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THE INNER STRUCTURE OF THE BLOCK JACOBI TYPE MATRIX RELATED TO THE COMPLEX MOMENT PROBLEM WITH THE MEASURE SUPPORTED ON THE SECOND ORDER CURVE

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Abstract. We present an exact inner structure of the block Jacobi type matrix related to the complex moment problem with the corresponding measure supported on an arbitrary second order curve in the complex plane. For completeness of the study we also present a solution of the direct and inverse spectral problems for such matrices. In this the way, we give a necessary and sufficient condition under which a matrix in the CMV-form generates a (pre)normal operator, namely, not obligatory a unitary one.

Надано точну внутрішню структуру блочної якоюієвої матриці, яка пов'язана з комплексною проблемою моментів і мірою, що міє носій на довільній кривій другого порядку в комплексній площині. Для повноти дослідження подаємо також розв'язок прямої та оберненої спектральних задачі для таких матриць. Ми також даємо необхідну і достатню умову зв якої CMV-матриця породжує (пре)нормальний оператор, а саме не обов'язково унітарний.

1. Introduction

By solving the classical Hamburger moment problem we obtain a usual three-diagonal matrix called a Jacobi matrix [1]. A construction of such a matrix poses no problem. But if we solve a trigonometric moment problem we get a three-diagonal block matrix (usually a five-diagonal matrix in the spacial form, a CMV-matrix [2, 3]). Necessary and sufficient conditions on elements of a matrix under which the matrix in the CMV-form generates a unitary operator are not simple but known [2, 3] (see also [4]) due to the fact that the inner structure of matrix elements are known. By the concept of the inner structure or a description, we mean a set of some given parameters and rules that are used to construct a matrix with the corresponding properties.

If we solve, for example, a strong Hamburger moment problem we meet two commuting three-diagonal block matrices called Jacobi-Laurent matrices (usually they are five-diagonal of a spacial form [5, 6]). Necessary and sufficient condition for elements of the matrix under which such matrices generate symmetric commuting operators are known [5, 6] since the inner structure of matrix elements is described.

The situation is much more complicated with the complex moment problem. By solving the complex moment problem we obtain a block three-diagonal Jacobi-type matrix corresponding to a (pre)normal operator (prenormal is a densely defined operator that has a normal extension in the same space) [7, 8]. In this case, the blocks of the matrix are growing. A description of the inner structure of such a matrix is, perhaps, a non-realistic task. We don't known conditions for elements of such a matrix under which the matrices would generate a (pre)normal operator. Only simple examples are known.

In this article we describe a particular case, namely an exact inner structure of a block Jacobi-type matrix related to the complex moment problem but with the correspondence measure supported only on an arbitrary second order curve in the complex plane. The

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description is given in form of a necessary and sufficient condition for elements of correspondence matrix. Such a matrix does not have growing blocks and has a form of a CMV-matrix. In this connection we answer an additional question, — under what conditions on the elements of the CMV-matrix it will generate a (pre)normal operator.

For completeness of the study we also present a solution of the direct and inverse spectral problems for such matrices. The inverse spectral problem is a construction of block three-diagonal Jacobi-type matrix using a given measure, and the direct spectral problem consists in recovering the measure using a given block three-diagonal Jacobi-type matrix.

We remark that, actually, the trigonometric and the strong Hamburger moment problems are two particular cases. In general, we do not have any description of elements of matrices that would correspond to the usual, the strong, the half-strong two-dimensional moment problems [9, 10, 11], the complex moment problem in the exponential form [13, 12]. These problems are still open.

A collection of the cited above results is also contained in the monograph [14].

2. Preliminaries

The complex moment problem consists in finding a condition on a given sequence of complex numbers $\{s_{m,n}\}$, $m, n \in \mathbb{N}_0$, $s_{m,n} \in \mathbb{C}$, that would imply existence of a nonnegative Borel measure $d\rho(z)$ on the complex plane $\mathbb{C}$ such that

$$s_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\rho(z), \qquad m, n \in \mathbb{N}_0. \tag{2.1}$$

For existence of representation (2.1) it is necessary that a condition of positive definiteness would be fulfilled [7, 8],

$$\sum_{j,k,m,n=0}^{\infty} f_{j,k} \bar{f}_{m,n} s_{j+n,k+m} \geq 0 \tag{2.2}$$

for all finite sequences of complex numbers $(f_{j,k})_{j,k=0}^{\infty}$, $f_{j,k} \in \mathbb{C}$.

For a given sequence of complex numbers $\{s_{m,n}\}_{m,n=0}^{\infty}$ there exists a representation (2.1) and is unique if it is positive definite and

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[2p]{s_{2p,2p}}} = \infty. \tag{2.3}$$

In this paper, we suppose that the measure $d\rho(z)$ is supported on an order two curve

$$\Gamma = \{\, z \,|\, z\bar{z} = \theta_0 \bar{z}^2 + \theta_1 z + \theta_2 \bar{z} + \theta_3,\ \theta_i \in \mathbb{C},\ i = 0, 1, 2, 3\}.$$

3. Construction of a three-diagonal block matrix

Consider a Borel probability measure $d\rho(z)$ on Γ and the space $L^2 = L^2(\Gamma, d\rho(z))$ of square integrable complex valued functions defined on Γ. We suppose that the support of this measure is an infinite set and the functions $z^n, \bar{z}^n$, $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$, are linearly independent on L^2.

Let us consider the sequence of functions

$$1; \quad z, \bar{z}; \quad z^2, \bar{z}^2; \quad \ldots \quad z^n, \bar{z}^n; \quad \ldots \tag{3.4}$$

and apply the Schmidt orthogonalization procedure (see, for example, [18] Ch. 7). As a result we obtain an orthonormal system of polynomials (each one ia a polynomial of z

and $\bar{z}$) which we denote in the following way

$$\begin{aligned} &P_0(z) = 1; \\ &P_{1;1}(z) = k_{1;1}z + \cdots,\ P_{2;1}(z) = k_{2;1}z^2 + \cdots,\ \cdots;\ P_{n;1}(z) = k_{n;1}z^n + \cdots,\quad \cdots \\ &P_{1;2}(z) = k_{1;2}\bar{z} + \cdots;\ P_{2;2}(z) = k_{2;2}\bar{z}^2 + \cdots;\quad \cdots\ P_{n;2}(z) = k_{n;2}e\bar{z}^n + \cdots;\ \cdots \end{aligned} \tag{3.5}$$

where $k_{n;1} > 0$, $k_{n;2} > 0$ and "$+\cdots$" denotes the next part of the corresponding polynomial. In such a way $P_{n;1}$ is a linear combination of $\{1, z, \bar{z}, \ldots, z^{n-1}, \bar{z}^{n-1}, z^n\}$ and $P_{n;2}$ is a linear combination of $\{1, z, \bar{z}, \ldots, z^{n-1}, \bar{z}^{n-1}, z^n, \bar{z}^n\}$.

Let us denote $\mathcal{P}_{n;1}$ and $\mathcal{P}_{n;2}$ two spaces spanned by $\{1, z, \bar{z}, \ldots, z^{n-1}, \bar{z}^{n-1}, z^n\}$ and $\{1, z, \bar{z}, \ldots, z^{n-1}, \bar{z}^{n-1}, z^n, \bar{z}^n\}$, respectively, $n \in \mathbb{N}$. Hence, one can say that $P_{n;1}(z)$ ($P_{n;2}(z)$) is orthogonal to $\mathcal{P}_{n-1;2}$ ($\mathcal{P}_{n;1}$). It is clear that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} &\mathcal{P}_0 \subset \mathcal{P}_{1;1} \subset \mathcal{P}_{1;2} \subset \cdots \subset \mathcal{P}_{n;1} \subset \mathcal{P}_{n;2} \subset \cdots, \\ &\mathcal{P}_{n;1} = \{P_0(z)\} \oplus \{P_{1;1}(z)\} \oplus \cdots \oplus \{P_{n-1;1}(z)\} \oplus \{P_{n-1;2}(z)\} \oplus \{P_{n;1}(z)\}, \\ &\mathcal{P}_{n;2} = \mathcal{P}_{n;1} \oplus \{P_{n;2}(z)\} = \{P_0(z)\} \oplus \{P_{1;1}(z)\} \oplus \cdots \oplus \{P_{n;1}(z)\} \oplus \{P_{n;2}(z)\}, \end{aligned} \tag{3.6}$$

where $\{P_{n;\alpha}(z)\}$, $n \in \mathbb{N}, \alpha = 1, 2$, denotes a one dimensional space spanned by $P_{n;\alpha}(z)$; $\mathcal{P}_0 = \mathbb{C}$.

In the sequel, we need the Hilbert space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathbb{C}^2. \tag{3.7}$$

Each vector $f \in \mathbf{l}_2$ has the form $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$, and consequently for all $f, g \in \mathbf{l}_2$,

$$\|f\|^2_{\mathbf{l}_2} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} < \infty, \quad (f,g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}.$$

For $n \in \mathbb{N}$, coordinates of a vector $f_n \in \mathcal{H}$, with respect to the standard orthonormal basis $\{e_{n;1}, e_{n;2}\}$ in the space $\mathbb{C}^2$, are denoted by $(f_{n;1}, f_{n;2})$ and, hence, we have $f_n = (f_{n;1}, f_{n;2})$.

Using the orthonormal system (3.5) one can define a mapping from $\mathbf{l}_2$ into L^2. We put $P_n(z) = (P_{n;1}(z), P_{n;2}(z)) \in \mathcal{H}_n$, then

$$\mathbf{l_2} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} (f_n, P_n(z))_{\mathcal{H}_n} \in L^2. \tag{3.8}$$

Since for $n \in \mathbb{N}$ $(f_n, P_n(z))_{\mathcal{H}_n} = f_{n;1}\overline{P_{n;1}(z)} + f_{n;2}\overline{P_{n;2}(z)}$ and $\|f\|^2_{\mathbf{l}_2} = \|(f_0, f_{1;1}, f_{1;2}, f_{2,1}, \ldots)\|^2_{l_2}$, we see that (3.8) is a mapping of the usual space l_2 into L^2 via the orthonormal system (3.5) and, hence, this mapping is isometric. The image of $\mathbf{l}_2$ under the mapping (3.8) is equal to the space L^2. This is indeed so, since linear combinations of (3.4) uniformly approximate an arbitrary continuous function on Γ and such a set of functions is dense in L^2; hence, the system (3.5) is total in L^2. Therefore the mapping (3.8) is a unitary transformation (denoted by I) that maps $\mathbf{l}_2$ onto L^2.

Let A be a bounded linear operator defined on the space $\mathbf{l}_2$. It is possible to construct an operator matrix $(a_{j,k})_{j,k=0}^{\infty}$, where, for each $j, k \in \mathbb{N}_0$, the element $a_{j,k}$ is an operator from $\mathcal{H}_k$ into $\mathcal{H}_j$, so that for all $f, g \in \mathbf{l}_2$ we have

$$(Af)_j = \sum_{k=0}^{\infty} a_{j,k} f_k,\ j \in \mathbb{N}_0, \quad (Af, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (a_{j,k} f_k, g_j)_{\mathcal{H}_j}. \tag{3.9}$$

To prove (3.9) we only need to write the usual matrix of the operator A in the space l_2 using the basis $(e_0 = 1, e_{1;1}, e_{1;2}, e_{2;1}, \ldots)$. Then $a_{j,k}$ for each $j, k \in \mathbb{N}$ is an operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ that has a matrix representation $(a_{j,k;\alpha,\beta})_{\alpha,\beta=1}^{2}$, so that

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{l_2}. \tag{3.10}$$

If $j=0, k=1,2,\dots$, then $(a_{0,k;\beta})_{\beta=1}^{2}$ is a 1×2-matrix operator $\mathcal{H}_k \longrightarrow \mathcal{H}_0$, where $a_{0,k;\beta}=(Ae_{k;\beta},e_0)_{l_2}$; if $k=0, j=1,2,\dots$, then $(a_{j,0;\alpha})_{\alpha=1}^{2}$ is a 2×1-matrix operator $\mathcal{H}_0 \longrightarrow \mathcal{H}_j$, where $a_{j,0;\alpha}=(Ae_0,e_{j;\alpha})_{l_2}$. And $a_{0,0}$ is a 1×1-matrix operator (scalar) $\mathcal{H}_0 \longrightarrow \mathcal{H}_0$, where $a_{0,0}=(Ae_0,e_0)_{l_2}$.

Let us consider the image $\hat{A}=IAI^{-1} : L^2 \longrightarrow L^2$ of the above operator A defined by the mapping (3.8). Its matrix in the basis (3.5), $(P_0(z),P_{1;1}(z),P_{1;2}(z),P_{2;1}(z),\dots)$, is equal to the usual matrix of the operator A: $l_2 \longrightarrow l_2$ in the corresponding basis (e_0, $e_{1;1}$, $e_{1;2}$, $e_{2;1}$, $\dots$). Using (3.10) and the above mentioned procedure, we get an operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $A: l_2 \longrightarrow l_2$. *By the definition this matrix is also the operator matrix of* $\hat{A}: L^2 \longrightarrow L^2$.

It is clear that $\hat{A}$ can be an arbitrary linear bounded operator in L^2.

Lemma 3.1. *Let $\hat{A}$ be the bounded normal operator of multiplication on z in the space L^2,*

$$L^2 \ni \varphi(z) \longmapsto (\hat{A}\varphi)(z)=z\varphi(z)\in L^2.$$

The operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $\hat{A}$ (i.e. of $A=I^{-1}\hat{A}I$) has a three-diagonal structure: $a_{j,k}=0$ for $|j-k|>1$.

Proof. Using (3.10) for $e_{n;\gamma}=I^{-1}P_{n;\gamma}(z)$, $n\in\mathbb{N}$, $\gamma=1,2$, we have, in the case $j,k\in\mathbb{N}$,

$$a_{j,k;\alpha,\beta}=(Ae_{k;\beta},e_{j;\alpha})_{l_2}=\int_{\Gamma} zP_{k;\beta}(z)\overline{P_{j;\alpha}(z)}d\rho(z), \qquad \alpha,\beta=1,2. \tag{3.11}$$

From (3.5) we have $zP_{k;1}(z)\in\mathcal{P}_{k+1;1}$ and $zP_{k;2}(z)\in\mathcal{P}_{k+1;1}$. According to (3.6) the integral in (3.11) is equal to zero for $j>k+1$ and for each $\alpha=1,2$.

On the other hand, the integral in (3.11) has the form

$$a_{j,k;\alpha,\beta}=\overline{\int_{\Gamma} \bar{z}P_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\,d\rho(z)}.$$

From (3.5) we have now that $\bar{z}P_{j;1}(z)\in\mathcal{P}_{j;2}$ and $\bar{z}P_{j;2}(z)\in\mathcal{P}_{j+1;2}$. According to (3.6) the last integral is equal to zero for $k>j+1$ and each $\beta=1,2$.

As a result, the integral in (3.11), i.e., the elements $a_{j,k;\alpha,\beta}$, $j,k\in\mathbb{N}$, are equal to zero for $|j-k|>1$, $\alpha,\beta=1,2$. The cases $j=1,2,\dots$, $k=0$ and $j=0$, $k=1,2,\dots$ are considered analogously (it is necessary to take into account that $e_0=I^{-1}P_0(z)$, $P_0(z)=1$). $\square$

In such a way the matrix $(a_{j,k})_{j,k=0}^{\infty}$ of the operator $\hat{A}$ has a three diagonal block structure

$$\left(\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \dots \\ \hline 0 & a_{0,0} & a_{0,1} & 0 & 0 & 0 & \dots \\ 1 & a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 & \dots \\ 2 & 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & \dots \\ 3 & 0 & 0 & a_{3,2} & a_{3,3} & a_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{3.12}$$

A further detailed analysis of the expressions in (3.11) gives a possibility to know about the zero and non zero elements of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=1}^{2}$ in each case for $|j-k|\le 1$. We describe the permutation properties of the matrix indexes j,k, and α,β. Let us remark that from (3.11) it follows that the norms $\|a_{j,k}\|$ are uniformly bounded with respect to $j,k\in\mathbb{N}_0$.

Lemma 3.2. *For the polynomials $P_{n;\alpha}(z)$ and the subspaces $\mathcal{P}_{m,\beta}$, $n,m\in\mathbb{N}_0$, $\alpha,\beta=1,2$, the following relations hold:*

$$\begin{aligned} zP_{n;1}(z)&\in\mathcal{P}_{n+1;1}, &\qquad \bar zP_{n;1}(z)&\in\mathcal{P}_{n;2};\\ zP_{n;2}(z)&\in\mathcal{P}_{n+1;1}, &\qquad \bar zP_{n;2}(z)&\in\mathcal{P}_{n+1;2};\\ zP_0(z)&\in\mathcal{P}_{1;1}, &\qquad \bar zP_0(z)&\in\mathcal{P}_{1;2}.\end{aligned}\tag{3.13}$$

Proof. According to (3.5) the polynomial $P_{n;1}(z)$, $n\in\mathbb{N}$, is equal to some linear combination of $\{1,\, z,\, \bar z,\, \ldots,\, z^{n-1},\, \bar z^{n-1},\, z^n\}$. Hence, by the multiplication on z we obtain a linear combination of $\{z,\, z^2,\, z\bar z,\, \ldots,\, z^n,\, \bar z^{n-1}z,\, z^{n+1}\}$, and such a linear combination belongs to $\mathcal{P}_{n+1;1}$. Similarly, by multiplication on $\bar z$ we obtain a linear combination of $\{\bar z,\, z\bar z,\, \bar z^2,\, \ldots,\, z^{n-1}\bar z,\, \bar z^n,\, z^n\bar z\}$, and such a linear combination belongs to $\mathcal{P}_{n;2}$.

According to (3.5), the polynomial $P_{n;2}(z)$, $n\in\mathbb{N}$, is equal to some linear combination of $\{1,\, z,\, \bar z,\, \ldots,\, z^{n-1},\, \bar z^{n-1},\, z^n,\, \bar z^n\}$. Hence, by multiplication on z we obtain a linear combination of $\{z,\, z^2,\, z\bar z,\, \ldots,\, z^n,\, \bar z^{n-1}z,\, z^{n+1},\, \bar z^n z\}$, and such a linear combination belongs to $\mathcal{P}_{n+1;1}$. In the same way, by multiplication on $\bar z$ we obtain a linear combination of $\{\bar z,\, z\bar z,\, \bar z^2,\, \ldots,\, z^n\bar z,\, \bar z^{n+1}\}$, and this linear combination belongs to $\mathcal{P}_{n+1;2}$.

Thus the relations (3.13) are proved since the case $n=0$ is obvious. □

Let us denote $((a^*)_{j,k})_{j,k=0}^{\infty}$ the operator matrix of the operator $(\hat A)^*$ adjoint to $\hat A$. Note that $(\hat A)^*$ is an operator of multiplication by $\bar z$. Taking into account the expression (3.11) for $j,k\in\mathbb{N}$ we have

$$(a^*)_{j,k;\alpha,\beta}=\int_\Gamma \bar zP_{k;\beta}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z)=\overline{\int_\Gamma zP_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\,d\rho(z)}=\overline{a_{k,j;\beta,\alpha}},\quad \alpha,\beta=1,2.\tag{3.14}$$

In cases $j=0$, $k\in\mathbb{N}$, $k=0$, $j\in\mathbb{N}$, $j=0$, $k=0$, instead of (3.11), we have

$$\begin{aligned} a_{0,k;\beta}&=\int_\Gamma zP_{k;\beta}(z)\,d\rho(z),\quad k\in\mathbb{N},\quad \beta=1,2;\\ a_{j,0;\alpha}&=\int_\Gamma z\overline{P_{j;\alpha}(z)}\,d\rho(z),\quad j\in\mathbb{N},\quad \alpha=1,2;\\ a_{0,0}&=\int_\Gamma z d\rho(z).\end{aligned}\tag{3.15}$$

In these cases the equality (3.14) has the form

$$(a^*)_{0,k;\beta}=\bar a_{k,0;\beta},\quad (a^*)_{j,0;\alpha}=\bar a_{0,j;\alpha},\quad a_{0,0}=\bar a_{0,0},\quad j,k=\mathbb{N},\quad \alpha,\beta=1,2.\tag{3.16}$$

Lemma 3.3. *Let $(a_{j,k})_{j,k=0}^{\infty}$ be an operator matrix of multiplication by z in L^2, where $a_{j,k}:\mathcal{H}_k\longrightarrow\mathcal{H}_j$; $a_{0,0}$, $a_{0,k}=(a_{0,k;\beta})_{\beta=1}^2$, $a_{j,0}=(a_{j,0;\alpha})_{\alpha=1}^2$, $a_{j,k}=(a_{j,k;\alpha,\beta})_{\alpha,\beta=1}^2$ are matrices of the operators $a_{j,k}$ with respect to the standard basis. Then*

$$a_{j,0;2}=a_{j,j+1;1,1}=a_{j,j+1;1,2}=a_{j+1,j;2,1}=a_{j+1,j;2,2}=0,\quad j\in\mathbb{N}.\tag{3.17}$$

Proof. According to (3.15), for $j\in\mathbb{N}$, we have

$$a_{j,0;2}=\int_\Gamma z\overline{P_{j,2}(z)}\,d\rho(z).\tag{3.18}$$

Due to (3.13) $z=zP_0(z)\in\mathcal{P}_{1;1}$, but $P_{j;2}(z)$ is orthogonal to $\mathcal{P}_{1;1}$ for $j\in\mathbb{N}$ (see (3.6)). Hence, in this case, the integral in (3.18) is equal to zero.

According to (3.11) and (3.13), for $j\in\mathbb{N}$, we have

$$a_{j,j+1;1,1}=\int_\Gamma zP_{j+1,1}(z)\overline{P_{j;1}(z)}\,d\rho(z)=\overline{\int_\Gamma \bar zP_{j,1}(z)\overline{P_{j+1;1}(z)}\,d\rho(z)},$$

where $\bar zP_{j;1}(z)\in\mathcal{P}_{j;2}$. But according to (3.6), $P_{j+1;1}(z)$ is orthogonal to $\mathcal{P}_{j;2}$ and, hence, the last integral is equal to zero.

Analogously, from (3.11), (3.13), and (3.6) for $j \in \mathbb{N}$ we have

$$a_{j,j+1;1,2} = \int_\Gamma z P_{j+1,2}(z) \overline{P_{j;1}(z)}\, d\rho(z) = \overline{\int_\Gamma \bar{z} P_{j,1}(z) \overline{P_{j+1;2}(z)}\, d\rho(z)},$$

where $\bar{z}P_{j;1}(z) \in \mathcal{P}_{j;2}$. But according to (3.6) $P_{j+1;2}(z)$ is orthogonal to $\mathcal{P}_{j;2}$ and, hence, the last integral is equal to zero.

Also, from (3.11), (3.13), and (3.6) for $j \in \mathbb{N}$ we have

$$a_{j+1,j;2,1} = \int_\Gamma z P_{j;1}(z) \overline{P_{j+1;2}(z)}\, d\rho(z),$$

$$a_{j+1,j;2,2} = \int_\Gamma z P_{j;2}(z) \overline{P_{j+1;2}(z)}\, d\rho(z),$$

where $zP_{j;1}(z), zP_{j;2}(z) \in \mathcal{P}_{j+1;1}$. But according to (3.6) $P_{j+1;2}(z)$ is orthogonal to $\mathcal{P}_{j+1;1}$, and, hence, both latter integrals are also equal to zero. ☐

So, after the previous investigations we conclude that the line under the 2×1-matrix $a_{1,0}$ and the line over (resp., under) the 2×2-matrices $a_{j,j+1}$ (resp., $a_{j+1,j}$) consists of zero elements. Taking into account (3.12) we can conclude that the normal matrix of the multiplication operator by z is a five diagonal usual scalar matrix.

Lemma 3.4. *The matrix $(a_{j,k})_{j,k=0}^{\infty}$ from Lemma 3.3 has the following positive elements:*

$$a_{0,1;2}, a_{1,0;1}; \quad a_{j,j+1;2,2}, a_{j+1,j;1,1}, \quad j \in \mathbb{N}. \tag{3.19}$$

Proof. Let us denote by $P'_{1;1}(z)$ the non-normalized vector $P_{1;1}(z)$, which is obtained from the Schmidt orthogonalization procedure for the sequence (3.4): $P'_{1;1}(z) = z - (z,1)_{L^2}$. Then, according to (3.15), we get

$$\begin{aligned} a_{1,0;1} &= \int_\Gamma z \overline{P_{1;1}(z)}\, d\rho(z) = \|P'_{1;1}(z)\|_{L^2}^{-1} \int_\Gamma z \overline{P'_{1;1}(z)}\, d\rho(z) \\ &= \|P'_{1;1}(z)\|_{L^2}^{-1} \int_\Gamma z \overline{(z - (z,1)_{L^2})}\, d\rho(z) = \|P'_{1;1}(z)\|_{L^2}^{-1} \left(1 - |(\bar{z},1)_{L^2}|^2\right). \end{aligned} \tag{3.20}$$

Later we will show that

$$|(\bar{z},1)_{L^2}|^2 + |(\bar{z}, P_{1;1}(z))_{L^2}|^2 < 1. \tag{3.21}$$

Hence, from (3.20) we get $a_{1,0;1} > 0$.

Let us consider $a_{0,1;2}$. As before, we denote by $P'_{1;2}(z)$ the non-normalized vector $P_{1;2}(z)$ obtained from the Schmidt orthogonalization procedure for the sequence (3.4), $P'_{1;2}(z) = \bar{z} - (\bar{z}, P_{1;1}(z))_{L^2} P_{1;1}(z) - (\bar{z},1)_{L^2}$. Then according to (3.15) and (3.21) we get

$$\begin{aligned} a_{0,1;2} &= \int_\Gamma z P_{1;2}(z)\, d\rho(z) \\ &= \|P'_{1;2}(z)\|_{L^2}^{-1} \int_\Gamma z \left(\bar{z} - (\bar{z}, P_{1;1}(z))_{L^2} P_{1;1}(z) - (\bar{z},1)_{L^2}\right) d\rho(z) \\ &= \|P'_{1;2}(z)\|_{L^2}^{-1} \left(1 - |(\bar{z}, P_{1;1}(z))_{L^2}|^2 - |(\bar{z},1)_{L^2}|^2\right) > 0. \end{aligned}$$

At the next step we consider $a_{j,j+1;2,2}$ and $a_{j+1,j;1,1}$, $j \in \mathbb{N}$. By (3.11), we have

$$a_{j,j+1;2,2} = \int_\Gamma z P_{j+1;2}(z) \overline{P_{j;2}(z)}\, d\rho(z) = \overline{\int_\Gamma \bar{z} P_{j;2}(z) \overline{P_{j+1;2}(z)}\, d\rho(z)}, \tag{3.22}$$

$$a_{j+1,j;1,1} = \int_\Gamma z P_{j;1}(z) \overline{P_{j+1;1}(z)}\, d\rho(z). \tag{3.23}$$

In the expression (3.22), for $a_{j,j+1;2,2}$ we have that $\bar{z}P_{j;2}(z) \in \mathcal{P}_{j+1;2}$ (see (3.13)) and that the coefficient of this function at $P_{j+1;2}(z)$ in decomposition (3.6) is positive. This fact follows from (3.4) and (3.5), since

$$\bar{z}P_{j;2}(z) = k_{j;2}e^{-(j+1)i\theta} + \bar{z}(\cdots) = \frac{k_{j;2}}{k_{j+1;2}}P_{j+1;2}(z) + Q_j(z), \tag{3.24}$$

where we denote by dots the terms from the correspondence expression for $P_{j;2}(z)$ in (3.5). In (3.24) $Q_j(z)$ is a linear combination of the terms z^l, $l = -j, -(j-1), \ldots, (j+1)$, that are orthogonal to $P_{j+1;2}(z)$. Substituting (3.24) into (3.22) we obtain the relation $a_{j,j+1;2,2} = k_{j;2}(k_{j+1,2})^{-1} > 0$.

Positivity of $a_{j+1;1,1}$ is obtained from (3.23) analogously. There one can use that according to (3.13) $zP_{j;1}(z) \in \mathcal{P}_{j+1;1}$; the coefficient of this function at $P_{j+1;1}(z)$ is positive as it follows from relation similar to (3.24).

Let us prove estimate (3.21). Actually it follows from the Parseval equality by decomposing the function $\bar{z}$ with respect to the orthonormal basis (3.5),

$$|(\bar{z}, P_0(z))_{L^2}|^2 + |(\bar{z}, P_{1;1}(z))_{L^2}|^2 + |(\bar{z}, P_{1;2}(z))_{L^2}|^2 + \cdots = \|\bar{z}\|^2_{L^2} = 1. \tag{3.25}$$

$\square$

In what follows we will use the usual well known notations for the elements $a_{j,k}$ of the Jacobi matrix,

$$a_n = a_{n+1,n} : \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \quad b_n = a_{n,n} : \mathcal{H}_n \longrightarrow \mathcal{H}_n,$$
$$c_n = a_{n,n+1} : \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \tag{3.26}$$

All the previous results are summarized in the following theorem.

Theorem 3.5. *The bounded normal operator $\hat{A}$ of multiplication by z in the space L^2, with respect to the orthonormal basis (3.5) of polynomials, has the form of a three-diagonal block Jacobi type unitary matrix $J = (a_{j,k})_{j,k=0}^{\infty}$ which acts on the space (3.7),*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \dim\mathcal{H}_0 = 1, \quad \dim\mathcal{H}_1 = \dim\mathcal{H}_2 = \cdots = 2. \tag{3.27}$$

The norms of all the operators $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$. In notation (3.26), this matrix has the form

$$J = \begin{pmatrix} b_0 & c_0 & 0 & 0 & 0 & \ldots \\ a_0 & b_1 & c_1 & 0 & 0 & \ldots \\ 0 & a_1 & b_2 & c_2 & 0 & \ldots \\ 0 & 0 & a_2 & b_3 & c_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \left(\begin{array}{c|ccc|ccc|ccc|ccc|c}
*b_0 & * & c_0 & * & & & & & & & & & & \cdots \\
\hline
* & * & & * & 0 & & 0 & & & & & & & \\
a_0 & & b_1 & & & c_1 & & & & 0 & & & & \cdots \\
0 & * & & * & * & & * & & & & & & & \\
\hline
 & * & & * & * & & * & 0 & & 0 & & & & \\
 & & a_1 & & & b_2 & & & c_2 & & & & & \cdots \\
 & 0 & & 0 & * & & * & * & & * & & & & \\
\hline
 & & & & * & & * & * & & * & 0 & & 0 & \\
 & & 0 & & & a_2 & & & b_3 & & & c_3 & & \cdots \\
 & & & & 0 & & 0 & * & & * & * & & * & \\
\hline
\vdots & & \vdots & & & \vdots & & & \vdots & & & \vdots & & \ddots
\end{array}\right). \tag{3.28}$$

In (3.28), b_0 is a 1×1-matrix (i.e., a scalar), a_0 is a 2×1-matrix, $a_0=(a_{0;\alpha})_{\alpha=1}^2$, c_0 is a 1×2-matrix, $c_0=(c_{0;\beta})_{\beta=1}^2$; for $j\in\mathbb{N}$ the elements $a_j=(a_{j;\alpha,\beta})_{\alpha,\beta=1}^2$, $b_j=(b_{j;\alpha,\beta})_{\alpha,\beta=1}^2$, $c_j=(c_{j;\alpha,\beta})_{\alpha,\beta=1}^2$ are 2×2-matrices. The matrices a_j, b_j, and c_j have elements either equal to zero or positive,

$$\begin{aligned}&a_{0;1}>0, \quad a_{0;2}=0; && c_{0;2}>0;\\ &a_{n;2,1}=a_{n;2,2}=0, \quad a_{n;1,1}>0; && c_{n;1,1}=c_{n;1,2}=0, \quad c_{n;2,2}>0; \quad n\in\mathbb{N}.\end{aligned}\tag{3.29}$$

The matrix (3.28), in the scalar form, is five-diagonals of the indicated structure.

The adjoint operator $(\hat{A})^*$, with respect to basis (3.5), has the form of a similar three-diagonal block Jacobi type matrix J^+,

$$\begin{aligned}(a^*)_n&=(a^*)_{n+1,n}:\mathcal{H}_n\longrightarrow\mathcal{H}_{n+1},\\ (b^*)_n&=(a^*)_{n,n}:\mathcal{H}_n\longrightarrow\mathcal{H}_n,\\ (c^*)_n&=(a^*)_{n,n+1}:\mathcal{H}_{n+1}\longrightarrow\mathcal{H}_n, \quad n\in\mathbb{N}_0.\end{aligned}\tag{3.30}$$

Also, the following equalities holds:

$$\begin{aligned}&(a^*)_{0;\alpha}=\bar{c}_{0;\alpha}, && (b^*)_0=\bar{b}_0, && (c^*)_{0;\beta}=\bar{a}_{0;\beta}, && \alpha,\beta=1,2;\\ &(a^*)_{n;\alpha,\beta}=\bar{c}_{n;\beta,\alpha}, && (b^*)_{n;\alpha,\beta}=\bar{b}_{n;\beta,\alpha}, && (c^*)_{n;\alpha,\beta}=\bar{a}_{n;\beta,\alpha}, && \alpha,\beta=1,2, \quad n\in\mathbb{N}.\end{aligned}\tag{3.31}$$

These matrices J, J^+ act as follows: $\forall f=(f_n)_{n=0}^\infty\in\mathbf{l}_2$

$$\begin{aligned}(Jf)_n&=a_{n-1}f_{n-1}+b_nf_n+c_nf_{n+1},\\ (J^+f)_n&=c_{n-1}^*f_{n-1}+b_n^*f_n+a_n^*f_{n+1}, \quad n\in\mathbb{N}_0, \quad f_{-1}=0\end{aligned}\tag{3.32}$$

*(here * denotes the usual adjoint matrix).*

The form of elements in the expression for J^+ follows from (3.14), (3.16), and (3.26).

4. The corresponding direct and inverse spectral problems

For this section we refer to [15], since a similar section is written in [15]. Although it suffices to replace only the circle Γ with an arbitrary second order curve Γ, we decided to include it here in full, since [15] is not available today.

In the previous section, we have actually given a solution of the inverse problem.

Now we will consider operators on the space $\mathbf{l}_2$ of the form (3.7). Additionally to the space $\mathbf{l}_2$ we consider its rigging

$$(\mathbf{l}_{\text{fin}})'\supset\mathbf{l}_2(p^{-1})\supset\mathbf{l}_2\supset\mathbf{l}_2(p)\supset\mathbf{l}_{\text{fin}},\tag{4.33}$$

where $\mathbf{l}_2(p)$ is the weighted $\mathbf{l}_2$ space with a weight $p=(p_n)_{n=0}^\infty$, $p_n\geq 1$, $(p^{-1}=(p_n^{-1})_{n=0}^\infty)$. In our case $\mathbf{l}_2(p)$ is the Hilbert space of sequences $f=(f_n)_{n=0}^\infty$, $f_n\in\mathcal{H}_n$, which satisfy

$$\|f\|_{\mathbf{l}_2(p)}^2=\sum_{n=0}^\infty\|f_n\|_{\mathcal{H}_n}^2p_n, \quad (f,g)_{\mathbf{l}_2(p)}=\sum_{n=0}^\infty(f_n,g_n)_{\mathcal{H}_n}p_n.\tag{4.34}$$

The space $\mathbf{l}_2(p^{-1})$ is defined analogously; $\mathbf{l}_{\text{fin}}$ is the space of finite sequences, and $(\mathbf{l}_{\text{fin}})'$ is the space conjugate to $\mathbf{l}_{\text{fin}}$. It is easy to show that the embedding $\mathbf{l}_2(p)\hookrightarrow\mathbf{l}_2$ is quasinuclear if $\sum_{n=0}^\infty p_n^{-1}<\infty$ (see for example [16] Ch. 7; [18] Ch. 15).

Let A be a normal operator standardly connected with the chain (4.33). According to the projection spectral theorem (see [17] Ch. 3, Theorem 2.7; [16] Ch. 5; [18], Ch. 15) such an operator has a representation

$$Af=\int_\Gamma z\Phi(z)\,d\sigma(z)f, \quad f\in\mathbf{l}_2,\tag{4.35}$$

where $\Phi(z):\mathbf{l}_2(p)\longrightarrow\mathbf{l}_2(p^{-1})$ is the generalized projection operator, and $d\sigma(z)$ is a spectral measure. The operator A^*, which is adjoint to A, has the same representation

(3.13) where $z\Phi(z)$ is replaced with $\bar{z}\Phi(z)$. For every $f \in \mathbf{l}_{\mathrm{fin}}$, the projection $\Phi(z)f \in \mathbf{l}_2(p^{-1})$ is a generalized eigenvector of the operators A and A^* with corresponding eigenvalues z and $\bar{z}$. For all $f, g \in \mathbf{l}_{\mathrm{fin}}$ we have the Parseval equality

$$(f,g)_{\mathbf{l}_2} = \int_\Gamma (\Phi(z)f,g)_{\mathbf{l}_2} d\sigma(z); \tag{4.36}$$

being extended by continuity, the equality (4.36) holds for all for $f, g \in \mathbf{l}_2$.

Let us denote by π_n the operator of orthogonal projection from $\mathbf{l}_2$ onto $\mathcal{H}_n$, $n \in \mathbb{N}_0$. Hence, $\forall f = (f_n)_{n=0}^\infty \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously in the spaces $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$ but possibly with the norm which is not equal to one.

Let us consider the operator matrix $(\Phi_{j,k}(z))_{j,k=0}^\infty$, where

$$\Phi_{j,k}(z) = \pi_j \Phi(z) \pi_k \;:\; \mathbf{l}_2 \longrightarrow \mathcal{H}_j, \; (\text{or } \mathcal{H}_k \longrightarrow \mathcal{H}_j). \tag{4.37}$$

The Parseval equality (4.36) can be rewritten as follows: $\forall f, g \in \mathbf{l}_2$

$$\begin{aligned}(f,g)_{\mathbf{l}_2} &= \sum_{j,k=0}^\infty \int_\Gamma (\Phi(z)\pi_k f, \pi_j g)_{\mathbf{l}_2} d\sigma(z) = \sum_{j,k=0}^\infty \int_\Gamma (\pi_j \Phi(z) \pi_k f, g)_{\mathbf{l}_2} d\sigma(z) \\ &= \sum_{j,k=0}^\infty \int_\Gamma (\Phi_{j,k}(z) f_k, g_j)_{\mathbf{l}_2} d\sigma(z).\end{aligned} \tag{4.38}$$

We will now consider a more special bounded operator A acting on the space $\mathbf{l}_2$. Namely, let it be generated by a matrix J which has a three-diagonal block structure of the form (3.28), with uniformly bounded norms of the elements a_n, b_n and c_n, with respect to $n \in \mathbb{N}_0$. So, this operator A is defined by the first expression in (3.32), the adjoint operator is defined analogously by the second expression in (3.32).

In what follows *we suppose that conditions (3.29) are fulfilled and, additionally, the operator A given by (3.28) on $\mathbf{l}_2$ is normal.* The conditions that imply that this operator is normal will be given in the next section. Here we only note that the operator A is normal if and only if its elements a_n, b_n and c_n satisfy some simple recursion relation obtained from the equality $AA^* = A^*A$.

In the next step we will rewrite the Parseval equality (4.38) in terms of generalized eigenvectors of the operator A. At first we prove the following lemma.

Lemma 4.1. *Let $\varphi(z) = (\varphi_n(z))_{n=0}^\infty$, $\varphi_n(z) \in \mathcal{H}_n$, $z \in \mathbb{C}$, be a fixed solution in $(\mathbf{l}_{\mathrm{fin}})'$ of the following system with the initial condition $\varphi_0(z) = \varphi_0 \in \mathbb{C}$:*

$$\begin{gathered}(J\varphi(z))_n = a_{n-1}\varphi_{n-1}(z) + b_n\varphi_n(z) + c_n\varphi_{n+1}(z) = z\varphi_n(z), \\ (J^+\varphi(z))_n = c_{n-1}^*\varphi_{n-1}(z) + b_n^*\varphi_n(z) + a_n^*\varphi_{n+1}(z) = \bar{z}\varphi_n(z), \\ n \in \mathbb{N}_0, \quad \varphi_{-1}(z) = 0.\end{gathered} \tag{4.39}$$

Then this solution exist $\forall \varphi_0$ and has the form: $\forall n \in \mathbb{N}$

$$\varphi_n(z) = Q_n(z)\varphi_0 = (Q_{n;1}, Q_{n;2})\varphi_0, \tag{4.40}$$

where $Q_{n;1}$ and $Q_{n;2}$ are polynomials in z and $\bar{z}$ and these polynomials have the form

$$Q_{n;1}(z) = l_{n;1}\bar{z}^n + q_{n;1}(z), \quad Q_{n;2}(z) = l_{n;2}z^n + q_{n;2}(z). \tag{4.41}$$

Here $l_{n;1} > 0$, $l_{n;2} > 0$ and $q_{n;1}(z)$, $q_{n;2}(z)$ are linear combinations of $z^j\bar{z}^k$ for $0 \le j+k \le n-1$; $Q_0(z) = 1$, $z \in \mathbb{C}$.

Proof. For $n = 0$, system (4.39) has the form

$$\begin{aligned}b_0\varphi_0 + c_0\varphi_1 &= z\varphi_0, \\ b_0^*\varphi_0 + a_0^*\varphi_1 &= \bar{z}\varphi_0,\end{aligned} \quad \Longrightarrow \quad \begin{aligned}c_{0;1}\varphi_{1;1} + c_{0;2}\varphi_{1;2} &= (z - b_0)\varphi_0, \\ a_{0;1}^*\varphi_{1;1} + a_{0;2}^*\varphi_{1;2} &= (\bar{z} - \bar{b}_0)\varphi_0.\end{aligned} \tag{4.42}$$

Here and in what follows we denote $\varphi_n(z) = (\varphi_{n;1}(z), \varphi_{n;2}(z)) \in \mathcal{H}_n$, $n \in \mathbb{N}$. Using the assumption (3.29) for the matrix (3.28) we rewrite the last two equalities in (4.42) in the form

$$d_0\varphi_1(z) = ((z - b_0)\varphi_0, (\bar{z} - \bar{b}_0)\varphi_0); \quad d_0 = \begin{pmatrix} c_{0;1} & c_{0;2} \\ a_{0;1} & 0 \end{pmatrix}, \quad c_{0;2} > 0, \quad a_{0;1} > 0.$$

Hence, there exists an inverse matrix d_0^{-1} and, consequently,

$$(\varphi_{1;1}(z), \varphi_{1;2}(z)) = \varphi_1(z) = d_0^{-1}((z - b_0)\varphi_0, (\bar{z} - \bar{b}_0)\varphi_0).$$

Therefore

$$\begin{aligned} \varphi_{1;1}(z) &= \frac{1}{a_{0;1}}(\bar{z} - \bar{b}_0)\varphi_0 = Q_{1;1}(z)\varphi_0, \\ \varphi_{1;2}(z) &= (r_1(z - b_0) + r_2(\bar{z} - \bar{b}_0) + r_3)\varphi_0 = Q_{1;2}(z)\varphi_0, \end{aligned} \tag{4.43}$$

where $r_1 > 0$, r_2 and r_3 some constants. In another words the solution $\varphi_n(z)$ of (4.39) for $n = 1$ has a form of (4.40) and (4.41).

Using induction suppose that, for $n \in \mathbb{N}$, there exists $\varphi_n(z)$ and it has the form (4.40) and (4.41). Then it will follow that $\varphi_{n+1}(z)$ also has the form (4.40) and (4.41).

From (4.39) we get

$$\begin{aligned} c_n\varphi_{n+1}(z) &= (z1 - b_n)\varphi_n(z) - a_{n-1}\varphi_{n-1}(z), \\ a_n^*\varphi_{n+1}(z) &= (\bar{z}1 - b_n^*)\varphi_n(z) - c_{n-1}^*\varphi_{n-1}(z). \end{aligned}$$

Summing the two last equalities and taking into account (3.29), (4.40), and (4.41) we obtain

$$\begin{aligned} d_n\varphi_{n+1}(z) &= ((z + \bar{z})1 - (b_n + b_n^*))\varphi_n(z) - (a_{n-1} + c_{n-1}^*)\varphi_{n-1}(z) \\ &= ((z + \bar{z})1 - (b_n + b_n^*))(l_{n;1}\bar{z}^n + q_{n;1}(z), l_{n;2}z^n + q_{n;2}(z))\varphi_0 \\ &\quad - (a_{n-1} + c_{n-1}^*)(l_{n-1;1}\bar{z}^{n-1} + q_{n-1;1}(z), l_{n-1;2}z^{n-1} + q_{n-1;2}(z))\varphi_0 \\ &= (s_{n;1}(z), s_{n;2}(z))\varphi_0; \\ d_n &= \begin{pmatrix} a_{n;1,1} & 0 \\ c_{n;2,1} + a_{n;1,2} & c_{n;2,2} \end{pmatrix}, \quad a_{n;1,1} > 0, \quad c_{n;2,2} > 0. \end{aligned} \tag{4.44}$$

From (4.44) it follows that the matrix d_n^{-1} exists. In such a way one can recover expressions for $\varphi_{n+1;1}(z)$ and $\varphi_{n+1;2}(z)$ using the right-hand size of (4.44), which we have denoted by $(s_{n;1}(z), s_{n;2}(z))\varphi_0$, i.e.,

$$\begin{aligned} (\varphi_{n+1;1}(z), \varphi_{n+1;2}(z)) &= d_n^{-1}(s_{n;1}(z), s_{n;2}(z))\varphi_0, \\ \varphi_{n+1;1}(z) &= \frac{1}{a_{n;1,1}}s_{n;1}\varphi_0, \\ \varphi_{n+1;2}(z) &= (d_n^{-1})_{2,1}s_{n;1}(z) + (d_n^{-1})_{2,2}s_{n;2}(z). \end{aligned} \tag{4.45}$$

From (4.44) and (4.45) we find the growth of the powers of z and $\bar{z}$ in the right-hand sides of these expressions. This is so since we have in each step a multiplication by the matrix $(z - \bar{z})1$. It gives that $s_{n;1}(z)$ and $s_{n;2}(z)$ have the form $l_{n+1;1}\bar{z}^{n+1} + q_{n+1;1}(z)$ and $l_{n+1;2}z^{n+1} + q_{n+1;2}(z)$, correspondingly.

It is not difficult to also see that the new higher terms $l_{n+1;1}$ and $l_{n+1;2}$ are also positive. For $l_{n+1,1}$ this fact follows from the second expression of (4.45) and for $l_{n+1,2}$ it follows from the last expression of (4.45), i.e, from positivity of diagonal elements of the matrix d_n and the location of the term $l_{n;2}z^n$ in the second coordinate of the vector $(l_{n;1}\bar{z}^n + q_{n;1}(z), l_{n;2}z^n + q_{n;2}(z))$. Hence, we complete the induction and a use of (4.43) finishes the proof. $\square$

Next we will consider $Q_n(z)$ for fixed z as a linear operator that acts from $\mathcal{H}_0$ into $\mathcal{H}_n$, i.e., $\mathcal{H}_0 \ni \varphi_0 \longmapsto Q_n(z)\varphi_0 \in \mathcal{H}_n$. We also regard $Q_n(z)$ as an operator valued polynomial of $z, \bar{z} \in \mathbb{C}$; hence, for the adjoint operator we have $Q_n^*(z) = (Q_n(z))^* : \mathcal{H}_n \longrightarrow \mathcal{H}_0$. Using the polynomials $Q_n(z)$ we construct the following representation for $\Phi_{j,k}(z)$.

Lemma 4.2. *The operator $\Phi_{j,k}(z)$, has the following representation for all $z \in T$:*

$$\Phi_{j,k}(z) = Q_j(z)\Phi_{0,0}(z)Q_k^*(z) : \mathcal{H}_k \longrightarrow \mathcal{H}_j, \quad j,k \in \mathbb{N}_0, \tag{4.46}$$

where $\Phi_{0,0}(z) \geq 0$ is a scalar.

Proof. For a fixed $k \in \mathbb{N}_0$ the vector $\varphi = \varphi(z) = (\varphi_j(z))_{j=0}^{\infty}$, where

$$\varphi_j(z) = \Phi_{j,k}(z) = \pi_j\Phi(z)\pi_k \in \mathcal{H}_j, \quad z \in \Gamma, \tag{4.47}$$

is a generalized solution in $(\mathbf{l}_{\mathrm{fin}})'$ of the equation $J\varphi(z) = z\varphi(z)$, since $\Phi(z)$ is a projector acting on generalized eigenvectors of the operator A with the corresponding generalized eigenvalues z. Therefore $\forall g \in \mathbf{l}_{\mathrm{fin}}$ we have $(\varphi, J^+g)_{\mathbf{l}_2} = z(\varphi, g)_{\mathbf{l}_2}$. Transfer here of the finite difference expression J^+ on φ, we get $(J\varphi, g)_{\mathbf{l}_2} = z(\varphi, g)_{\mathbf{l}_2}$. Hence, it follows that $\varphi = \varphi(z) \in \mathbf{l}_2(p^{-1})$ is a usual solution of the equation $J\varphi = z\varphi$ with the initial condition $\varphi_0 = \pi_0\Phi(z)\pi_k \in \mathcal{H}_0$.

Since $\forall f \in \mathbf{l}_{\mathrm{fin}}$, the vector $\Phi(z)f \in \mathbf{l}_2(p^{-1})$ is also a generalized eigenvector of the operator A^* with the corresponding eigenvalue $\bar{z}$ (because A is a unitary operator), the same $\varphi = \varphi(z)$ in (4.47) is also a solution of the equation $J^+\varphi = \bar{z}\varphi$ with the same initial condition, $\varphi_0 = \pi_0\Phi(z)\pi_k$.

Using Lemma 4.1 and due to (4.40) we obtain

$$\Phi_{j,k}(z) = Q_j(z)(\Phi_{0,k}(z)), \quad j \in \mathbb{N}_0. \tag{4.48}$$

The operator $\Phi(z) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is formally selfadjoint in $\mathbf{l}_2$ (as a derivation with respect to the corresponding resolution of identity in $\mathbf{l}_2$ of the operator A on the spectral measure). Hence, according to (4.46) we get

$$(\Phi_{j,k}(z))^* = (\pi_j\Phi(z)\pi_k)^* = \pi_k\Phi(z)\pi_j = \Phi_{k,j}(z), \quad j,k \in \mathbb{N}_0. \tag{4.49}$$

For a fixed $j \in \mathbb{N}_0$ from (4.49) and previous considerations it follows that the vector

$$\psi = \psi(z) = (\psi_k(z))_{k=0}^{\infty}, \quad \psi_k(z) = \Phi_{k,j}(z) = (\Phi_{j,k}(z))^*$$

is a usual solution of the equations $J\psi = z\psi$ and $J^+\psi = \bar{z}\psi$ with the initial condition $\psi_0 = \Phi_{0,j}(z) = (\Phi_{j,0}(z))^*$.

Using again Lemma 4.1 we obtain a representation of type (4.48),

$$\Phi_{k,j}(z) = Q_k(z)(\Phi_{0,j}(z)), \quad k \in \mathbb{N}_0. \tag{4.50}$$

Taking into account (4.49) and (4.50) we get

$$\Phi_{0,k}(z) = (\Phi_{k,0}(z))^* = (Q_k(z)\Phi_{0,0}(z))^* = \Phi_{0,0}(z)(Q_k(z))^*, \qquad k \in \mathbb{N}_0 \tag{4.51}$$

(here we used that $\Phi_{0,0}(z) \geq 0$, which follows from (4.36) and (4.37)). Substituting (4.51) into (4.48) we obtain (4.46). ☐

Now we obtain the possibility to rewrite the Parseval equality (4.38) in a more concrete form. To this end we substitute the expression (4.46) for $\Phi_{j,k}(z)$ into (4.38) to get

$\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$\begin{aligned}(f,g)_{\mathbf{l}_2} &= \sum_{j,k=0}^{\infty} \int_{\Gamma} (\Phi_{j,k}(z) f_k, g_j)_{\mathbf{l}_2} d\sigma(z) = \sum_{j,k=0}^{\infty} \int_{\Gamma} (Q_j(z)\Phi_{0,0}(z) Q_k^*(z) f_k, g_j)_{\mathbf{l}_2} d\sigma(z) \\ &= \sum_{j,k=0}^{\infty} \int_{\Gamma} (Q_k^*(z) f_k, Q_j^*(z) g_j)_{\mathbf{l}_2} d\rho(z) = \int_{\Gamma} \Big(\sum_{k=0}^{\infty} Q_k^*(z) f_k \Big) \overline{\Big(\sum_{j=0}^{\infty} Q_j^*(z) g_j \Big)} d\rho(z), \\ d\rho(z) &= \Phi_{0,0}(z)\, d\sigma(z).\end{aligned} \tag{4.52}$$

Introduce the Fourier transform $\widehat{\ }$ which is generated by the unitary operator A in the space $\mathbf{l}_2$, $\forall f \in \mathbf{l}_{\mathrm{fin}}$

$$\mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} Q_n^*(z) f_n \in L^2(\Gamma, d\rho(z)). \tag{4.53}$$

Hence, (4.52) gives the Parseval equality in a final form: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$(f,g)_{\mathbf{l}_2} = \int_{\Gamma} \hat{f}(z) \overline{\hat{g}(z)}\, d\rho(z). \tag{4.54}$$

After extending by continuity we see that (4.54) holds for all $f, g \in \mathbf{l}_2$.

Orthogonality of polynomials $Q_n^*(z)$ follows from (4.53) and (4.54). This follows from (4.53) and (4.54) by taking $f = (0, \ldots, 0, f_k, 0, \ldots)$, $f_k \in \mathcal{H}_k$, $g = (0, \ldots, 0, g_j, 0, \ldots)$, $g_j \in \mathcal{H}_j$. Then $\forall k, j \in \mathbb{N}_0$

$$\int_{\Gamma} (Q_k^*(z) f_k) \overline{(Q_j^*(z) g_j)}\, d\rho(z) = \delta_{j,k} (f_j, g_j)_{\mathcal{H}_j}. \tag{4.55}$$

Using representation (4.40) for these polynomials we can rewrite (4.55) in a more classical scalar form. For this reason we remark that $Q_0^*(z) = \bar{Q}_0(z)$ and for $n \in \mathbb{N}$, according to (4.40), $Q_n(z) = (Q_{n;1}(z), Q_{n;2}(z)) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$. Hence, for the adjoint operator $Q_n^*(z) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ we have: $\forall x \in \mathcal{H}_0$, $y = (y_1, y_2) \in \mathcal{H}_n$

$$\begin{aligned}(Q_n(z)x, y)_{\mathcal{H}_n} &= ((Q_{n;1}(z)x, Q_{n;2}(z)x), (y_1, y_2))_{\mathcal{H}_n} = Q_{n;1}(z) x \bar{y}_1 + Q_{n;2}(z) x \bar{y}_2 \\ &= x \overline{(\overline{Q_{n;1}(z)} y_1 + \overline{Q_{n;2}(z)} y_2)} = (x, Q_n^*(z) y)_{\mathcal{H}_0},\end{aligned}$$

that is, $Q_n^*(z) y = \overline{Q_{n;1}(z)} y_1 + \overline{Q_{n;2}(z)} y_2$.

Due to the last equality for $n \in \mathbb{N}$ and $f_n = (f_{n,1}, f_{n_2}) \in \mathcal{H}_n$, $z \in \Gamma$ we obtain

$$Q_n^*(z) f_n = \overline{Q_{n;1}(z)} f_{n;1} + \overline{Q_{n;2}(z)} f_{n;2}, \quad Q_0^*(z) = 1. \tag{4.56}$$

Therefore (4.55) takes the form: $\forall f_0, g_0, f_{k;1}, f_{k;2}, g_{j;1}, g_{j;2} \in \mathbb{C}$, $j, k \in \mathbb{N}$

$$\int_{\Gamma} (\overline{Q_{k;1}(z)} f_{k;1} + \overline{Q_{k;2}(z)} f_{k;2}) \overline{(\overline{Q_{j;1}(z)} f_{j;1} + \overline{Q_{j;2}(z)} f_{j;2})}\, d\rho(z) = \delta_{j,k} (f_{j;1} \bar{g}_{j;1} + f_{j;2} \bar{g}_{j;2}).$$

This equality is equivalent to the following relation of orthogonality in the usual classical form: $\forall j, k \in \mathbb{N}_0$, $\forall \alpha, \beta = 1, 2$

$$\int_{\Gamma} \overline{Q_{k;\beta}^*(z)} Q_{j;\alpha} d\rho(z) = \delta_{j,k} \delta_{\alpha,\beta} \quad (Q_{0;\alpha} = Q_0(z)). \tag{4.57}$$

Let us remark that due to (4.56) the Fourier transform (4.53) can be rewritten in the form: $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$

$$\hat{f}(z) = f_0 + \sum_{n=1}^{\infty} (\overline{Q_{n;1}(z)} f_{n;1} + \overline{Q_{n;2}(z)} f_{n;2}), \quad z \in \Gamma. \tag{4.58}$$

Using the above stated results of this section, we can formulate the following spectral theorem for our unitary operator A.

Theorem 4.3. *Consider the space (3.7)*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus, \cdots, \quad \mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathbb{C}^2, \tag{4.59}$$

and the linear operator A which is defined on finite vectors $\mathbf{l}_{\mathrm{fin}}$ by a block three-diagonal Jacobi type matrix J of the form (3.28) via the first expression in (3.32). We suppose that all its elements a_n, b_n and c_n are uniformly bounded, some elements of these matrices are equal to zero or positive according to (3.29) and the extension of A by continuity is a normal operator on this space.

The eigenfunction expansion of the operator A has the following form. According to Lemma 4.1 we construct from $\varphi_0 \in \mathbb{C}$ a solution $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$, of equations (4.39) for $z \in \Gamma$: $\varphi_n(z) = Q_n(z)\varphi_0 = (Q_{n;1}(z), Q_{n;2}(z))\varphi_0$, where $Q_{n;1}(z)$ and $Q_{n;2}(z)$ are polynomials of z and $\bar{z}$. Then the Fourier transform has the form

$$\mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = f_0 + \sum_{n=1}^{\infty} (\overline{Q_{n;1}(z)} f_{n;1} + \overline{Q_{n;2}(z)} f_{n;2}) \in L^2(\Gamma, d\rho(z)). \tag{4.60}$$

Here $Q_n^*(z) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ is adjoint to the operator $Q_n(z) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$, $d\rho(z)$ is the probability spectral measure of A.

The Parseval equality has the form: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$(f,g)_{\mathbf{l}_2} = \int_\Gamma \hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z), \quad (Jf,g)_{\mathbf{l}_2} = \int_\Gamma z\hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z); \tag{4.61}$$

after extending by continuity, formulas (4.60) and (4.61) hold for all $f, g \in \mathbf{l}_2$. The operator (4.60) is unitary that maps $\mathbf{l}_2$ onto the whole $L^2(\Gamma, d\rho(z))$.

The polynomials $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 1, 2$, and $Q_{0,\alpha}(z) = 1$, form an orthonormal system in $L^2(\Gamma, d\rho(z))$ in the sense of (4.57) and it is total in this space.

Proof. It is only necessary to show that the orthogonal polynomials $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 1, 2$, and $Q_{0,\alpha}(z) = 1$ form a total set in the space $L^2(\Gamma, d\rho(z))$. For this reason we remark at first that, due to the choice of the measure $d\rho(z)$ on Γ, the elements $z^j, \bar{z}^k$, $j, k \in \mathbb{N}_0$, form a total set in $L^2(\Gamma, d\rho(z))$.

Let us suppose the contrary, i.e., that our system of polynomials is not total. Then there exist non zero function $h(z) \in L^2(\Gamma, d\rho(z))$ that is orthogonal to all of these polynomials and hence according to (4.41) to all $z^j, \bar{z}^k$, $j, k \in \mathbb{N}_0$. But then $h(z) = 0$. □

The latter theorem solves the direct problem for a unitary operator A which is generated on the space $\mathbf{l}_2$ by a matrix J of the form (3.28).

The inverse problem consists in constructing, from a given measure $d\rho(z)$ on Γ, a normal matrix J of the form (3.28) that has its spectral measure equal to $d\rho(z)$. *This construction is carried out according to Theorem 3.5 with a use of the Schmidt orthogonalization procedure applied to system (3.4). For the constructed, from $d\rho(z)$, the matrix J of the form (3.28) has the spectral measure of the corresponding unitary operator A that coincides with the initial measure.*

Proof. This is true, since the system of orthogonal polynomials, corresponding to A, $\overline{Q_{n,1}(z)}$ and $\overline{Q_{n,2}(z)}$, $n \in \mathbb{N}$, and $Q_0(z)$, are orthonormal in $L^2(\Gamma, d\rho(z))$ and according to Lemma 4.1 are constructed from $\bar{z}^j, z^k$, $z \in \Gamma$, in th same way as system (3.5) is constructed from z^j, $\bar{z}^k$, $j, k \in \mathbb{N}_0$. In such a way, $\forall n \in \mathbb{N}$

$$Q_0(z) = 1 = P_0(z), \quad \overline{Q_{n,1}(z)} = P_{n;2}(z), \quad \overline{Q_{n,2}(z)} = P_{n;1}(z). \tag{4.62}$$

Since both systems of polynomials form a total set in $L^2(\Gamma, d\rho(z))$, (4.62) establishes equality of the spectral measure constructed from the operator and the given one. □

Let us remark that the expressions (3.11) and (3.15) (as it was known in the classical theory of Jacobi matrices) reestablish the initial matrix (3.28) from the spectral measure $d\rho(z)$ of the operator constructed from J in $\mathbf{l}_2$.

5. The inner structure of the corresponding matrices

To present our result in a convenient form, we redefine elements of matrix (3.28) in the following unusual form:

$$J=\begin{pmatrix} a_{0;00} & a_{1;00} & a_{1;01} & & & \\ a_{0;10} & a_{1;10} & a_{1;11} & & & \\ & a_{2;00} & a_{2;10} & a_{3;00} & a_{3;01} & \\ & a_{2;01} & a_{2;11} & a_{3;10} & a_{3;11} & \\ & & & a_{4;00} & a_{4;10} & \cdots \\ & & & a_{4;01} & a_{4;11} & \cdots \end{pmatrix} \tag{5.63}$$

Takin into account real (positive) elements J, the adjoint matrix has a form

$$J^+=\begin{pmatrix} \bar a_{0;00} & a_{0;10} & & & & \\ \bar a_{1;00} & \bar a_{1;10} & \bar a_{2;00} & a_{2;01} & & \\ a_{1;01} & \bar a_{1;11} & \bar a_{2;10} & \bar a_{2;11} & & \\ & & \bar a_{3;00} & \bar a_{3;10} & \bar a_{4;00} & \bar a_{4;01} \\ & & a_{3;01} & \bar a_{3;11} & \bar a_{4;10} & \bar a_{4;11} \\ & & & & \cdots & \cdots \end{pmatrix} \tag{5.64}$$

Now we find conditions that would guarantee the equality $JJ^+ = J^+J$.

Using the unusual form of the matrix elements we can formulate the theorem in a concise manner.

Theorem 5.1. *The matrix (5.65) is normal, i.e., satisfies the equality $JJ^+ = J^+J$ (and hence, generates pre(normal operator)) if and only if its elements satisfy the following conditions. Some of elements are arbitrary (basic parameters) and other are calculated by recurrence:*

$$\forall a_{1;00}\in\mathbb{C}, a_{1;01}>0, \quad a_{0;10}:=\sqrt{|a_{1;00}|^2+a_{1;01}^2}, \tag{5.65}$$

$$a_{1;11}:=\frac{1}{a_{1;01}}[\bar a_{1;00}(a_{0;00}-a_{1;10})-a_{1;01}(\bar a_{0;00}-\bar a_{1;10})]; \tag{5.66}$$

and, for $n\in\mathbb{N}:=\{1,2,...\}$, we take an arbitrary element $a_{n+1;00}\in\mathbb{C}$ satisfying

$$|a_{n+1;00}|^2<|a_{n;11}|^2+a_{n;01}^2, \tag{5.67}$$

hence the technical parameter

$$k_n:=\frac{1}{a_{n;01}}\sqrt{|a_{n;11}|^2+a_{n;01}^2-|a_{n+1;00}|^2}, \tag{5.68}$$

gives

$$a_{n+1;11}:=-k_n\bar a_{n;00}, \quad a_{n+1;01}:=k_n a_{n;01}=a_{1;01}k_1k_2...k_n, \tag{5.69}$$

and

$$a_{n+1;10}:=\frac{1}{a_{n;01}}[a_{n-1;10}a_{n;00}+a_{n-1;11}\bar a_{n;11}-\bar a_{n;00}a_{n+1;00}]. \tag{5.70}$$

Proof. We will denote elements of th matrices JJ^+ and J^+J by $(j:k)$, $j,k\in\mathbb{N}$. We also use such symbols for separate equalities of the matrix equation $JJ^+ = J^+J$. Due to the symmetry of JJ^+ and J^+J it is sufficient to consider only the cases $(j:k)$, $j \le k$, $j,k \in \mathbb{N}$. Hence we have:

$$\begin{aligned}
&(1:1) \quad a_{0;00}\bar a_{0;00} + a_{1;00}\bar a_{1;00} + a_{1;01}a_{1;01} = \bar a_{0;00}a_{0;00} + a_{0;10}a_{0;10},\\
&(1:2) \quad a_{0;00}a_{0;10} + a_{1;00}\bar a_{1;10} + a_{1;01}\bar a_{1;11} = \bar a_{0;00}a_{1;00} + a_{0;10}a_{1;10},\\
&(1:3) \quad a_{1;00}\bar a_{2;00} + a_{1;01}\bar a_{2;10} = \bar a_{0;00}a_{1;01} + a_{0;10}a_{1;11},\\
&(1:4) \quad a_{1;00}a_{2;01} + a_{1;01}\bar a_{2;11} = 0,\\
&(2:2) \quad a_{0;10}a_{0;10} + a_{1;10}\bar a_{1;10} + a_{1;11}\bar a_{1;11}\\
&\qquad\quad = \bar a_{1;00}a_{1;00} + \bar a_{1;10}a_{1;10} + \bar a_{2;00}a_{2;00} + a_{2;01}a_{2;01},\\
&(2:3) \quad a_{1;10}\bar a_{2;00} + a_{1;11}\bar a_{2;10} = \bar a_{1;00}a_{1;01} + \bar a_{1;10}a_{1;11} + \bar a_{2;00}a_{2;10} + a_{2;01}a_{2;01},\\
&(2:4) \quad a_{1;10}a_{2;01} + a_{1;11}\bar a_{2;11} = \bar a_{2;00}a_{3;00} + a_{2;01}a_{3;10},\\
&(2:5) \quad 0 = \bar a_{2;00}a_{3;01} + a_{2;01}\bar a_{3;11};
\end{aligned} \tag{5.71}$$

and so on for $n \ge 2$:

$$\begin{aligned}
(2n-1:2n-1) \quad & a_{2n-2;00}\bar a_{2n-2;00} + a_{2n-2;10}\bar a_{2n-2;10} + a_{2n-1;00}\bar a_{2n-1;00}\\
& + a_{2n-1;01}a_{2n-1;01} = a_{2n-3;01}a_{2n-3;01} + \bar a_{2n-3;11}a_{2n-3;11}\\
& + \bar a_{2n-2;10}a_{2n-2;10} + \bar a_{2n-3;11}a_{2n-3;11},\\
(2n-1:2n) \quad & a_{2n-2;00}\bar a_{2n-2;01} + a_{2n-2;10}\bar a_{2n-2;11} + a_{2n-1;01}\bar a_{2n-1;11}\\
& = \bar a_{2n-2;10}a_{2n-1;00} + \bar a_{2n-2;11}a_{2n-1;01},\\
(2n-1:2n+1) \quad & a_{2n-1;00}\bar a_{2n;00} + a_{2n-1;01}\bar a_{2n;10} = \bar a_{2n-2;10}a_{2n-1;01} + \bar a_{2n-2;11}a_{2n-1;11},\\
(2n-1:2n+2) \quad & a_{2n-1;00}a_{2n;01} + a_{2n-1;01}\bar a_{2n;11} = 0,\\
(2n:2n) \quad & a_{2n-2;01}\bar a_{2n-2;01} + a_{2n-2;11}\bar a_{2n-2;11} + a_{2n-1;10}\bar a_{2n-1;10}\\
& + a_{2n-1;11}\bar a_{2n-1;11} = \bar a_{2n-1;00}a_{2n-1;00} + \bar a_{2n-1;10}a_{2n-1;10}\\
& + \bar a_{2n;00}a_{2n;00} + \bar a_{2n;01}a_{2n;01},\\
(2n:2n+1) \quad & a_{2n-1;00}\bar a_{2n;00} + a_{2n-1;11}\bar a_{2n;10}\\
& = \bar a_{2n-1;00}a_{2n-1;01} + \bar a_{2n-1;10}a_{2n-1;11} + \bar a_{2n;00}a_{2n;10} + a_{2n;01}a_{2n;11},\\
(2n:2n+2) \quad & a_{2n-1;10}\bar a_{2n;01} + a_{2n-1;11}\bar a_{2n;11} = \bar a_{2n;00}a_{2n+1;00} + a_{2n;01}a_{2n+1;10},\\
(2n:2n+3) \quad & 0 = \bar a_{2n;00}a_{2n+1;01} + a_{2n;01}a_{2n+1;11}.
\end{aligned} \tag{5.72}$$

Thus, we conclude that $(1:1)$ (5.71) gives (5.65), $(1:2)$ (5.71) gives (5.66) and $(1:3)$ (5.71) gives (5.70) for $n=1$. Take some coefficient $k_1 > 0$ and put $a_{2;11} := -k_1\bar a_{1;00}$, $a_{2;01} := k_1 a_{1;01}$. So we obtain (5.69) for $n=1$. The line $(1:2)$ in (5.71) gives $|a_{2;00}|^2 - |a_{1;11}|^2 = (1-k_1^2)a_{1;01}^2$, i.e., (5.67) for $n=1$ and here we calculate (5.68) for $n=1$.

The most difficult is $(2:3)$ (5.71). Let us show that $(2:3)$ follows from $(1:2)$, $(1:3)$, and $(2:2)$ (5.71).

Using $(2:2)$ in (5.71) we substitute

$$a_{1;01}^2(k_1^2-1) = |a_{1;11}|^2 - |a_{2;00}|^2$$

into $(2:3)$ (5.71)

$$\bar a_{2;;00}(a_{1;10} - a_{2;10}) - a_{1;11}(\bar a_{1;10} - \bar a_{2;10}) = \bar a_{1;00}a_{1;01}(1-k_1^2).$$

Hence, multiplying the latter equality by $a_{1;00}$ and $a_{1;01}^2$ we have

$$\begin{aligned}
&a_{1;00}\bar a_{2;;00}a_{1;01}^3(a_{1;10} - a_{2;10}) - a_{1;11}a_{1;00}a_{101}^2(\bar a_{1;10} - \bar a_{2;10})\\
&= |a_{1;00}|^2 a_{1;01}^2|a_{2;00}|^2 - |a_{1;00}|^2 a_{1;01}^2|a_{1;11}|^2.
\end{aligned} \tag{5.73}$$

From $(1:3)$ (5.71) we have

$$\begin{aligned} a_{2;00}\bar{a}_{1;00} &= a_{0;10}\bar{a}_{1;11} - a_{1;01}(a_{2;10} - a_{0;00}), \\ \bar{a}_{2;00}a_{1;00} &= a_{0;10}a_{1;11} - a_{1;01}(\bar{a}_{2;10} - \bar{a}_{0;00}), \end{aligned} \tag{5.74}$$

and from $(1:2)$ (5.71) we have

$$\begin{aligned} a_{1;01}a_{1;11} &= \bar{a}_{1;00}(a_{0;00} - a_{1;00}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;00}), \\ a_{1;01}\bar{a}_{1;11} &= a_{1;00}(\bar{a}_{0;00} - \bar{a}_{1;00}) - a_{0;10}(a_{0;00} - a_{1;00}). \end{aligned} \tag{5.75}$$

Then we substitute (5.74) and (5.75) into (5.73) in several steps. Namely,

$$\begin{aligned} & a_{1;01}^3[a_{0;10}a_{1;11} - a_{1;01}(\bar{a}_{2;10} - \bar{a}_{0;00})](a_{1;10} - a_{2;10}) \\ & - a_{1;00}a_{1;01}^2[\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})](\bar{a}_{1;10} - \bar{a}_{2;10}) \\ & = a_{1;01}^2[a_{0;10}a_{1;11} - a_{1;01}(\bar{a}_{2;10} - \bar{a}_{0;00})][a_{0;10}\bar{a}_{1;11} - a_{1;01}(a_{2;10} - a_{0;00})] \\ & - |a_{1;00}|^2[a_{1;00}(\bar{a}_{0;00} - \bar{a}_{1;10}) - a_{0;10}(a_{0;00} - a_{1;10})] \\ & \cdot [\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})], \end{aligned}$$

$$\begin{aligned} & a_{1;01}^2[a_{0;10}a_{1;11}a_{1;01} - a_{1;01}^2(\bar{a}_{2;10} - \bar{a}_{0;00})](a_{1;10} - a_{2;10}) \\ & - a_{1;00}a_{1;01}^2[\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})](\bar{a}_{1;10} - \bar{a}_{2;10}) \\ & = [a_{0;10}a_{1;11}\bar{a}_{1;01} - a_{1;01}^2(\bar{a}_{2;10} - \bar{a}_{0;00})][a_{0;10}\bar{a}_{1;11}a_{1;01} - a_{1;01}^2(a_{2;10} - a_{0;00})] \\ & - |a_{1;00}|^2[a_{1;00}(\bar{a}_{0;00} - \bar{a}_{1;10}) - a_{0;10}(a_{0;00} - a_{1;10})] \\ & \cdot [\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})], \end{aligned}$$

$$\begin{aligned} & a_{1;01}^2[a_{0;10}(\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})) - a_{1;01}^2(\bar{a}_{2;10} - \bar{a}_{0;00})](a_{1;10} - a_{2;10}) \\ & - a_{1;00}a_{1;01}^2[\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})](\bar{a}_{1;10} - \bar{a}_{2;10}) \\ & = [a_{0;10}(\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})) - a_{1;01}^2(\bar{a}_{2;10} - \bar{a}_{0;00})] \\ & \cdot [a_{0;10} - (a_{1;00}(\bar{a}_{0;00} - \bar{a}_{1;10}) - a_{0;10}(a_{0;00} - a_{1;10})) - a_{1;01}^2(a_{2;10} - a_{0;00})] \\ & - |a_{1;00}|^2[a_{1;00}(\bar{a}_{0;00} - \bar{a}_{1;10}) - a_{0;10}(a_{0;00} - a_{1;10})] \\ & \cdot [\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})], \end{aligned}$$

$$\begin{aligned} & [a_{1;01}^2a_{0;10}(\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})) - a_{1;01}^4(\bar{a}_{2;10} - \bar{a}_{0;00})](a_{1;10} - a_{2;10}) \\ & - [|a_{1;00}|^2a_{1;01}^2(a_{0;00} - a_{1;10}) - a_{1;00}a_{1;01}^2a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})](\bar{a}_{1;10} - \bar{a}_{2;10}) \\ & = [\bar{a}_{1;00}a_{0;10}(a_{0;00} - a_{1;10}) - a_{0;10}^2(\bar{a}_{0;00} - \bar{a}_{1;10})) - a_{1;01}^2(\bar{a}_{2;10} - \bar{a}_{0;00})] \\ & \cdot [a_{1;00}a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10}) - a_{0;10}^2(a_{0;00} - a_{1;10})) - a_{1;01}^2(a_{2;10} - a_{0;00})] \\ & - |a_{1;00}|^2[a_{1;00}(\bar{a}_{0;00} - \bar{a}_{1;10}) - a_{0;10}(a_{0;00} - a_{1;10})] \\ & \cdot [\bar{a}_{1;00}(a_{0;00} - a_{1;10}) - a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})], \end{aligned}$$

$$\begin{aligned} & \bar{a}_{1;00}a_{1;01}^2a_{0;10}(a_{0;00} - a_{1;10})(a_{1;10} - a_{2;10}) - a_{1;01}^2a_{0;10}^2(\bar{a}_{0;00} - \bar{a}_{1;10})(a_{1;10} - a_{2;10}) \\ & - a_{1;01}^4(\bar{a}_{2;10} - \bar{a}_{0;00})(a_{1;10} - a_{2;10}) - |a_{1;00}|^2a_{1;01}^2(a_{0;00} - a_{1;10})(\bar{a}_{1;10} - \bar{a}_{2;10}) \\ & + a_{1;00}a_{1;01}^2a_{0;10})(\bar{a}_{0;00} - \bar{a}_{1;10})(\bar{a}_{1;10} - \bar{a}_{2;10}) = |\bar{a}_{1;00}|^2a_{0;10}^2(a_{0;00} - a_{1;10})(\bar{a}_{0;00} - \bar{a}_{1;10}) \\ & - \bar{a}_{1;00}a_{0;10}^3(a_{0;00} - a_{1;10})^2 - \bar{a}_{1;00}a_{1;01}^2a_{0;10}(a_{0;00} - a_{1;10})(a_{2;10} - a_{0;00}) \\ & - a_{1;00}a_{0;10}^3(\bar{a}_{0;00} - \bar{a}_{1;10})^2 + a_{0;10}^4(a_{0;00} - a_{1;10})(\bar{a}_{0;00} - \bar{a}_{1;10}) \\ & + a_{0;10}^2a_{1;01}^2(a_{0;00} - a_{1;10})(a_{2;10} - a_{0;00})] - a_{1;00}a_{0;10}a_{1;01}^2(\bar{a}_{0;00} - \bar{a}_{1;10})(a_{2;10} - a_{0;00}) \\ & - a_{1;00}a_{0;10}a_{1;01}^2(\bar{a}_{0;00} - \bar{a}_{1;10})(\bar{a}_{2;10} - \bar{a}_{0;00}) - a_{1;01}^4(a_{2;10} - a_{0;00})(\bar{a}_{2;10} - \bar{a}_{0;00}) \\ & + |a_{1;00}|^4(a_{0;00} - a_{1;10})(\bar{a}_{0;00} - \bar{a}_{1;10}) + a_{1;00}|a_{1;00}|^2a_{0;10}(\bar{a}_{0;00} - \bar{a}_{1;10})^2 \\ & + \bar{a}_{1;00}|a_{1;00}|^2a_{0;10}(a_{0;00} - a_{1;10})^2 - |a_{1;00}|^2a_{0;10}^2(a_{0;00} - a_{1;10})(\bar{a}_{0;00} - \bar{a}_{1;10}), \end{aligned}$$

$$
\begin{aligned}
&\bar{a}_{1;00}a_{1;01}^2a_{0;10}[(a_{0;00}-a_{1;10})(a_{1;10}-a_{2;10})+(a_{0;00}-a_{1;10})(a_{2;10}-a_{0;00})]\\
&+a_{1;00}a_{1;01}^2a_{0;10}[(\bar{a}_{0;00}-\bar{a}_{1;10})(\bar{a}_{1;10}-\bar{a}_{2;10})+(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{2;10}-a_{0;00})]\\
&-a_{1;01}^2a_{0;10}^2[(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{1;10}-a_{2;10})+(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{2;10}-a_{0;00})\\
&+(a_{0;00}-a_{1;10})(\bar{a}_{2;10}-\bar{a}_{0;00})]\\
&-a_{1;01}^4[(\bar{a}_{2;10}-\bar{a}_{0;00})(a_{1;10}-a_{2;10})+(a_{2;10}-a_{0;00})(\bar{a}_{2;10}-\bar{a}_{0;00})\\
&+(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})]\\
&-|a_{1;00}|^2a_{1;01}^2[(a_{0;00}-a_{1;10})(\bar{a}_{1;10}-\bar{a}_{2;10})+2(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})]\\
&=-|\bar{a}_{1;00}|^2a_{0;10}^3(a_{0;00}-a_{1;10})^2-\bar{a}_{1;00}a_{0;10}^3(\bar{a}_{0;00}-\bar{a}_{1;10})^2\\
&+a_{1;00}|a_{1;00}|^2[(\bar{a}_{0;00}-\bar{a}_{1;10})^2+(a_{0;00}-a_{1;10})^2]\\
&+|a_{1;00}|^4(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})+2|a_{1;00}|^2a_{1;01^2}(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})\\
&+a_{1;01}^4(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})-|a_{1;00}|^4(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})\\
&+|a_{1;00}|^2a_{1;00}a_{0;10}(\bar{a}_{0;00}-\bar{a}_{1;10})^2+|a_{1;00}|^2\bar{a}_{1;00}a_{1;01}(a_{0;00}-a_{1;10})^2,
\end{aligned}
$$

$$
\begin{aligned}
&-\bar{a}_{1;00}a_{1;01}^2a_{0;10}(a_{0;00}-a_{1;10})^2-a_{1;00}a_{1;01}^2a_{0;10}(\bar{a}_{0;00}-\bar{a}_{1;10})^2\\
&-a_{1;01}^2a_{1;00}^2[(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{1;10}-a_{2;10})+(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{2;10}-a_{0;00})\\
&+(a_{0;00}-a_{1;10})(\bar{a}_{2;10}-\bar{a}_{0;00})+(a_{0;00}-a_{1;10})(\bar{a}_{1;10}-\bar{a}_{2;10})\\
&+2(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})]\\
&-a_{1;01}^4[(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{1;10}-a_{2;10})+(\bar{a}_{0;00}-\bar{a}_{1;10})(a_{2;10}-a_{0;00})\\
&+(\bar{a}_{0;00}-\bar{a}_{1;10})(\bar{a}_{2;10}-\bar{a}_{0;00})+(\bar{a}_{2;10}-\bar{a}_{0;00})(a_{1;10}-a_{2;10})\\
&+(a_{2;10}-a_{0;00})(\bar{a}_{2;10}-\bar{a}_{0;00})+(a_{0;00}-a_{1;10})(\bar{a}_{0;00}-\bar{a}_{1;10})\\
&=-\bar{a}_{1;00}a_{0;10}(|a_{1;00}|^2+a_{1;01}^2)(a_{0;00}-a_{1;10})^2-a_{1;00}a_{0;10}(|a_{1;00}|^2+a_{1;01}^2)(\bar{a}_{0;00}-\bar{a}_{1;10})^2\\
&+a_{1;00}|a_{1;00}|^2a_{1;01}(\bar{a}_{0;00}-\bar{a}_{1;10})^2+a_{1;00}|a_{1;00}|^2a_{1;01}(a_{0;00}-a_{1;10})^2,
\end{aligned}
$$

where we used (5.65).

$$
\begin{aligned}
&-\bar{a}_{1;00}a_{1;01}^2a_{0;10}(a_{0;00}-a_{1;10})^2-a_{1;00}a_{1;01}^2a_{0;10}(\bar{a}_{0;00}-\bar{a}_{1;10})^2-a_{1;01}^2a_{1;00}^2\cdot 0-a_{1;01}^4\cdot 0\\
&=-\bar{a}_{1;00}a_{0;10}|a_{1;00}|^2(a_{0;00}-a_{1;10})^2-\bar{a}_{1;00}a_{0;10}a_{1;01}^2(a_{0;00}-a_{1;10})^2\\
&-a_{1;10}|a_{1;00}|^2a_{0,10}(\bar{a}_{0;00}-\bar{a}_{1;10})^2-a_{1;00}a_{0;10}a_{1;01}^2(\bar{a}_{0;00}-\bar{a}_{1;10})^2\\
&+a_{1;00}|a_{1;00}|^2a_{0,10}(\bar{a}_{0;00}-\bar{a}_{1;10})^2+\bar{a}_{1;00}|a_{1;00}|^2a_{0,10}(a_{0;00}-a_{1;10})^2.
\end{aligned}
$$

And we obtain the identity.

In such a long way we proved the first step of induction, i.e., using the result for $k=1$ we complete the proof for $k=2$. A transition from the step $k=n$ to $k=n+1$ is the same as described above, with substitutions $a_{n-1;xy}$, $a_{n;xy}$ and $a_{n;xy}$, $(x,y=0,1)$ for $a_{0;xy}$, $a_{1;xy}$ and $a_{2;xy}$ $(x,y=0,1)$, correspondingly. Such substitution is valid due to the special form of indices of the elements in (3.28). $\square$

Remark 5.2. It is obvious that we can choose anther elements of matrix (5.65) as basic parameters and calculate the rest.

Example 5.3. As a simple example we propose the normal matrix

$$J = \begin{pmatrix} a_{0;00} & a_{1;00} & a_{1;01} & & & \\ a_{0;10} & a_{1;10} & a_{1;11} & & & \\ & \bar{a}_{1;11} & a_{1;10} & -a_{1;00} & a_{1;01} & \\ & a_{1;01} & -\bar{a}_{1;00} & a_{1;10} & -a_{1;11} & \\ & & & -\bar{a}_{1;11} & a_{1;10} & \cdots \\ & & & a_{1;01} & \bar{a}_{1;00} & \cdots \end{pmatrix}$$

that we obtain by taking $k_i = 1$, $i \in \mathbb{N}$.

Remark 5.4. In our opinion such kind of matrices with periodic numbers can be of a special interest.

Also it is of interest to look at spectral properties of the operators generated by such matrices with regard to their dependences on its elements.

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