb{T}^3}\frac{\cos p_i\psi(\mathbf{p})d\mathbf{p}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}}\right|^2\le 0.\end{aligned}$$

The lemma is proved. □

Lemma 6.8. *Let $\gamma>0$ and $\mu>3(1+\gamma)$. Then the identity*

$$\frac{1}{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}=\frac{(3+3\gamma-z_{\mu,\gamma}(\mathbf{p}))(6+3\gamma-z)}{\mu(z_{\mu,\gamma}(\mathbf{p})+\varepsilon(\mathbf{p})-z)}\frac{1}{1+Q(\mu,\gamma,z;\mathbf{p})}$$

holds for all $\mathbf{p}\in\mathbb{T}^3$, where $z_{\mu,\gamma}(\mathbf{p})$ is the eigenvalue of the two-particle operator $h_{\mu,\gamma}(\mathbf{p})$ and

$$\begin{aligned}Q(\mu,\gamma,z;\mathbf{p}) = \int_{\mathbb{T}^3}\Bigg[&\frac{\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})}+\frac{\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})+\xi(\mathbf{p})}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z}\\ &+\frac{(\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s}))(\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})+\xi(\mathbf{p}))}{(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p}))(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z)}\Bigg]d\mathbf{s}.\end{aligned}$$

Proof. From Lemma 3.1 we obtain

$$\mu\int_{\mathbb{T}^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})}=1. \tag{6.10}$$

Using this equality we get

$$\begin{aligned}&\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))\\ &\quad=\mu\int_{\mathbb{T}^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})}-\mu\int_{T^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z}\\ &\quad=\mu\int_{T^3}\frac{(z_{\mu,\gamma}(\mathbf{p})+\varepsilon(\mathbf{p})-z)d\mathbf{s}}{[\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})]\,[\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z]}.\end{aligned}$$

Then taking into account the equalities

$$\frac{1}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})}=\frac{1}{3+3\gamma-z_{\mu,\gamma}(\mathbf{p})}\left(1+\frac{\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})}\right)$$

and

$$\frac{1}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z}=\frac{1}{6+3\gamma-z}\left(1+\frac{\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})+\xi(\mathbf{p})}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z}\right)$$

we have

$$\begin{aligned}\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))&=\mu\frac{(z_{\mu,\gamma}(\mathbf{p})+\varepsilon(\mathbf{p})-z)}{[3+3\gamma-z_{\mu,\gamma}(\mathbf{p})]\,[6+3\gamma-z]}\\ &\times\Bigg[1+\int_{\mathbb{T}^3}\frac{\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})}d\mathbf{s}+\int_{\mathbb{T}^3}\frac{\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})+\xi(\mathbf{p})}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z}d\mathbf{s}+\\ &+\int_{\mathbb{T}^3}\frac{(\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s}))(\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})+\xi(\mathbf{p}))}{(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p}))(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z)}d\mathbf{s}\Bigg]\\ &=\mu\frac{z_{\mu,\gamma}(\mathbf{p})+\varepsilon(\mathbf{p})-z}{(3+3\gamma-z_{\mu,\gamma}(\mathbf{p}))(6+3\gamma-z)}\Bigg[1+Q(\mu,\gamma,z;\mathbf{p})\Bigg],\end{aligned}$$

whence the required equality follows.

The lemma is proved. □

Lemma 6.9. *Let $\gamma > 0$. Then there exists $\mu_\gamma > 0$ such that, for any $\mu > \mu_\gamma$, the estimates*

$$\frac{\xi(\mathbf{p})}{\mu} - \frac{C}{\mu^2} \le Q(\mu, \gamma, z; \mathbf{p}) \le \frac{\xi(\mathbf{p})}{\mu} + \frac{C}{\mu^2}$$

hold for all $\mathbf{p} \in \mathbb{T}^3$, where C is a positive constant depending only on γ.

Proof. We write $Q(\mu, \gamma, z; \mathbf{p}) = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \int_{\mathbb{T}^3} \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p})} d\mathbf{s}, \quad I_2 = \int_{\mathbb{T}^3} \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s}) + \xi(\mathbf{p})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z} d\mathbf{s}, \\ I_3 &= \int_{\mathbb{T}^3} \frac{(\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s}))(\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s}) + \xi(\mathbf{p}))}{(\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p}))(\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z)} d\mathbf{s}. \end{aligned}$$

Since $\int_{\mathbb{T}^3} [\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})]^{2k-1} d\mathbf{s} = 0, \quad k = 1, 2, \ldots$, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{T}^3} \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{(3 + 3\gamma - z_{\mu,\gamma}(\mathbf{p}))\left[1 - \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{3 + 3\gamma - z_{\mu,\gamma}(\mathbf{p})}\right]} d\mathbf{s} = \int_{\mathbb{T}^3} \sum_{n=1}^{\infty} \left[\frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{3 + 3\gamma - z_{\mu,\gamma}(\mathbf{p})}\right]^n d\mathbf{s} \\ &= \int_{\mathbb{T}^3} \sum_{k=1}^{\infty} \left[\frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{3 + 3\gamma - z_{\mu,\gamma}(\mathbf{p})}\right]^{2k} d\mathbf{s} = \int_{\mathbb{T}^3} \frac{[\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})]^2}{[3 + 3\gamma - z_{\mu,\gamma}(\mathbf{p})]^2 - [\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})]^2} d\mathbf{s} \\ &\ge 0. \end{aligned}$$

For the second integral I_2, we get

$$I_2 = \int_{\mathbb{T}^3} \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z} d\mathbf{s} + \int_{\mathbb{T}^3} \frac{\xi(\mathbf{p})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z} d\mathbf{s} = I_2^1 + I_2^2.$$

Similarly to I_1 we obtain

$$I_2^1 = \int_{\mathbb{T}^3} \frac{[\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})]^2}{[3 + 3\gamma + \varepsilon(\mathbf{p}) - z]^2 - [\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})]^2} d\mathbf{s} \ge 0.$$

Taking into account (6.10) and positivity of $\varepsilon(\mathbf{p})$ we have

$$\begin{aligned} I_2^2 &= \int_{\mathbb{T}^3} \frac{\xi(\mathbf{p})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p})} d\mathbf{s} \\ &\quad + \int_{\mathbb{T}^3} \left[\frac{\xi(\mathbf{p})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z} - \frac{\xi(\mathbf{p})}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p})}\right] d\mathbf{s} \\ &= \frac{\xi(\mathbf{p})}{\mu} + \int_{\mathbb{T}^3} \frac{\xi(\mathbf{p})(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))}{(\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z)(\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p}))} d\mathbf{s} \\ &\ge \frac{\xi(\mathbf{p})}{\mu} - \int_{\mathbb{T}^3} \frac{|\xi(\mathbf{p})(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))| d\mathbf{s}}{(\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) + \varepsilon(\mathbf{p}) - z)(\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p}))} \\ &\ge \frac{\xi(\mathbf{p})}{\mu} + \frac{|\xi(\mathbf{p})(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))|}{z} \int_{\mathbb{T}^3} \frac{d\mathbf{s}}{\varepsilon(\mathbf{s}) + \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - z_{\mu,\gamma}(\mathbf{p})} \\ &= \frac{\xi(\mathbf{p})}{\mu} + \frac{|\xi(\mathbf{p})(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))|}{z\mu}. \end{aligned}$$

By similar reasoning we obtain

$$\begin{aligned}I_3 = &\int_{\mathbb{T}^3} \frac{(\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})+\frac{\xi(\mathbf{p})}{2})^2 ds}{(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p}))(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z)}\\
&-\frac{\xi^2(\mathbf{p})}{4}\int_{\mathbb{T}^3} \frac{ds}{(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p}))(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z)}\\
\geq &-\frac{\xi^2(\mathbf{p})}{4}\int_{\mathbb{T}^3} \frac{ds}{(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p}))(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z)} \geq \frac{\xi^2(\mathbf{p})}{4z\mu}.\end{aligned}$$

Using the above estimates, and Lemmas 3.4 and 3.6, we get

$$\begin{aligned}Q(\mu,\gamma;\mathbf{p}) &\geq \frac{\xi(\mathbf{p})}{\mu} + \frac{|\xi(\mathbf{p})(z-z_{\mu,\gamma}(\mathbf{p})-\varepsilon(\mathbf{p}))|}{z\mu} + \frac{\xi^2(\mathbf{p})}{4z\mu}\\
&\geq \frac{\xi(\mathbf{p})}{\mu} + \frac{|\xi(\mathbf{p})||z-z_{\mu,\gamma}(\mathbf{p})|}{z\mu} + \frac{|\xi(\mathbf{p})||\varepsilon(\mathbf{p})|}{z\mu} + \frac{\xi^2(\mathbf{p})}{4z\mu}\\
&\geq \frac{\xi(\mathbf{p})}{\mu} + \frac{3}{z_{\mu,\gamma}(\mathbf{0})\mu} + \frac{18}{z_{\mu,\gamma}(\mathbf{0})\mu} + \frac{9}{4z_{\mu,\gamma}(\mathbf{0})\mu} \geq \frac{\xi(\mathbf{p})}{\mu} - \frac{C}{\mu^2}.\end{aligned}$$

Now we prove the second part of the estimates. By Lemma 3.5 we obtain

$$\begin{aligned}I_1 &= \int_{\mathbb{T}^3} \frac{[\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})]^2}{[3+3\gamma-z_{\mu,\gamma}(\mathbf{p})]^2-[\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})]^2} d\mathbf{s}\\
&\leq \frac{9(1+\gamma)^2}{[3+3\gamma-z_{\mu,\gamma}(\mathbf{p})]^2-9(1+\gamma)^2} \leq \frac{9(1+\gamma)^2}{\mu^2-9(1+\gamma)^2}.\end{aligned}$$

Analogously,

$$I_2^1 = \int_{\mathbb{T}^3} \frac{[\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})]^2}{[3+3\gamma+\varepsilon(\mathbf{p})-z]^2-[\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s})]^2} d\mathbf{s} \leq \frac{9(1+\gamma)^2}{\mu^2-9(1+\gamma)^2}$$

and

$$\begin{aligned}I_2^2 &\leq \frac{\xi(\mathbf{p})}{\mu} + \int_{\mathbb{T}^3} \frac{|\xi(\mathbf{p})(z-z_{\mu,\gamma}(\mathbf{p})-\varepsilon(\mathbf{p}))| d\mathbf{s}}{(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})+\varepsilon(\mathbf{p})-z)(\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p}))}\\
&\leq \frac{\xi(\mathbf{p})}{\mu} + \frac{|\xi(\mathbf{p})(z-z_{\mu,\gamma}(\mathbf{p})-\varepsilon(\mathbf{p}))|}{-z\mu},\\
I_3 &\leq \frac{9(1+\gamma)(2+\gamma)}{-z}\int_{\mathbb{T}^3} \frac{ds}{\varepsilon(\mathbf{s})+\gamma\varepsilon(\mathbf{p}+\mathbf{s})-z_{\mu,\gamma}(\mathbf{p})} = \frac{9(1+\gamma)(2+\gamma)}{-z\mu}.\end{aligned}$$

Taking into account the obtained above equalities and estimates, as well as the statements of Lemmas 3.4 and 3.6, i.e., $-z > -z_{\mu,\gamma}(\mathbf{0}) > \mu - 3(1+\gamma)$, we have

$$\begin{aligned}Q(\mu,\gamma,z;\mathbf{p}) &\leq \frac{18(1+\gamma)^2}{\mu^2-9(1+\gamma)^2} + \frac{\xi(\mathbf{p})}{\mu} + \frac{|\xi(\mathbf{p})(z-z_{\mu,\gamma}(\mathbf{p})-\varepsilon(\mathbf{p}))|}{-z\mu} + \frac{9(1+\gamma)(2+\gamma)}{-z\mu}\\
&\leq \frac{\xi(\mathbf{p})}{\mu} + \frac{C}{\mu^2}.\end{aligned}$$

The proof is complete. □

Lemma 6.10. *Let $\gamma > 0$. Then there exists $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ and $z \leq z_{\mu,\gamma}(\mathbf{0})$ the estimate*

$$\|A^{(1)}_{\mu,\gamma}(\mathbf{0},z)\| \leq \frac{C}{\mu}$$

holds, where C is a positive constant depending only on γ.

Proof. Without loss of generality we further denote various variables by C. Let $g \in D(A_{\mu,\gamma}(\mathbf{0},z))$ be a normalized element. Then from Lemmas 6.8, 6.9 and 3.4 we obtain

$$
\begin{aligned}
&|(A^{(1)}_{\mu,\gamma}(\mathbf{0},z)g,g)| \leq \\
&\leq \frac{\mu}{(6+3\gamma-z)^2}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\frac{(\xi(\mathbf{p})+\xi(\mathbf{s})+\gamma\xi(\mathbf{p}+\mathbf{s}))^2|g(\mathbf{s})||\overline{g(\mathbf{p})}|d\mathbf{s}d\mathbf{p}}{(E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})-z)\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}} \\
&\leq \frac{\mu(6+3\gamma)^2}{(6+3\gamma-z)^2(-z)}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\frac{|g(\mathbf{s})||g(\mathbf{p})|d\mathbf{s}d\mathbf{p}}{\sqrt{\Delta_{\mu,\gamma}(\mathbf{s},z-\varepsilon(\mathbf{s}))}\sqrt{\Delta_{\mu,\gamma}(\mathbf{p},z-\varepsilon(\mathbf{p}))}} \\
&= \frac{\mu(6+3\gamma)^2}{(6+3\gamma-z)^2(-z)}\left(\int_{\mathbb{T}^3}\sqrt{\frac{(3+3\gamma-z_{\mu,\gamma}(\mathbf{s}))(6+3\gamma-z)}{\mu(z_{\mu,\gamma}(\mathbf{s})+\varepsilon(\mathbf{s})-z)(1+Q(\mu,\gamma,z;\mathbf{s}))}}|g(\mathbf{s})|d\mathbf{s}\right)^2 \\
&\leq \frac{\mu(3+3\gamma-z_{\mu,\gamma}(\mathbf{0}))(6+3\gamma-z)(6+3\gamma)^2}{\mu(6+3\gamma-z)^2(-z)}\left(\int_{\mathbb{T}^3}\frac{|g(\mathbf{s})|d\mathbf{s}}{\sqrt{\varepsilon(\mathbf{s})\left(1+\frac{\xi(\mathbf{s})}{\mu}-\frac{C}{\mu^2}\right)}}\right)^2 \\
&\leq \frac{\left(\mu+\frac{9(1+\gamma)^2}{\mu}\right)(6+3\gamma)^2}{(6+3\gamma-z)(-z)}\int_{\mathbb{T}^3}|g(\mathbf{s})|^2d\mathbf{s}\int_{\mathbb{T}^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})(1+\frac{\xi(\mathbf{s})}{\mu}-\frac{C}{\mu^2})} \\
&\leq \frac{\left(\mu+\frac{9(1+\gamma)^2}{\mu}\right)(6+3\gamma)^2}{(6+3\gamma-z_{\mu,\gamma}(\mathbf{0}))(-z_{\mu,\gamma}(\mathbf{0}))}\int_{\mathbb{T}^3}|g(\mathbf{s})|^2d\mathbf{s}\int_{\mathbb{T}^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})(1+\frac{\xi(\mathbf{s})}{\mu}-\frac{C}{\mu^2})} \\
&\leq \frac{\left(\mu+\frac{9(1+\gamma)^2}{\mu}\right)(6+3\gamma)^2}{(3+\mu)(\mu-3(1+\gamma))}\int_{\mathbb{T}^3}|g(\mathbf{s})|^2d\mathbf{s}\int_{\mathbb{T}^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})(1+\frac{\xi(\mathbf{s})}{\mu}-\frac{C}{\mu^2})} \leq \frac{C}{\mu}. \qquad \square
\end{aligned}
$$

Lemma 6.11. *Let $\gamma > 0$. Then there exists $\mu_\gamma > 0$ such that, for any $\mu > \mu_\gamma$ the eigenvalue $\lambda_{\mu,\gamma}(z_{\mu,\gamma}(\mathbf{0}))$ of the operator $A^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))$ satisfies*

$$\frac{\gamma}{\gamma_0}-\frac{C}{\mu}\leq\lambda_{\mu,\gamma}(z_{\mu,\gamma}(\mathbf{0}))\leq\frac{\gamma}{\gamma_0}+\frac{C}{\mu},$$

where C is a positive constant depending only on γ.

Proof. Without loss of generality we further denote various variables by C. Using Lemmas 6.8, 6.9, and 3.4 we get

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\frac{\sin^2 p_1 d\mathbf{p}}{\Delta_{\mu,\gamma}(\mathbf{p},z_{\mu,\gamma}(\mathbf{0})-\varepsilon(\mathbf{p}))} = \\
&= \int_{\mathbb{T}^3}\sin^2 p_1\left[\frac{(3+3\gamma-z_{\mu,\gamma}(\mathbf{p}))(6+3\gamma-z_{\mu,\gamma}(\mathbf{0}))}{\mu(z_{\mu,\gamma}(\mathbf{p})+\varepsilon(\mathbf{p})-z_{\mu,\gamma}(\mathbf{0}))(1+Q(\mu,\gamma,\mathbf{p}))}\right]d\mathbf{p} \\
&\geq \frac{(3+3\gamma-z_{\mu,\gamma}(\boldsymbol{\pi}))(6+3\gamma-z_{\mu,\gamma}(\mathbf{0}))}{\mu}\int_{\mathbb{T}^3}\frac{\sin^2 p_1}{\varepsilon(\mathbf{p})\,(1+Q(\mu,\gamma,\mathbf{p}))}d\mathbf{p} \\
&\geq \frac{\mu(6+3\gamma-z_{\mu,\gamma}(\mathbf{0}))}{\mu}\int_{\mathbb{T}^3}\frac{\sin^2 p_1}{\varepsilon(\mathbf{p})\left(1+\frac{\xi(\mathbf{p})}{\mu}+\frac{C}{\mu^2}\right)}d\mathbf{p} \\
&\geq (6+3\gamma-z_{\mu,\gamma}(\mathbf{0}))\left[\int_{\mathbb{T}^3}\frac{\sin^2 p_1}{\varepsilon(\mathbf{p})}d\mathbf{p}-\frac{C}{\mu}\right].
\end{aligned}
$$

From this and Lemma 3.4 we have

$$\begin{aligned}\lambda_{\mu,\gamma}(z_{\mu,\gamma}(\mathbf{0})) &\geq \frac{\mu\gamma}{6+3\gamma-z_{\mu,\gamma}(\mathbf{0})}\left[\int_{\mathbb{T}^3}\frac{\sin^2 p_1}{\varepsilon(\mathbf{p})}d\mathbf{p}-\frac{C}{\mu}\right]\\ &\geq \frac{\mu\gamma}{\mu+3+\dfrac{9(1+\gamma)^2}{\mu}}\left[\frac{1}{\gamma_0}-\frac{C}{\mu}\right]\geq\frac{\gamma}{\gamma_0}-\frac{C}{\mu}.\end{aligned}$$

The upper estimate is proved analogously. The proof is complete. □

Lemma 6.12. *Let $\gamma>\gamma_0$. Then there exists $\mu_\gamma>0$ such that for any $\mu>\mu_\gamma$ the following identity holds true:*

$$n_+[1,\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]=3.$$

Proof. Let $\varepsilon\in(0,1)$. Then Weyl's inequality (see [13]) and Lemma 6.7 yield

$$\begin{aligned}n_+[1-\varepsilon,\mathrm{A}^{(0)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))] &= n_+[1-\varepsilon,\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))+\mathrm{A}^{(0-)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\\ &\leq n_+[1-\varepsilon,\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]+n_+[0,\mathrm{A}^{(0-)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\\ &= n_+[1-\varepsilon,\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))].\end{aligned}$$

From this and Weyl's inequality we get

$$\begin{aligned}n_+[1,\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))] &= n_+[1-\varepsilon+\varepsilon,\mathrm{A}^{(0)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))+\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\\ &\leq n_+[1-\varepsilon,\mathrm{A}^{(0)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]+n_+[\varepsilon,\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\\ &\leq n_+[1-\varepsilon,\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]+n_+[\varepsilon,\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))].\end{aligned}$$

By Lemma 6.10 we conclude that for any $\varepsilon>0$ there exists $\mu_\gamma>0$ such that

$$n_+[\varepsilon,\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]=0$$

for all $\mu>\mu_\gamma$. Hence Lemma 6.11 implies that

$$n_+[1,\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\leq n_+[1-\varepsilon,\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]=n_+[1,\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]=3.$$

Now we show that $n_+[1,\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\geq 3$. Let $\mathcal{H}_3=\{g_1,g_2,g_3\}$ be a three-dimensional subspace of $L_2^o(\mathbb{T}^3)$ spanned by the eigenfunctions g_1,g_2 and g_3 of the operator $\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))$. Then for an arbitrary normalized element $g\in\mathcal{H}_3$ we have

$$\begin{aligned}\big(\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))g,g\big) &= \big(\mathrm{A}^{(0)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))g,g\big)+\big(\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))g,g\big)\\ &= \big(\mathrm{A}^{(0+)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))g,g\big)+\big(\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))g,g\big).\end{aligned}$$

Hence, from Lemmas 6.11 and 6.10 we get

$$\big(\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))g,g\big)\geq\frac{\gamma}{\gamma_0}-\frac{C}{\mu}.$$

We choose the parameter μ_γ such that for any $\mu>\mu_\gamma$ the right-hand side of the last inequality is greater than one. Then by the definition of $n_+[1,\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]$ we get

$$n_+[1,\mathrm{A}_{\mu,\gamma}(\mathbf{0},z_{\mu,\gamma}(\mathbf{0}))]\geq dim\mathcal{H}_3=3.$$

The lemma is proved. □

Proof of Theorem 6.2. The proof follows directly from Lemmas 6.1 and 6.12. □

Proof of Theorem 6.3. Let $0 < \gamma < \gamma_0$. Then by Lemmas 6.10 and 6.11 we have the existence of $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ the following estimates hold true:

$$\begin{aligned}
&\sup_{\|\varphi\|=1} (\mathrm{A}_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0}))\varphi, \varphi) \\
&\quad \le \sup_{\|\varphi\|=1} \left(\mathrm{A}^{(0)}_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0}))\varphi, \varphi\right) + \sup_{\|\varphi\|=1} \left(\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0}))\varphi, \varphi\right) \\
&\quad = \lambda_{\mu,\gamma}(z_{\mu,\gamma}(\mathbf{0})) + \|\mathrm{A}^{(1)}_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0}))\| \le \frac{\gamma}{\gamma_0} + \frac{C}{\mu} + \frac{C}{\mu}.
\end{aligned}$$

Choosing μ_γ sufficiently large we get that

$$\sup_{\|\varphi\|=1} (\mathrm{A}_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0}))\varphi, \varphi) \le 1,$$

i.e.,

$$n_+[1, \mathrm{A}_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0}))] = 0.$$

Then the proof of the theorem follows from Lemma 6.1. □

Proof of Theorem 2.1. By Lemma 5.4 we get that

$$\lim_{\mathbf{K}\to\mathbf{0}} \|A_{\mu,\gamma}(\mathbf{K}, \tau_{\min,\gamma}(\mu, \mathbf{K})) - A_{\mu,\gamma}(\mathbf{0}, z_{\mu,\gamma}(\mathbf{0})))\| = 0.$$

From this and Lemma 6.12 we have

$$n_+[1, \mathrm{A}_{\mu,\gamma}(\mathbf{K}, \tau_{\min,\gamma}(\mu, \mathbf{K}))] = 3$$

for all $\mathbf{K} \in V_\delta(\mathbf{0})$.

The proof is complete. □

Proof of Theorem 2.2. The proof follows from Theorem 6.3 and repeating the same procedure as in the proof of Theorem 2.1. □

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