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EXISTENCE OF CLASSICAL SOLUTIONS FOR INITIAL BOUNDARY VALUE PROBLEMS FOR NONLINEAR DISPERSIVE EQUATIONS OF ODD-ORDERS

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Abstract. In this paper we investigate a class of initial boundary value problems for a class of nonlinear dispersive equations of odd orders. We prove existence of at least one solution and existence of at least one nonnegative solution. Our method is based on a use of a fixed point theory for the sum of two operators.

У статті досліджено клас початкових граничних задач для класу нелінійних дисперсійних рівняння непарних порядків. Доведено існування принаймні одного розв'язоку і існування хоча б одного невід'ємного розв'язку. Наш метод базується на використанні теорії про нерухомі точки для суми двох операторів.

1. Introduction

In this paper, we investigate the following initial boundary value problem (IBVP)

$$\begin{aligned} v_t + vv_x + \sum_{j=1}^{l}(-1)^{j+1}\partial_x^{2j+1}v &= 0, \quad t\geq 0, \quad x\in[0,1],\\ v(0,x) &= v_0(x), \quad x\in[0,1],\\ \partial_x^i v(t,0)=\partial_x^i v(t,1) &= 0, \quad t\geq 0,\ i=0,1,...,l-1,\\ \partial_x^l v(t,1) &= 0, \quad t\geq 0, \end{aligned} \tag{1.1}$$

where, l belongs to the set of all nonzero natural numbers $\mathbb{N}^*$ and

(H1): $v_0\in\mathcal{C}^{2l+1}([0,1])$, $0\leq v_0(x)\leq B$, $x\in[0,1]$, for some constant $B>1$.

The first equation of IBVP (1.1) includes well-known classical Korteweg-de Vries and Kawahara equations ($l=1$ and $l=2$, respectively) which model the dynamics of long small-amplitude waves in various media, see for example, [1, 4, 11, 13].

Well-posedness of such kind of problems was studied in [22] in the linear case. In [25], general mixed problems for linear multi-dimensional $(2b+1)-$hyperbolic equations were studied by means of functional analysis methods. Boundary value problems in bounded domains for dispersive equations can be found in [3, 25]. Cauchy problem for dispersive equations of high orders was studied in [5, 9, 15, 19, 2, 20, 24]. The nonhomogeneous initial-boundary value problems for quasilinear one-dimensional odd-order equations posed on a bounded interval was studied in [10]. The authors prove existence and uniqueness of global weak and regular solutions for reasonable initial and boundary conditions. Solvability of initial-boundary value problems for one-dimensional generalized dispersive equations of higher orders posed on a bounded interval was studied in [17]. The article [18] was concerned by general mixed problems for nonlinear dispersive equations of any odd-orders posed on bounded intervals, in which results on existence, uniqueness and exponential decay of solutions were presented. Some other works on initial-boundary value problems for odd-order dispersive equations are [7, 8, 10, 14, 16].

The aim of this paper is to investigate the IBVP (1.1) for existence of global in time classical solutions. Here, by a classical solution u to the first equation of (1.1) we mean

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a solution at least $(2l+1)$ times continuously differentiable in x and once in t for any t ≥ 0. In other words, u belongs to the space $\mathcal{C}^1([0,\infty), \mathcal{C}^{2l+1}([0,1]))$ of continuously differentiable functions on $[0,\infty)$ with values in the Banach space $\mathcal{C}^{2l+1}([0,1])$. So, suppose that

(H2): there exist a constant $A > 0$ and a positive function $g \in \mathcal{C}((0,\infty))$ such that

$$(2l+1)!(1+t)\int_0^t g(t_1)dt_1 \leq A, \quad t \in [0,\infty).$$

In the last section, we will give an example for a function g that satisfies $(H2)$. Assume that the constants B and A which appear in the conditions $(H1)$ and $(H2)$, respectively, satisfy the following inequalities:

(H3): $AB_1 < B$, where $B_1 = (l+1)B + B^2$,

and

(H4): $AB_1 < \frac{L}{5}$, where $B_1 = (l+1)B + B^2$ and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leq B, \quad R_1 + \frac{A}{m}B_1 > \left(\frac{1}{5m} + 1\right)L,$$

with r and R_1 being positive constants and $m > 0$ is large enough.

Our main results are as follows.

Theorem 1.1. *Assume that the hypotheses* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. Then the IBVP* (1.1) *has at least one solution* $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^{2l+1}([0,1]))$.

Theorem 1.2. *Assume that the hypotheses* $(H1)$, $(H2)$, *and* $(H4)$ *are satisfied. Then the IBVP* (1.1) *has at least one nonnegative solution* $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^{2l+1}([0,1]))$.

The plan of this paper is as follows. In the next section, we give some auxiliary results used for the proof of our main results. Then in Section 3, we give some properties of solutions of problem (1.1), which include an integral representation and some estimates. In Section 4, we prove Theorem 1.1 and Theorem 1.2. Finally, in Section 5, two illustrative examples of our main results are given.

2. Fixed points and nonnegative fixed points for the sum of two operators

In this section, we will recall two results which concern the existence of fixed points and nonnegative fixed points for the sum of two operators. The proof of the following theorem can be found in [12].

Theorem 2.1. *Let* E *be a Banach space and*

$$E_1 = \{x \in E : \|x\| \leq R\},$$

with $R > 0$. *Consider two operators* T *and* S, *where*

$$Tx = -\epsilon x, \ x \in E_1,$$

with $\epsilon > 0$ *and* $S : E_1 \to E$ *be continuous and such that*

(i): $(I-S)(E_1)$ *resides in a compact subset of* E *and*

(ii): $\{x \in E : x = \lambda(I-S)x, \ \|x\| = R\} = \emptyset$, *for any* $\lambda \in \left(0, \frac{1}{\epsilon}\right)$.

Then there exists $x^* \in E_1$ *such that*

$$Tx^* + Sx^* = x^*.$$

Let X be a real Banach space.

Definition 2.2. A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Definition 2.4. A closed, convex set $\mathcal{P}$ in X is said to be cone if

(1) $\beta x \in \mathcal{P}$ for any $\beta \geq 0$ and for any $x \in \mathcal{P}$,
(2) $x, -x \in \mathcal{P}$ implies $x = 0$.

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$.

Proposition 2.5. *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping, $S : \overline{U} \to E$ is a completely continuous one and $S(\overline{U}) \subset (I - T)(\Omega)$.*

If $T + S$ has no fixed point on $\partial U \cap \Omega$ and there exists $\varepsilon > 0$ small enough such that

$$Sx \neq (I - T)(\lambda x) \text{ for all } \lambda \geq 1 + \varepsilon, \ x \in \partial U \text{ and } \lambda x \in \Omega,$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I - T)^{-1}S : \overline{U} \to \mathcal{P}$ is completely continuous without fixed point in the boundary ∂U and it is readily seen that the following condition is satisfied

$$(I - T)^{-1}Sx \neq \lambda x \quad \text{for all } x \in \partial U \text{ and } \lambda \geq 1 + \varepsilon.$$

Our claim then follows from the definition of i_* given in [6] and from [23, Lemma 2.3] □

The following result will be used to prove our main result.

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is completely continuous and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i): *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*
(ii): *there exists $\varepsilon > 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*
(iii): *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I - T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T + S$. Suppose that $Sx \neq (I - T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the conclusion is obvious. By [6, Proposition 2.16] and Proposition 2.5, we have

$$i_* (T + S, U_1 \cap \Omega, \mathcal{P}) = i_* (T + S, U_3 \cap \Omega, \mathcal{P}) = 0 \text{ and } i_* (T + S, U_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index yields

$$i_* (T + S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}) = 1 \text{ and } i_* (T + S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}) = -1.$$

Consequently, by the existence property of the index i_*, $T + S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

3. Integral representation and some estimates

Let $X = \underbrace{X_1 \times \ldots \times X_1}_{(2l+3) \text{ times}}$, where $X_1 = \mathcal{C}^1([0,\infty), \mathcal{C}^{2l+1}([0,1]))$. For $u = (u_1, \ldots, u_{2l+3}) \in X$, define the operators $S_{1j},\ j = 1, ..., 2l+3$ and S_1 as follows:

$$\begin{aligned}
S_{11}u(t,x) &= \partial_t u_1(t,x) + u_1 \partial_x u_1(t,x) + \sum_{k=1}^{l} (-1)^{k+1} \partial_x^{2k+1} u_1(t,x),\\
S_{12}u(t,x) &= u_1(0,x) - v_0(x),\\
S_{13}u(t,x) &= u_1(t,0),\\
S_{14}u(t,x) &= \partial_x u_1(t,0),\\
\ldots\ \ \ldots &\ \ \ldots\ldots\ldots,\\
S_{1(l+2)}u(t,x) &= \partial_x^{l-1} u_1(t,0),\\
S_{1(l+3)}u(t,x) &= \partial_x^{l} u_1(t,1),\\
S_{1(l+4)}u(t,x) &= u_1(t,1),\\
S_{1(l+5)}u(t,x) &= \partial_x u_1(t,1),\\
\ldots\ldots\ \ \ldots &\ \ \ldots\ldots\ldots\ldots\ldots,\\
S_{1(2l+3)}u(t,x) &= \partial_x^{l-1} u_1(t,1)
\end{aligned}$$

and

$$S_1 u(t,x) = (S_{11}u(t,x), S_{12}u(t,x), ..., S_{1(l+2)}u(t,x), S_{1(l+3)}u(t,x), ..., S_{1(2l+3)}u(t,x)),$$

$(t,x) \in [0,\infty) \times [0,1]$. Note that if $u = (u_1, \ldots, u_{2l+3}) \in X$ is so that

$$S_1 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times [0,1],$$

then u_1 satisfies the IBVP (1.1). In the sequel, the space $X = \underbrace{X_1 \times \ldots \times X_1}_{(2l+3) \text{ times}}$, where $X_1 = \mathcal{C}^1([0,\infty), \mathcal{C}^{2l+1}([0,1]))$, will be endowed with the norm

$$\|u\| = \max_{m \in \{1,\ldots,2l+3\}} \|u_m\|_1, \quad u = (u_1, \ldots, u_{2l+3}),$$

with $\|.\|_1$ is the norm of X_1, defined by

$$\begin{aligned}
\|v\|_1 &= \max\{ \sup_{(t,x)\in[0,\infty)\times[0,1]} |v(t,x)|, \sup_{(t,x)\in[0,\infty)\times[0,1]} |\partial_t v(t,x)|,\\
&\qquad \sup_{(t,x)\in[0,\infty)\times[0,1]} |\partial_x^j v(t,x)|, \quad j \in \{1, \ldots, 2l+1\}\},
\end{aligned}$$

provided it exists. For a function $u = (u_1, \ldots, u_{2l+3}) \in X$, we will write

$$|u(t,x)| = \max_{j \in \{1,\ldots,2l+3\}} |u_j(t,x)|, \quad (t,x) \in [0,\infty) \times [0,1].$$

Lemma 3.1. *Under hypothesis $(H1)$ and for $u \in X$ with $\|u\| \leq B$, the following estimates hold:*

$$|S_1 u(t,x)|, \quad |S_{1j} u(t,x)| \leq B_1, \quad (t,x) \in [0,\infty) \times [0,1], \quad j \in \{1, \ldots, (2l+3)\},$$

where $B_1 = (l+1)B + B^2$.

Proof. Suppose that $(H1)$ is satisfied and let $u \in X$ with $\|u\| \leq B$.

(i): The estimation of $|S_{11}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned} |S_{11}u(t,x)| &= |\partial_t u_1(t,x) + u_1(t,x)\partial_x u_1(t,x) + \sum_{j=1}^{l}(-1)^{j+1}\partial_x^{2j+1}u_1(t,x)| \\ &\leq |\partial_t u_1(t,x)| + |u_1(t,x)||\partial_x u_1(t,x)| + \sum_{j=1}^{l}|\partial_x^{2j+1}u_1(t,x)| \\ &\leq (l+1)B + B^2 \\ &= B_1. \end{aligned}$$

(ii): The estimation of $|S_{12}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{12}u(t,x)| = |u_1(0,x) - v_0(x)| \leq |u_1(0,x)| + |v_0(x)| \leq 2B \leq B_1.$$

(iii): The estimation of $|S_{13}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{13}u(t,x)| = |u_1(t,0)| \leq B \leq B_1.$$

(iv): The estimation of $|S_{14}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{14}u(t,x)| = |\partial_x u_1(t,0)| \leq B \leq B_1.$$

....

(v): The estimation of $|S_{1(l+2)}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{1(l+2)}u(t,x)| = |\partial_x^{l-1}u_1(t,0)| \leq B \leq B_1.$$

(vi): The estimation of $|S_{1(l+3)}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{1(l+3)}u(t,x)| = |\partial_x^{l}u_1(t,1)| \leq B \leq B_1.$$

(vii): The estimation of $|S_{1(l+4)}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{1(l+4)}u(t,x)| = |u_1(t,1)| \leq B \leq B_1.$$

(viii): The estimation of $|S_{1(l+5)}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{1(l+5)}u(t,x)| = |\partial_x u_1(t,1)| \leq B \leq B_1.$$

....

(ix): The estimation of $|S_{1(2l+3)}u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$|S_{1(2l+3)}u(t,x)| = |\partial_x^{l-1}u_1(t,1)| \leq B \leq B_1.$$

Hence,

$$|S_1u(t,x)| \leq B_1, \quad (t,x) \in [0,\infty) \times [0,1].$$

This completes the proof. □

Lemma 3.2. *Suppose that* $(H1)$ *is satisfied and let* $h \in \mathcal{C}([0,\infty))$ *be a positive function on* $(0,\infty)$. *If* $u = (u_1, \dots, u_{2l+3}) \in X$ *satisfies the following integral equation:*

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l+1}h(t_1)S_1u(t_1,x_1)dx_1dt_1 = 0, \quad (t,x) \in [0,\infty) \times [0,1], \quad (3.2)$$

then u_1 *satisfies the IBVP* (1.1).

Proof. We differentiate the equation (3.2) two times with respect to t and $(2l+2)$ times with respect to x and we find

$$h(t)S_1u(t,x) = 0, \quad (t,x) \in [0,\infty) \times [0,1].$$

Hence,

$$S_1u(t,x) = 0, \quad (t,x) \in (0,\infty) \times [0,1].$$

Because $S_1u(\cdot,\cdot)$ is a continuous function on $[0,\infty) \times [0,1]$, we get

$$0 = \lim_{t\to 0} S_1u(t,x) = S_1u(0,x), \quad (t,x) \in [0,\infty) \times [0,1].$$

Thus,

$$S_1u(t,x) = 0, \quad (t,x) \in [0,\infty) \times [0,1].$$

This completes the proof. □

For $u \in X$, define the operator S_2 as follows:

$$S_2u(t,x) = \int_0^t \int_0^x (t-t_1)(x-x_1)^{2l+1} g(t_1) S_1u(t_1,x_1)dx_1dt_1, \quad (t,x) \in [0,\infty) \times [0,1], \tag{3.3}$$

with g being the function which appears in the condition $(H2)$.

Lemma 3.3. *Under hypothesis* $(H1)$ *and* $(H2)$ *and for* $u \in X$, *with* $\|u\| \le B$, *the following estimate holds:*

$$\|S_2u\| \le AB_1.$$

where B_1 *is the constant defined in Lemma 3.1.*

Proof. Suppose that $(H1)$ and $(H2)$ are satisfied and let $u \in X$, with $\|u\| \le B$.

(i): The estimation of $|S_2u(t,x)|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned}
|S_2u(t,x)| &= \left| \int_0^t \int_0^x (t-t_1)(x-x_1)^{2l+1} g(t_1) S_1u(t_1,x_1)dx_1dt_1 \right| \\
&\le \int_0^t \int_0^x (t-t_1)(x-x_1)^{2l+1} g(t_1) |S_1u(t_1,x_1)|dx_1dt_1 \\
&\le B_1 t \int_0^t g(t_1)dt_1 \\
&\le AB_1.
\end{aligned}$$

(ii): The estimation of $\left|\frac{\partial}{\partial t} S_2u(t,x)\right|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned}
\left|\frac{\partial}{\partial t} S_2u(t,x)\right| &= \left| \int_0^t \int_0^x (x-x_1)^{2l+1} g(t_1) S_1u(t_1,x_1)dx_1dt_1 \right| \\
&\le \int_0^t \int_0^x (x-x_1)^{2l+1} g(t_1) |S_1u(t_1,x_1)|dx_1dt_1 \\
&\le B_1 \int_0^t g(t_1)dt_1 \\
&\le AB_1.
\end{aligned}$$

(iii): The estimation of $\left|\frac{\partial}{\partial x} S_2 u(t,x)\right|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned}
\left|\frac{\partial}{\partial x} S_2 u(t,x)\right| &= (2l+1)\left|\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l} g(t_1) S_1 u(t_1,x_1) dx_1 dt_1\right| \\
&\le (2l+1)\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l} g(t_1) |S_1 u(t_1,x_1)| dx_1 dt_1 \\
&\le (2l+1) B_1 t \int_0^t g(t_1) dt_1 \\
&\le AB_1.
\end{aligned}$$

(iv): The estimation of $\left|\frac{\partial^2}{\partial x^2} S_2 u(t,x)\right|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x^2} S_2 u(t,x)\right| &= (2l+1)\cdot 2l\left|\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l-1} g(t_1) S_1 u(t_1,x_1) dx_1 dt_1\right| \\
&\le (2l+1)\cdot 2l\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l-1} g(t_1) |S_1 u(t_1,x_1)| dx_1 dt_1 \\
&\le (2l+1)\cdot 2l B_1 t \int_0^t g(t_1) dt_1 \\
&\le AB_1.
\end{aligned}$$

(v): The estimation of $\left|\frac{\partial^3}{\partial x^3} S_2 u(t,x)\right|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned}
\left|\frac{\partial^3}{\partial x^3} S_2 u(t,x)\right| &= (2l+1)\cdot 2l\cdot(2l-1)\left|\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l-2} g(t_1) S_1 u(t_1,x_1) dx_1 dt_1\right| \\
&\le (2l+1)\cdot 2l\cdot(2l-1)\int_0^t \int_0^x (t-t_1)(x-x_1)^{2l-2} g(t_1) |S_1 u(t_1,x_1)| dt_1 \\
&\le (2l+1)\cdot 2l\cdot(2l-1) B_1 t \int_0^t g(t_1) dt_1 \\
&\le AB_1.
\end{aligned}$$

... ...

(vi): The estimation of $\left|\frac{\partial^{2l}}{\partial x^{2l}} S_2 u(t,x)\right|$, $(t,x) \in [0,\infty) \times [0,1]$:

$$\begin{aligned}
\left|\frac{\partial^{2l}}{\partial x^{2l}} S_2 u(t,x)\right| &= (2l+1)!\left|\int_0^t \int_0^x (t-t_1)(x-x_1) g(t_1) S_1 u(t_1,x_1) dx_1 dt_1\right| \\
&\le (2l+1)!\int_0^t \int_0^x (t-t_1)(x-x_1) g(t_1) |S_1 u(t_1,x_1)| dx_1 dt_1 \\
&\le (2l+1)! B_1 t \int_0^t g(t_1) dt_1 \\
&\le AB_1.
\end{aligned}$$

(vii): The estimation of $\left|\frac{\partial^{2l+1}}{\partial x^{2l+1}}S_2u(t,x)\right|$, $(t,x)\in[0,\infty)\times[0,1]$:

$$\begin{aligned}\left|\frac{\partial^{2l+1}}{\partial x^{2l+1}}S_2u(t,x)\right| &= (2l+1)!\left|\int_0^t\int_0^x (t-t_1)g(t_1)S_1u(t_1,x_1)dx_1dt_1\right|\\ &\le (2l+1)!\int_0^t\int_0^x (t-t_1)g(t_1)|S_1u(t_1,x_1)|dx_1dt_1\\ &\le (2l+1)!B_1t\int_0^t g(t_1)dt_1\\ &\le AB_1.\end{aligned}$$

Consequently

$$\|S_2u\|\le AB_1.$$

This completes the proof. □

4. Proof of the main results

4.1. **Proof of Theorem 1.1.** Let $X=\underbrace{X_1\times\ldots\times X_1}_{(2l+3)\text{ times}}$, where

$$X_1=\mathcal{C}^1([0,\infty),\mathcal{C}^{2l+1}([0,1])).$$

Assume that the hypotheses $(H1)$, $(H2)$ and $(H3)$ are satisfied. Choose $\epsilon\in(0,1)$, such that $\epsilon B_1(1+A)<B$. Let $\widetilde{\widetilde{Y}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\widetilde{Y}=\overline{\widetilde{\widetilde{Y}}}$ be the closure of $\widetilde{\widetilde{Y}}$,

$$Y=\{u\in\widetilde{Y}:\|u\|\le B\}.$$

Note that Y is a compact set in X. For $u\in X$, define the operators

$$\begin{aligned}Tu(t,x) &= -\epsilon u(t,x),\\ Su(t,x) &= u(t)+\epsilon u(t,x)+\epsilon S_2u(t,x),\quad (t,x)\in[0,\infty)\times[0,1],\end{aligned}$$

where S_2 is the operator given by formula (3.3). For $u\in Y$, we have

$$\|(I-S)u\|=\|\epsilon u-\epsilon S_2u\|\le\epsilon\|u\|+\epsilon\|S_2u\|\le\epsilon B_1+\epsilon AB_1=\epsilon B_1(1+A)<B.$$

Thus, $S:Y\to X$ is continuous and $(I-S)(Y)$ resides in a compact subset of X. Now, suppose that there is a $u\in X$ so that $\|u\|=B$ and

$$u=\lambda(I-S)u,$$

or

$$\frac{1}{\lambda}u=(I-S)u=-\epsilon u-\epsilon S_2u,$$

or

$$\left(\frac{1}{\lambda}+\epsilon\right)u=-\epsilon S_2u,$$

for some $\lambda\in\left(0,\frac{1}{\epsilon}\right)$. Hence, $\|S_2u\|\le AB_1<B$,

$$\epsilon B<\left(\frac{1}{\lambda}+\epsilon\right)B=\left(\frac{1}{\lambda}+\epsilon\right)\|u\|=\epsilon\|S_2u\|<\epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T+S$ has a fixed point $u^*=(u_1^*,...,u_{2l+3}^*)\in Y$. Therefore

$$\begin{aligned}u^*(t,x) &= Tu^*(t,x)+Su^*(t,x)\\ &= -\epsilon u^*(t,x)+u^*(t,x)+\epsilon u^*(t,x)+\epsilon S_2u^*(t,x),\quad (t,x)\in[0,\infty)\times[0,1],\end{aligned}$$

whereupon

$$0 = S_2 u^*(t,x), \quad (t,x) \in [0,\infty) \times [0,1].$$

Consequently, from Lemma 3.2 and from the definition of S_2 given in formula (3.3), it follows that u_1^* is a solution to the IBVP (1.1). This completes the proof of Theorem 1.1.

4.2. **Proof of Theorem 1.2.** Assume that the hypotheses $(H1)$, $(H2)$ and $(H4)$ are satisfied. Set $X = \underbrace{X_1 \times \ldots \times X_1}_{(2l+3)\text{ times}}$, where $X_1 = \mathcal{C}^1([0,\infty), \mathcal{C}^{2l+1}([0,1]))$. For $u = (u_1, \ldots, u_{2l+3}) \in X$, we will write $u \geq 0$ whenever $u_j \geq 0$, $j \in \{1, \ldots, 2l+3\}$. Let

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad [0,\infty) \times [0,1]\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t,x) &= (1+m\epsilon)v(t,x) - \epsilon \frac{L}{10}, \\ S_3 v(t,x) &= -\epsilon S_2 v(t,x) - m\epsilon v(t,x) - \epsilon \frac{L}{10}, \end{aligned}$$

$(t,x) \in [0,\infty) \times [0,1]$, where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (3.3). Note that if $v = (v_1, \ldots, v_{2l+3}) \in X$ is a fixed point of the operator $T_1 + S_3$, then v_1 is a solution to the IBVP(1.1). Let us define the following sets:

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}, \text{ with } R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m}. \end{aligned}$$

(1) For $u, v \in \Omega$, we have

$$\|T_1 u - T_1 v\| = (1+m\epsilon)\|u - v\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(2) For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\|S_3 v\| \leq \epsilon \|S_2 v\| + m\epsilon \|v\| + \epsilon \frac{L}{10} \leq \epsilon \left(AB_1 + mR_1 + \frac{L}{10} \right).$$

Therefore $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

(3) Let $u \in \overline{\mathcal{P}_{R_1}}$. Set

$$v = u + \frac{1}{m} S_2 u + \frac{L}{5m}.$$

Note that $S_2 u + \frac{L}{5} \geq 0$ on $[0,\infty) \times [0,1]$. We have $v \geq 0$ on $[0,\infty) \times [0,1]$ and

$$\|v\| \leq \|u\| + \frac{1}{m}\|S_2 u\| + \frac{L}{5m} \leq R_1 + \frac{A}{m}B_1 + \frac{L}{5m} = R_2.$$

Therefore $v \in \Omega$ and

$$-\epsilon m v = -\epsilon m u - \epsilon S_2 u - \epsilon \frac{L}{10} - \epsilon \frac{L}{10}$$

or

$$(I - T_1)v = -\epsilon m v + \epsilon \frac{L}{10} = S_3 u.$$

Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

(4) Assume that for any $u_0 \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}$ there exist $\lambda > 0$ and $u \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $u \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3 u = (I - T_1)(u - \lambda u_0).$$

Then

$$-\epsilon S_2 u - m\epsilon u - \epsilon \frac{L}{10} = -m\epsilon(u - \lambda u_0) + \epsilon \frac{L}{10}$$

or

$$-S_2 u = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 u\| = \left\| \lambda m u_0 + \frac{L}{5} \right\| > \frac{L}{5}.$$

This is a contradiction.

(5) Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $w \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \varepsilon_1$ such that $\lambda_1 w \in \overline{\mathcal{P}_{R_2}}$ and

$$S_3 w = (I - T_1)(\lambda_1 w).$$

Since $w \in \partial\mathcal{P}_L$ and $\lambda_1 w \in \overline{\mathcal{P}_{R_2}}$, it follows that

$$\left(\frac{2}{5m} + 1 \right) L < \lambda_1 L = \lambda_1 \|w\| \leq R_1 + \frac{A}{m} B_1 + \frac{L}{5m}.$$

Moreover,

$$-\epsilon S_2 w - m\epsilon w - \epsilon \frac{L}{10} = -\lambda_1 m\epsilon w + \epsilon \frac{L}{10},$$

or

$$S_2 w + \frac{L}{5} = (\lambda_1 - 1) m w.$$

From here,

$$2\frac{L}{5} > \left\| S_2 w + \frac{L}{5} \right\| = (\lambda_1 - 1) m \|w\| = (\lambda_1 - 1) m L$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Hence and Theorem 2.6, it follows that the operator $T_1 + S_3$ has at least two fixed points $u^* = (u_1^*, ..., u_{2l+3}^*)$ and $v^* = (v_1^*, ..., v_{2l+3}^*)$ so that

$$\|u^*\| = L < \|v^*\| \leq R_1$$

or

$$r \leq \|u^*\| < L < \|u^*\| \leq R_1.$$

We have u_1^* and v_1^* are solutions of the IBVP (1.1). Not that, $\|u^*\| \neq \|v^*\|$, but we can get $u_1^* = v_1^*$. Consequently, the IBVP (1.1) has at least one nonnegative solution. This completes the proof of Theorem 1.2.

5. Two illustrative examples

Let

$$h(s) = \log \frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty} (1+s+s^2+s^3+s^4+s^5+s^6)h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty} (1+s+s^2+s^3+s^4+s^5+s^6)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2+s^3+s^4+s^5+s^6) \times \left(\frac{1}{44\sqrt{2}} \log \frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1-s^{22}} \right) \leq C_1,$$

$s \in [0,\infty)$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [21] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in [0,\infty),$$

and

$$g_1(t) = Q(t), \quad t \in [0,\infty).$$

Then there exists a constant $C_2 > 0$ such that

$$(2l+1)!(1+t) \int_0^t g_1(t_1)dt_1 \leq C_2, \quad t \in [0,\infty),$$

where l belongs to the set of all nonzero natural numbers $\mathbb{N}^*$. Let

$$g(t) = \frac{A}{C_2} g_1(t), \quad t \in [0,\infty).$$

Then

$$(2l+1)!(1+t) \int_0^t g(t_1)dt_1 \leq A, \quad t \in [0,\infty),$$

i.e., $(H2)$ holds.

5.1. **Example 1.** Take $l = 3$ in (1.1) and consider the following initial boundary value problem

$$\begin{array}{rcl} v_t + vv_x + \sum_{j=1}^{3}(-1)^{j+1}\partial_x^{2j+1}v & = & 0, \quad t \geq 0, \quad x \in [0,1], \\ v(0,x) & = & \frac{x^5(x-1)^6}{1+x^8(x-1)^{10}}, \quad x \in [0,1], \\ \partial_x^i v(t,0) = \partial_x^i v(t,1) & = & 0, \quad t \geq 0, \; i = 0,1,2, \\ \partial_x^l v(t,1) & = & 0, \end{array} \tag{5.4}$$

so that $(H1)$ holds, with $B = 10$, for example. Take

$$B = 10, \quad \text{and } A = \frac{1}{10^4}.$$

Then

$$AB_1 = A((l+1)B + B^2) = \frac{1}{10^4} \cdot (4 \times 10 + 10^2) < B.$$

So, Condition $(H3)$ is fulfilled. Thus, the conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied. Hence, by Theorem 1.1, it follows that Problem (5.4) has at least one solution $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^7([0,1]))$.

In the sequel, take

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.$$

Clearly,

$$r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 + \frac{A}{m}B_1 > \left(\frac{1}{5m} + 1\right)L, \quad AB_1 < \frac{L}{5},$$

i.e., $(H4)$ holds. Hence, by Theorem 1.2, it follows that the initial-boundary value problem (5.4) has at least one nonnegative solution $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^7([0,1]))$.

5.2. **Example 2.** Take $l = 5$ in (1.1) and consider the following initial boundary value problem

$$\begin{array}{rcl} v_t + vv_x + \sum_{j=1}^{5}(-1)^{j+1}\partial_x^{2j+1}v & = & 0, \quad t \geq 0, \quad x \in [0,1], \\ v(0,x) & = & \frac{x^9(x-1)^{10}}{1+x^{12}(x-1)^{14}}, \quad x \in [0,1], \\ \partial_x^i v(t,0) = \partial_x^i v(t,1) & = & 0, \quad t \geq 0, \; i = 0,1,2,3,4, \\ \partial_x^l v(t,1) & = & 0, \end{array} \tag{5.5}$$

so that $(H1)$ holds, with $B = 10$, for example. Take

$$B = 10, \quad \text{and } A = \frac{1}{10^4}.$$

Then

$$AB_1 = A[(l+1)B + B^2] = \frac{1}{10^4} \cdot (6 \times 10 + 10^2) < B.$$

So, condition $(H3)$ is fulfilled. Thus, the conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied. Hence, by Theorem 1.1, it follows that Problem (5.5) has at least one solution $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^{11}([0,1]))$.

In the sequel, take

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.$$

Clearly,

$$r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 + \frac{A}{m}B_1 > \left(\frac{1}{5m} + 1\right)L, \quad AB_1 < \frac{L}{5},$$

i.e., $(H4)$ holds. Hence, by Theorem 1.2, it follows that the initial-boundary value problem (5.5) has at least one nonnegative solution $u \in \mathcal{C}^1([0,\infty), \mathcal{C}^{11}([0,1]))$.

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