Methods of Functional Analysis and Topology
Vol. 28 (2022), no. 3, pp. 242–258
https://doi.org/10.31392/MFAT-npu26_3.2022.04

METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

ON STOCHASTIC COSURFACES AND TOPOLOGICAL QUANTUM FIELD THEORIES

JEAN-PIERRE MAGNOT

Abstract. We analyze the notion of a stochastic cosurface and show the following: the obstructions to the construction of non-abelian stochastic cosurfaces previously highlighted can be overcome by an ordering choice; the presence of an underlying manifold is not mandatory and stochastic cosurfaces can be defined in more general CW-complexes. We also describe a dimension extension procedure, in which any $d-$stochastic cosurface can be extended to a $(d+k)$-stochastic cosurface if the underlying CW-complex has $(d+k)$-faces.

We finish with a link of stochastic cosurfaces with topological quantum field theories and with an analog of deformation algebra indexed by a non-linear set of formal variables.

Аналізується поняття стохастичної коповерхні та доводиться наступне: перешкоди для побудови неабелевих стохастичниз коповерхонь, про які йшлось раніше, можуть бути подолані за разунок вибору порядку; наявність базового многовиду не є обов'язковим і стохастичним коповерхні можуть бути визначені в більш загальних CW-комплексах. Також описано процедуру розширення розмірності, де будь-яку $d-$стохастичну поверхню можна продовжити до $(d+k)$-стохастичній поверхні, якщо базовий CW-комплекс має $(d+k)$-граней.

Також розглядається зв'язок між стохастичними коповерхонями та топологічною квантовою теорією поля і з аналогом деформованою алгебра, що індексована нелінійним набором формальних змінних.

Introduction

Topological quantum field theories rely on the so called "cobordism hypothesis" formulated by Baez and Dolan in [6]. This hypothesis, formally worked out in the long preprint [11], remains a long-standing idea for a classification of topological quantum field theories (see e.g. [7]) and has ramifications in algebra, topology and representation theory. A good review on the subject is [8]. In this reference, one can see that a topological quantum field theory is a morphism from cobordisms to abelian categories. Several examples and constructions are sketched.

The present work develops, in a constructive way, a class of series indexed by manifolds with boundary, carrying so called "cut-and paste" properties of composition in the spirit of cobordism composition, that are derived from stochastic models in quantum and statistical physics. Such series can be understood as topological quantum field theories in a large sense. Our main starting point is a quite old series of works on so-called stochastic cosurfaces [1, 2, 3, 4, 5] where, in a heuristic way, time is considered as taking values in a non-linear manifold of dimension $d \geq 2$, and where time slices act through piecewise smooth hypersurfaces, in other words, $(d-1)-$dimensional skeletons with particular properties in a more general way than in topological quantum field theories but in the same spirit. Such models have applications in fields of mathematical physics such as lattice models and Higgs fields among others, including 2D Yang-Mills theories, in the fully developed work [10].

2020 *Mathematics Subject Classification.* 60G20,22E65,22E66,58B25,70G65.
Keywords. Stochastic cosurfaces, topological quantum field theory.

Analyzing the conditions that are technically necessary to develop cobordism-like composition that fit with these works, it appears that the key elements are very weak: one only needs associativity of the composition law (which needs not totality as in the case of cobordism examples) and a $\mathbb{N}^*-$ grading on the required set of indexes, that is, our pieces of smooth hypersurfaces.

From these preliminary considerations, we review existing results on stochastic cosurfaces and extend them, reaching some topological quantum field theory-like developments.

More precisely, we start from works [1, 2, 3, 4, 5] that introduced a so-called stochastic cosurface. We enlarge the settings of the previous references, and adapt them to build families of measures indexed by cobordism, such that, if γ and γ' are two morphisms of cobordism that can be composed into $\gamma\gamma'$, then for the corresponding measures, we get $\mu(\gamma\gamma') = \mu(\gamma)\mu(\gamma')$ (convolution product). By the way, the mapping $\gamma \mapsto \mu(\gamma)$ can be understood as a formal series over a family Γ of morphisms for cobordisms.

These results show that the initial investigations of [1, 2, 3, 4, 5], carried out for $d = 2$ stochastic cosurfaces with an arbitrary Lie group, or abelian cosurfaces of any dimension carry, with mild considerations, some possible generalizations for not necessarily abelian stochastic cosurfaces at any dimension. Even if the cut-and-paste formulas require more attention, the effects due to the presence of non-abelian groups required the introduction of an order in the slice of the non-linear time, the series of measures that we produce offer more usual expressions, in terms of series of measures, of the complex effects of non-abelian theories. Heuristically speaking, series of measures furnish classical treatise and expression of non-abelian (non-linear?) effects that seem to have been ignored in previous works.

1. Markov Cosurfaces in codimension 1

1.1. **Settings.** Let M be a $d-$dimensional connected oriented Riemannian manifold. Let $\mathcal{H}_\vee$ be the set of embedded, oriented, smooth, closed, connected hypersurfaces (codimension 1 submanifolds) of M with piecewise smooth border. What we call hypersurface is mostly smooth hypersurfaces on the manifold M, but since we need piecewise smooth oriented hypersurfaces, we need to build them by induction, gluing together the smooth components. What we get at the end is a space of oriented piecewise smooth hypersurfaces, with piecewise smooth border.

Definition 1.1. We set $\mathcal{H}_\vee^{(1)} = \mathcal{H}_\vee$. For $d \geq 2$, we define by induction:

- Let $(s_1, s_2) \in \mathcal{H}_\vee \times \mathcal{H}_\vee$. If
  (1) $s_1 \cap s_2 \subset \partial s_1 \cap \partial s_2$ is a $(d-2)$ piecewise smooth manifold and
  (2) the orientations induced on $s_1 \cap s_2$ by s_1 and s_2 are opposite,

then we define $s_1 \vee s_2$ to be the oriented piecewise smooth hypersurface of M obtained by gluing s_1 and s_2 along their common border. The orientation of $s_1 \vee s_2$ is the one induced by s_1 and s_2. The set of all such hypersurfaces is denoted by $\mathcal{H}_\vee^{(2)}$.

- Let $(s_1, s_2) \in \mathcal{H}_\vee^{(n-1)} \times \mathcal{H}_\vee$. If
  (1) $s_1 \cap s_2 \subset \partial s_1 \cap \partial s_2$ is a $(d-2)$ piecewise smooth manifold and
  (2) the orientations induced on $s_1 \cap s_2$ by s_1 and s_2 are opposite,

then we define in the same way $s_1 \vee s_2$. The set of such hypersurfaces is denoted by $\mathcal{H}_\vee^{(n)}$.

- We set $\Sigma_\vee = \bigcup_{n\in\mathbb{N}^*} \mathcal{H}_\vee^{(n)}$.

In all the article, we shall also assume that the connected components of ∂s are in $\Sigma_{\vee,n-1}$ if $s \in \Sigma_{\vee,n}$, for $n \in \mathbb{N}^*$.

Remark 1.1. If M is $2-$dimensional, it might seem that definition 1.1 generalizes the composition of non-parameterized piecewise smooth paths, setting $\mathcal{H}$ to be the set of smooth paths and $\vee$ the groupoïd composition law of oriented piecewise smooth paths. In fact, we need to reformulate the definition for $d = 2$ in order to fit with the usual composition of paths. Let us look at the following example. Let $M = \mathbb{R}^2$, and let s_1 and s_2 the paths parameterized by $s_1(t) = (\cos(\pi t), \sin(\pi t))$ and $s_2(t) = (-\cos(\pi t), -\sin(\pi t))$ for $t \in [0, 1]$. We have $\partial s_1 = \partial s_2 = \{(-1; 0); (1; 0)\}$, with "opposite orientations" (i.e. the endpoint of s_1 (resp. s_2) is the initial point of s_2 (resp. s_1)) so that paths s_1 and s_2 can be composed. But in order to have a loop, one has to determine which point among $\{(-1; 0); (1; 0)\}$ will be the initial point. The choice comes with the order in the composition of paths: $s_1 * s_2$ or $s_2 * s_1$. Such a choice cannot be done with Definition 1.1 because the law $\vee$ is obviously commutative in $\mathcal{H}_\vee \times \mathcal{H}_\vee$.

Now, we define $\mathcal{H}_*$ as the set of (non-parameterized, but oriented) smooth hypersurfaces s on the oriented manifold M, equipped in addition with a prescribed orientation of smooth components of ∂s. *Initial parts* of ∂s, denoted by $\alpha(s)$, are those for which the prescribed orientation is opposite to the one induced by s, and the *final parts*, denoted by $\beta(s)$, are the ones for which they coincide. For $d = 2$, the orientation of paths can naturally prescribe the initial and final points. This is the (apparently natural) choice that has been made in [2] but we remark here that this choice is not necessary. The picture of the following definition will merely be the same as the one of definition 1.1, but each smooth component of the border of the hypersurface is assigned to be either initial or final. In order to keep the coherence with the loop composition,

- we can glue together a final part with an initial part,
- and the final parts and the initial parts can be the same set-theoretically, just as in the case of a loop starting and finishing at the same point.

Here is the construction:

Definition 1.2. We set $\mathcal{H}_*^{(1)} = \mathcal{H}_*$. By induction:

• Let $(s_1, s_2) \in \mathcal{H}_* \times \mathcal{H}_*$. Let $a = \alpha(s_1) \cap \beta(s_2)$. We define $s_1 * s_2$ as the oriented piecewise smooth hypersurface of M obtained by gluing s_1 and s_2 on a and denoted by $s_1 \cup_a s_2$. The orientation of $s_1 * s_2$ is the one induced by s_1 and s_2 on $s_1 \cup_a s_2$ By the way, we have $\alpha(s_1 * s_2) = \alpha(s_2) \cup (\alpha(s_1) - a)$, $\beta(s_1 * s_2) = \beta(s_1) \cup (\beta(s_2) - a)$, and $\partial(s_1 * s_2) = \alpha(s_1 * s_2) \coprod \beta(s_1 * s_2)$. The set of such hypersurfaces is denoted by $\mathcal{H}_*^{(2)}$.

• Let $(s_1, s_2) \in \mathcal{H}_*^{(n-1)} \times \mathcal{H}_*$. Then we define $s_1 * s_2$ in the same way. The set of such hypersurfaces is denoted by $\mathcal{H}_*^{(n)}$.

• We set $\Sigma_* = \bigcup_{n \in \mathbb{N}^*} \mathcal{H}_*^{(n)}$.

Notice that there is a forgetful map $\Sigma_* \to \Sigma_\vee$ only for the hypersurfaces $s \in \Sigma^*$ that have no self-intersection. The following example, based on the Möbius band, shows that this restriction is needed.

Example 1.1. Let us fix $M = \mathbb{R}^3$ and let

$$s_1 = \left\{ \left(\cos(\pi t), \sin(\pi t), s - \frac{1}{2} \right) | (t, s) \in [0; 1]^2 \right\}$$

such that $\alpha(s_1) = \left\{ (\cos(\pi t), \sin(\pi t), s) | (t, s) \in (\partial[0; 1])^2, t = 0 \right\}$, $\beta(s_1) = \partial s_1 - \alpha(s_1)$ and let

$$s_2 = \left\{ (-\cos(\pi t), -\sin(\pi t)(1 + cos(\pi t)), s - cos(\pi t)) | (t, s) \in [0; 1]^2 \right\}$$

such that

$$\alpha(s_2) = \left\{ \left(-\cos(\pi t), -\sin(\pi t) \left(1 + \frac{sin(\pi t)}{2} \right), s - cos(\pi t) - \frac{1}{2} \right) | (t, s) \in (\partial[0; 1])^2, t = 0 \right\},$$

$\beta(s_2) = \partial s_2 - \alpha(s_2)$. If one glues topologically s_1 and s_2, we get the Möbius band which is non-orientable. So that $s_1 \vee s_2$ is not defined in this case. By our choices of initial and final parts, $s_1 * s_2$ and $s_2 * s_1$ both exist, because they can be represented by the "cut" of the Möbius band, with

$$\alpha(s_1 * s_2) = \alpha(s_2) = \Big\{(1;0;s)| - \frac{1}{2} \leq s \leq \frac{1}{2}\Big\},$$
$$\beta(s_1 * s_2) = \beta(s_1) = \Big\{(1;0;s)| - \frac{1}{2} \leq s \leq \frac{1}{2}\Big\},$$

and with

$$\alpha(s_2 * s_1) = \alpha(s_1) = \Big\{(-1;0;s)| - \frac{1}{2} \leq s \leq \frac{1}{2}\Big\},$$
$$\beta(s_2 * s_1) = \beta(s_2) = \Big\{(-1;0;s)| - \frac{1}{2} \leq s \leq \frac{1}{2}\Big\}.$$

One can say that this is not natural, since, e.g., $\alpha(s_1 * s_2)$ is not in the border of the underlying C^0-manifold. This is one of the reasons why we discuss this example in details. Moreover, this fits with the natural composition of paths: ignoring the third coordinate, we get back the classical composition of paths, for which loops are topologically without border but have a starting point and an endpoint.

In what follow, Σ_M represents either $\Sigma_\vee$ or Σ_* with an adequate choice of the Lie group G (we choose G to be abelian for $\Sigma_\vee$). Anyway, we denote the group law of G by the operation of multiplication, and we denote $s_1 s_2$ for $s_1 \vee s_2$ or $s_1 * s_2$.

Definition 1.3. A $G-$valued *cosurface* is a map

$$c : \Sigma_M \to G$$

such that

(1) $\forall (s_1, s_2) \in \Sigma_M \times \mathcal{H}, c(s_1 s_2) = c(s_1)c(s_2)$ and
(2) we denote by $\tilde{s}$ the same cosurface as $s \in \Sigma_M$ with the opposite orientation. Then $\forall s \in \Sigma_M, c(\tilde{s}) = c(s)^{-1}$.

Let $\tau_s(c) = c(s)$ and $\Gamma_{M,G}$ be the set of $G-$valued cosurfaces of M equipped with the the smallest $\sigma-$ algebra making measurable the collection of maps

$$\{\tau_s : \Gamma_{M,G} \to G | s \in \Sigma_M\}.$$

Let $(\Omega, \mathcal{B}, p)$ be any probability space.

Definition 1.4. A *stochastic cosurface* is a map

$$C : \Omega \times \Sigma_M \to G$$

such that

(1) $\forall \omega \in \Omega$, $C(\omega, .) \in \Gamma_{M,G}$.
(2) the map $\omega \in \Omega \mapsto C(\omega, .)$ is a $\Gamma_{M,G}-$valued measurable map.

For a subset $\Lambda \subset M$ we consider the $\sigma-$algebra $\mathfrak{T}(\Lambda)$ generated by stochastic cosurfaces $C(s)$ where $s \subset \Lambda$. In other words,

$$\mathfrak{T}(\Lambda) = \sigma \left\{ \{C \in \Gamma_{M,G} | C(s) \in B\} | s \subset \Lambda ; B \text{ is a Borel subset of } G \right\}.$$

Now, we have to define finite sequences of hypersurfaces that we consider as a *complex* (of hypersurfaces).

Definition 1.5. Let $n \in \mathbb{N}^*$. An $n-$*complex* on M is a $n-$uple $K = (s_1, ...s_n) \in (\Sigma_M)^n$ such that $s_i \neq s_j$ for $i \neq j$. We define

$$C(K) = (C_1(s_1), ..., C_n(s_n))$$

where each $C_i \in \Gamma(M; G)$. We denote by $\mathcal{K}$ the set of complexes of any length n.

Notice that a complex is an *ordered* sequence, and related with this order there is a natural notion of *subcomplex* of a complex K. If $K = (s_1, \dots, s_n)$, a subcomplex L is a subsequence of K, that is

$$\exists l < m \leq n, \qquad L = (s_l, \dots, s_m) = (s_i)_{l \leq i \leq m}.$$

Now, we need to recognize the complexes that define skeletons of a partition of the manifold M.

Definition 1.6. An n-complex $K = (s_1, \dots, s_n)$ is *regular* if $\forall (i,j) \in {\mathbb{N}_n}^2$,

$$i \neq j \Rightarrow s_i \cap s_j \subset \partial s_i \cap \partial s_j.$$

Notice that the definition does not consider initial and final parts of the borders. In the sequel, since many ways to understand complexes can be useful (set-theory, topological spaces, sequences, oriented manifolds) we shall use the standard notations in these various fields and we shall specify in what sense we use them only if the notations carry any ambiguity.

Definition 1.7. Let K be a regular n-complex. K is called *saturated* if and only if $\bigcup_{i \in \mathbb{N}_n} s_i$ defines the borders of a covering of M by connected and simply connected closed subsets. In other words, there is a family $(A_k)_k$ of closed connected and simply connected subsets of M such that

(1) $\bigcup_k A_k = M$,
(2) for two any indexes k and k', if $k \neq k'$,

$$A_k \cap A_{k'} \subset \partial A_k \cap \partial A_{k'} \subset \bigcup_{i \in \mathbb{N}_n} s_i.$$

We say that a regular n-complex K *splits* M through the subcomplex $L = (s_i)_{l \leq i \leq m}$ if

(1) $\bigcup_{s \in L} s$ splits M into two connected components M^+ and M^- and
(2) $\bigcup_{i<l} s_i \subset Adh(M^-)$; we denote $K^- = (s_i)''_{i<l} = K \cap M^-$",
(3) $\bigcup_{i>m} s_i \subset Adh(M^+)$; we denote $K^+ = (s_i)''_{i>m} = K \cap M^+$".

(here, Adh means topological closure)

Example 1.2 (around the 2-cube)**.** Let $ABCDEFGH \subset \mathbb{R}^3$ be the 2-cube, and we assume that each coordinate of A, B, C, D, E, F, G and H is equal to ± 1.

Let us consider the (empty) 2-cube $ABCDEFGH$ as a piecewise smooth hypersurface in $\mathbb{R}^3$. By the orientation of $\mathbb{R}^3$, and since the cube divides $\mathbb{R}^3$ into an inside part and an outside part, each face is oriented so that $ABCDEFGH \in \Sigma_\vee$. The 2-cube divides $\mathbb{R}^3$ into two parts, that we recognize as interior and exterior, which are connected and simply connected (but one is not contractible). So that, it splits $\mathbb{R}^3$. We can also say that we have a regular 6-complex made of the faces of $ABCDEFGH$ (where we have to choose an order which is non canonical).

Let us now project $ABCDEFGH$ into S^2 radially. Then the segments of $ABCDEFGH$ define a class of regular complexes on S^2. The complex of the segments is not uniquely defined because of the order that we have to choose, and also because of the orientations of the segments that we have to choose. This will yield different possible splittings. For example,if we consider the the complex $K = (s_1, ... s_{12})$ defined by

$$\begin{aligned} K &= \big((A;B),(B;E),(B;C),(C;D),(C;G),(A;E),(E;F),\\ &\quad (F;G),(G;H),(H;D),(D;A),(E;H)\big), \end{aligned}$$

we have the subcomplex

$$L = ((A;E),(E;F),(F;G),(G;H),(H;D),(D;A))$$

that splits K. We have $(S^2)^+ = AEHD \cup EFGH$ and $(S^2)^- = ABCD \cup ABFE \cup BCGF \cup CDGH$ (which are both contractible) and finally $K^+ = (E;H)$ and $K^- = ((A;B),(B;E),(B;C),(C;D),(C;G))$.

Definition 1.8. Let C be a stochastic cosurface. C is said to be a *Markov cosurface* if for each $n \in \mathbb{N}^*$ and for each regular n-complex K which splits through a subcomplex $L = (s_i)_{l \leq i \leq m}$, for each couple of maps (f^+, f^-) for which the above expectations exists and f^+ (resp. f^-) is $\mathfrak{T}(M^+ \cup \bigcup_{l \leq i \leq m} s_i)$-measurable (resp. $\mathfrak{T}(M^- \cup \bigcup_{l \leq i \leq m} s_i)$-measurable), we have

$$\mathbb{E}(f^+ f^- | \mathcal{T}(\bigcup_{l \leq i \leq m} s_i)) = \mathbb{E}(f^+ | \mathfrak{T}(\bigcup_{l \leq i \leq m} s_i)) \mathbb{E}(f^- | \mathfrak{T}(\bigcup_{l \leq i \leq m} s_i)).$$

1.2. **Markov cosurfaces and Markov semigroups.** Let λ be the Haar measure on G which is now assumed unimodular. We introduce a projective system of probability measures on $\{G^K; K \in \mathcal{K}\}$. For this, we use a partial order on $\mathcal{K}$.

Proposition 1.1. *Let $(K, K') \in \mathcal{K}^2$ such that $K = K'$ in the set-theoretic sense. We write $K \prec K'$ if $\forall s \in K$, there exists a subcomplex L' of K' such that s is a composition of elements of L', ordered by indexes. Here $\prec$ is an order on $\mathcal{K}$.*

Proof. Comparing this proposition with [5], we already have that $\prec$ is only a preorder. So that we need only to check reflexivity. Let $s \in K$. Taking $L' = \{s\}$, we get $K \prec K$. Moreover, let $(K, K') \in \mathcal{K}^2$, if $K \prec K'$ and $K' \prec K$, $\forall s \in K, s \in K'$ and $\forall s' \in K', s \in K$ and hence K and K' have the same hypersurfaces, indexed with respect to the same order. □

We now recall the standard definition of filters for the order $\prec$.

Definition 1.9. A filter $P \subset \mathcal{K}$ is such that:

(1) $\forall (K, K') \in P^2, \exists K'' \in P, (K \prec K'' \wedge K' \prec K'')$.
(2) $(\forall K \in \mathcal{K}, \exists K' \in P, K \prec K') \Rightarrow K \in P$.

Let Q_t be a convolution semigroup of probability measures on G with densities, i.e., $Q_t = q_t.\lambda$ satisfies

(1) $Q_0 = \delta_e$ (the Dirac measure at the unit element)
(2) $\forall s, t \in (\mathbb{R}_+^*)^2$, $\forall x \in G$, $(q_t q_s)(x) = \int_G q_s(xy^{-1}) q_t(y) d\lambda(y) = q_{t+s}(x)$
(3) $\lim_{t \to 0} Q_t = \delta_e$ weakly
(4) $\forall (x, y) \in G^2, q_{(.)}(xy) = q_{(.)}(yx)$

Now, we need to separate the exposition among the two approaches of cosurfaces, one on $\Sigma_\vee$ and the other on Σ_*. In both cases, we fix $K \in P$, a regular saturated complex with associated domains $D = \{A_1, ... A_m\}$. Each $A \in D$ is oriented through the orientation of M.

• On $\Sigma_\vee$. We define

$$\varphi_A(s) = \begin{cases} s, & \text{if } s \subset \partial A \text{ has the same orientation as } \partial A, \\ \tilde{s}, & \text{if } s \subset \partial A \text{ has the opposite orientation from } \partial A, \\ \emptyset, & \text{if } s \not\subset \partial A. \end{cases}$$

We set

$$\phi_A(C(K)) = \prod_{s \in K} C \circ \varphi_A(s)$$

(this product is a convolution product of measures.)

• On Σ_*. Here, $K = (s_1, ..., s_m)$ is ordered by indexation. Then, we have to work by induction to define φ_A.

- Let $s_j \in K$ be the first element in K such that $s_j \subset \partial A$. For $i < j$, we set $\varphi_A(s_i) = \emptyset$. Then we compare the orientation of ∂A with the one of s_j as in the case of $\Sigma_M^\vee$ to define $\varphi_A(s_j)$.
- Assume now that we have determined φ_A up to an index j. Take l to be the first index after j such that $s_l \subset \partial A$. As before, for $j < i < l$, we set $\varphi_A(s_i) = \emptyset$. First, compare the orientation of s_l with the one of ∂A and change s_l into $\tilde{s}_l$, if necessary, as before. Notice that final parts of s_j and initial parts of s_l are not considered here. This enables anyway to define

$$\phi_A(C(K)) = C \circ \varphi_A(s_1) ... C \circ \varphi_A(s_n) \tag{1.1}$$

Both in the case of $\Sigma_\vee$ and in the case of Σ_*, we set

Definition 1.10.

$$\mu_K^Q(C) = \prod_{i=1}^{k} q_{|A_i|} \left(\phi_{A_i} \circ C(K) \right).$$

Remark 1.2. When $d = 2$, changing the orientation of the path $s \in \mathcal{H}$ is the same as permuting its initial and its final points. Then, the procedure described for Σ_* makes also final parts and initial parts coinciding.

1.3. **Action of the symmetric group.** Looking at Definition 1.10 of μ_K, we easily see that the value of μ_K is independent of the order of the sequences $A = (A_1, \ldots, A_k)$, since the group G is unimodular. Unlikely, there is no invariance under reordering K in the nonabelian case (see the definition of ϕ_A in equation 1.1). Here, ϕ_A depends on the order of the saturated complex $K = (s_1, \ldots, s_n)$. So that, the action of the n-symmetric group $\mathfrak{S}_n$ on indexes of n-saturated complexes

$$(\sigma, K = (s_1, \ldots, s_n)) \longmapsto \sigma.K = \left(s_{\sigma(1)}, \ldots, s_{\sigma(n)}\right)$$

generates an action $\sigma.\mu_K = \mu_{\sigma.K}$. Setting $\mathcal{KS}$ to be the set of saturated complexes on M, denoting by $\mathfrak{S}_\infty$ the group of bijections on $\mathbb{N}^*$, we get an action $\mathfrak{S}_\infty \times \mathcal{KS} \to \mathcal{KS}$ in the following way: completing $K = (s_1, \ldots, s_n) \in \mathcal{KS}$ into $\hat{K} = (s_1, \ldots, s_n, \emptyset, \ldots) \in \Sigma_*^{\mathbb{N}^*}$, a bijection $\sigma \in \mathfrak{S}_\infty$ on indexes gives a sequence $\sigma.\hat{K}$ with only n elements different from $\emptyset$. We define $\sigma.K$ to be the n-saturated complex (indexed by $\mathbb{N}_n = \{1, 2, \ldots, n\}$) as the collection $\{s_1, \ldots, s_n\}$, ordered by $\sigma.\hat{K}$ index-wise.

1.4. **Examples.** This selection of examples is based on earlier works [1, 2, 3, 4, 5], where only $d = 2$ examples on Σ_* or examples on $\Sigma_\vee$ were considered. An example on Σ_* with $d = 3$ will be given later because the tools needed have to be much clarified.

1.4.1. *The $d = 2$ holonomy cosurface.* ([2], compare with the settings described in [10]) Let M be a 2-manifold. Let $\mathcal{P}(M)$ be the space of piecewise oriented smooth paths, with canonical initial and final points ("canonical" means induced by the path orientation). By the way, open paths in $\mathcal{P}(M)$ can be identified with a subset of $\Sigma_\vee$ and there is a map $\Sigma_* \to \mathcal{P}(M)$ which coincides with the forgetful map $\Sigma_* \to \Sigma_\vee$ on open paths and with a map on loops that is changing the initial and final parts if necessary. Let G be a Lie group and let $P = M \times G$ the trivial principal bundle over M with structure group G. Let θ be a connection on P and we note by Hol_θ the holonomy mapping $\mathcal{P}(M) \to G$ for which the horizontal lift starts at $(\alpha(p), e_G)$. Let $s \in \Sigma_*$, which we identify with the corresponding path that is also denoted by s. We define the holonomy cosurface c by

$$c(s) = Hol_\theta(\tilde{s}).$$

We need here to invert the orientation of the path because of the right action of the holonomy group on the principal bundle P.

Remark 1.3. Since the definition of (non stochastic) cosurfaces does not require any measure, one can take for G any Lie group on which the notion of horizontal lift of a path with respect to a connection is well defined. At this level, the construction is valid for any (finite dimensional) Lie group, but for any Banach Lie group, or regular Fréchet Lie group, regular c_∞-Lie group [9], as well as for regular frölicher Lie group [12].

Remark 1.4. This approach is quite similar to the approach of gauge theories via quantum loop gravity approach, see e.g. [13] for an up-to date paper. However, many open questions remain when one wishes to work along the lines of this viewpoint.

Then choosing Q as the heat semi-group on G, we get a stochastic cosurface picture of $2d$-Yang-Mills fields (see, e.g., [10, 14] and references therein for an extensive work in the case where M is a 2-dimensional manifold and the topology is non trivial).

1.4.2. *Markov cosurfaces and lattice models* [2, 5]. Let $L_\epsilon = \mathbb{Z}^d$. Let U be an invariant function on a compact group G and a "coupling constant" $\beta > 0$. Let Λ be a bounded subset of L_ϵ and let us define the (normalized) probability measure

$$\mu_\epsilon^\Lambda = \frac{1}{Z_{\Lambda,\epsilon}} \exp\left(-\beta \sum_{\gamma\subset\Lambda} U(C(\partial\gamma))\right) \prod_{\gamma\subset\Lambda} dC(\partial\gamma)$$

where γ is an elementary cell, $\partial\gamma$ the boundary, $C(\partial\gamma)$ a variable associated to $\partial\gamma$ with values in G. In the sense of projective limits of measures, the limit $\Lambda \to L_\epsilon$ exists and defines a Gibbs-like lattice cosurface. In the cases $G = U(1), SU(2)$, the continuum limit $\epsilon \to 0$ for μ_ϵ has been shown to exist for appropriate U and by a suitable choice of $\beta(\epsilon)$ such that $\lim_{\epsilon\to 0}\beta(\epsilon) = +\infty$.

1.4.3. *Markov cosurfaces and Higgs fields in 2-dimensional space time* [5, 1]. Let Λ be a bounded non empty subset of $\mathbb{Z}^2$. Cosurfaces C are defined along the edges of $\mathbb{Z}^2$. Let G be a compact Lie group, equipped with a representation ρ on a Euclidean space V with a scalar product $< .;. >$ and the norm $|.|$. Let $(\lambda,\mu) \in \left(\mathbb{R}_+^*\right)^2$, and let φ be a V- valued random field over $\mathbb{Z}^2$. We define a probability measure$\mu_{\lambda,\Lambda}$ on $\mathbb{Z}^2$ with support in Λ by

$$\mu_{\lambda,\Lambda}(d\phi) = \frac{1}{Z_\Lambda} e^{-\frac{\lambda}{2}\sum_{x\in\Lambda}\left(B+\frac{\mu^2}{\lambda}\right)|\varphi(x)|^2} e^{-\frac{\lambda}{2}\sum_{x,y\in\Lambda}<\varphi(x),\rho\circ C(xy)\varphi(y)>} \prod_{x\in\Lambda} d\varphi(x),$$

where $C(xy)$ is the evaluation of the Markov cosurface C on the path xy. Replacing the lattice $\mathbb{Z}^2$ with $\epsilon\mathbb{Z}^2$, with a suitable choice of $\lambda(\epsilon,\mu(\epsilon)$, yields the continuum limit Higgs models as $\epsilon \to 0$. In the cases $G = U(1), SU(2)$, with Q_t the heat semi-group, our Higgs fields coincide with the Higgs fields in the physics literature. For a description of the mathematical construction of the continuum limit, keeping C fixed, see [5].

2. Cosurfaces without underlying manifolds

Let us now consider a Hilbert space H. Mimicking section 1, we show how the notions can be extended without the codimension 1 assumption.

2.1. **Settings.** Let M be a d-dimensional connected oriented Riemannian manifold. Let $\mathcal{H}_{\vee,n}$ (resp. $\mathcal{H}_{*,n}$) be the set of embedded, oriented, smooth, closed, connected n-submanifolds of H with piecewise smooth border (resp. the set of embedded, oriented, smooth, closed, connected n-submanifolds of H with piecewise smooth border and with initial and final parts). Let us recall the assumptions for the composition $\vee$.

Let $(s_1, s_2) \in \mathcal{H}_{\vee,n} \times \mathcal{H}_{\vee,n}$. If

(1) $s_1 \cap s_2 \subset \partial s_1 \cap \partial s_2$ is a non empty $(d-2)$ piecewise smooth manifold and
(2) the orientations induced on $s_1 \cap s_2$ by s_1 and s_2 are opposite.

then we define $s_1 \vee s_2$ by gluing s_1 and s_2 along their common border. The orientation of $s_1 \vee s_2$ is the one induced by s_1 and s_2, and by the gluing conditions, if $s_1 \vee s_2$ is itself a (non piecewise) smooth submanifold of H, then there is an orientation on $s_1 \vee s_2$ that generates the orientation of s_1 and of s_2.

We denote by $\Sigma_{\vee,n}$ the corresponding space of piecewise smooth n-submanifolds of H, and set

$$\Sigma_{\vee,H} = \coprod_{n\in\mathbb{N}} \Sigma_{\vee,n}.$$

Notice that we can extend $\vee$ component-wise to $\Sigma_{\vee,H}$.

Let us do the same for $*$. $\mathcal{H}_{*,n}$ is the set of (non-parameterized, but oriented) n-submanifolds s of H equipped in addition with a prescribed orientation of smooth components of ∂s. Initial parts of ∂s, denoted by $\alpha(s)$, are those for which the prescribed orientation is opposite to the one induced by s, and the final parts, denoted by $\beta(s)$, are the ones for which they coincide. These are exactly the definitions given in Section 1, we can define $\Sigma_{*,n}$ the corresponding space of piecewise smooth submanifolds and we set

$$\Sigma_{*,H} = \coprod_{n\in\mathbb{N}} \Sigma_{*,n}$$

with extension of $*$ component-wise. In what follows, when it causes no ambiguity, we omit the notations $\vee$ or $*$ and denote the composition rule by multiplication, with an adequate choice of the Lie group G (we choose G to be abelian for $\Sigma_{\vee,H}$). Anyway, we denote the group law of G by multiplication too. The following notions can be extended in a straightforward way; the only conceptual difference is that the orientation of a surface s cannot be compared with the orientation of an underlying manifold M. The manifold M is replaced with the CW-complex obtained by gluing the domains A_k along the borders.

Definition 2.1. A G-valued cosurface is a map

$$c : \Sigma_H \to G$$

such that

(1) $\forall (s_1, s_2) \in \Sigma_H \times \mathcal{H}, c(s_1 s_2) = c(s_1)c(s_2)$ and
(2) We denote by $\tilde{s}$ the same n-submanifold as $s \in \Sigma_M$ with opposite orientation on s and ∂s. Then $\forall s \in \Sigma_M, c(\tilde{s}) = c(s)^{-1}$.

We denote by $\Gamma(G)$ the set of all G-valued cosurfaces.

Let $m \in \mathbb{N}^*$. An *(m, n)-complex* on H is an m-uple $K = (s_1, \ldots, s_m) \in (\Sigma_n)^m$ such that $s_i \neq s_j$ for $i \neq j$. We define

$$C(K) = (C(s_1), \ldots, C(s_n))$$

where each $C_i \in \Gamma(M; G)$. We denote by $\mathcal{K}_n$ the set of complexes of dimension n and of any length m.

An n-complex $K = (s_1, \ldots, s_n)$ is *regular* if $\forall (i, j) \in {\mathbb{N}_n}^2$,

$$i \neq j \Rightarrow s_i \cap s_j \subset \partial s_i \cap \partial s_j.$$

There is a natural notion of subcomplex of a complex K. If $K = (s_1, \ldots, s_n)$, a subcomplex L is a subsequence of K, that is,

$$\exists l < m \leq n, L = (s_l, \ldots, s_m) = (s_i)_{l\leq i\leq m}.$$

Let K be a regular n-complex. K is called *saturated* if and only if there is a $(n+1)$-complex A such that $\bigcup_{i\in\mathbb{N}_n} s_i$ defines the borders of the $(n+1)$-surfaces of A by connected and simply connected closed subsets. In other words, a family $(A_k)_k$ of closed connected and simply connected subsets of H defines, by gluing along K, a CW-complex also noted M such that

(1) $\bigcup_k A_k = M$

(2) for any two indexes k and k', if $k \neq k'$,

$$A_k \cap A_{k'} \subset \partial A_k \cap \partial A_{k'} \subset \bigcup_{i \in \mathbb{N}_n} s_i.$$

We say that a complex K *splits* through a subcomplex $L = (s_i)_{l \leq i \leq l'}$ if $\bigcup_{s \in L} s$ splits M (as a subset of H) into two topologically connected components M^+ and M^- such that $K \cap M^+ = (s_i)_{i > m}$ and $K \cap M^- = (s_i)_{i > l'}$

2.2. **Dimension extension.** We use here an idea of the previous definition, gluing together simply connected $(n+1)$-surfaces A_i along an n-complex K in order to get, by induction on the dimension N of the surfaces, a construction of n-cosurfaces from lower dimensions.

Definition 2.2. A complex $K \in \mathcal{K}_n$ is called *saturated* if it can be embedded into an $(n+1)$-submanifold M of H, for which it is a complex for cobordism. M is called a *saturation* of K.

In this definition, we then obtain M by gluing, along K, a family of elements of Σ_{n+1} which are connected and simply connected.

Definition 2.3. A complex $K \in \mathcal{K}_{\vee,n}$ is called *weakly saturated* if there is a complex $A = (A_1, \dots, A_k) \in \mathcal{K}_{*,n+1}$ obtained

- topologically by gluing each set A_i on $L \subset K$ with respect to the borders ∂A_i such that there is a bijective map $\partial A_i \to L$.
- the orientations of the borders on ∂A_i correspond to the orientations of the submanifolds in the sequence L.
- any $s \in K$ is at least glued once.

A is called a *weak saturation* of K.

Notice that with this definition, the orientation of each A_i is left free of choice. Moreover, the definition can be extended straightaway by replacing $\vee$ by $*$ since the initial and final parts of $s \in K$ do not interfere with the gluing.

Definition 2.4. Let c_K be defined on the surfaces $A \in \Sigma_{*,n+1}$ that are smooth with piecewise smooth border $\partial A \subset K$. Then we define

$$c_K(A) = \prod_{s \in \partial A \subset K} c \circ \varphi_A(s)$$

(the product is with respect to the order in K).

Proposition 2.1. *If G is abelian, then c_K is the restriction of a cosurface c' on $\Sigma_{*,n+1}$ that coincides with c_K on any A where it is defined.*

Proof. First, for a complex K_1 such that $K \prec K_1$, we can get the values of c on K_1. Then, all we have to show is that, given A an $(n+1)$-submanifold of H with piecewise smooth border along K and taking $A' = (A_1, \dots, A_l)$ such that $A \prec A'$, in other words $A = A_1 * \dots * A_l$, taking K' the (unordered) skeleton of the gluing $A_1 * \dots * A_l$, one has

$$c_K(A) = c_{K'}(A_1) \dots c_{K'}(A_l).$$

There is of course an ambiguity on the order of K' but since G is abelian, for each $s \in K'$, we have only to count the number of indexes $j \in \mathbb{N}_l$ such that $\varphi_{A_j}(s) = s$ and compare it to the number of indexes such that $\varphi_{A_j}(s) = \tilde{s}$. Since we are in the $*$-composition, we have at each edge $s \not\subset K$ only one index of each type, which are the A_j's for which s is in the initial part and the final part, respectively. So that the contributions that are "interior" compensate. □

Definition 2.5. Let $K \in \mathcal{K}_n$. An *overcomplex* K' of K is a weakly saturated complex such that there exists a complex K_1 with $K \prec K_1$ and $K_1 \subset K'$ with preserved order.

Proposition 2.2. *If G is non abelian, c_K is the restriction of a cosurface c' on $\Sigma_{*,n+1}$ such that for any overcomplex K' of K, for any $s \in K' \backslash K$, there exists c'' a cosurface on $\Sigma_{*,n+1}$,*

$$c'(s) = c''_K(s) \in Z(G).$$

Proof. Let A be a complex as in Definition 2.4. The main points are to know

- what happens for K_1 such that $K \prec K_1$? Here again, we can make use of the complex c (we recall that if $K \prec K_1$, the order is preserved).

- What happens for an overcomplex K' that the skeleton of a complex A' such that A? what to do with the "interiors"? Let K_1 be the complex made of elements $s \in K'$. Let us assign arbitrarily a value $c''(s) = g \in Z(G)$ (e.g. take $g = e$) if $s \in K' - K_1$. We have

$$\begin{aligned}\prod_{i=1}^{l} c_K(A_i) &= \prod_{i=1}^{l} \prod_{s \in \partial A_i \subset K} c \circ \varphi_{A_i}(s) \\ &= \prod_{i=1}^{l} \prod_{s \in \partial A \subset K_1} c \circ \varphi_{A_i}(s)\end{aligned}$$

on one hand. And on the other hand,

$$\begin{aligned}\prod_{j=1}^{l'} c_K(A'_j) &= \prod_{i=1}^{l'} \prod_{s \in \partial A'_j \subset K} c \circ \varphi_{A_j}(s) \\ &= \left(\prod_{j=1}^{l'} \prod_{s \in \partial A \subset K_1} c \circ \varphi_{A_i}(s)\right)\left(\prod_{j=1}^{l'} \prod_{s \in \partial A \subset K - K_1} c \circ \varphi_{A_i}(s)\right) \\ &= \left(\prod_{j=1}^{l'} \prod_{s \in \partial A \subset K_1} c \circ \varphi_{A_i}(s)\right).e\end{aligned}$$

for the same reasons as in the last proof. □

Example 2.1 (Dimension extension of the holonomy cosurface)**.** We have given a way to extend cosurfaces to higher dimensions. Let us now use it to show that the notion of a non abelian cosurface is not void on $\Sigma_{*,n}$ for $n > 1$. For this, let us consider the infinite lattice

$$\mathbb{Z}^\infty = \left\{(u_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} | u_n \neq 0 \text{for a finite number of indexes } \right\}$$

Let G be a unimudular Lie group, and let θ be a connection on $H \times G$. Along the edges of this lattice in particular and more generally on any path, we can define the holonomy cosurface as in Section 1.4.1. Let K be a weakly saturated 1-complex along the edges of $\mathbb{Z}^\infty$. Let us now consider the squares that are described by this lattice, and more specifically those that are gluing along K. They are 2-submanifolds on H and we can define a cosurface $c_{2,K}$ on $\Sigma_{2,*}$ that reads as in Theorem 2.2. Then, for K "large enough", we can find a weakly saturated complex of squares $A = (A_1, \ldots, A_n)$ to reproduce the procedure to get a non abelian cosurface on cubes, and this until we reach the dimension d for which there no longer exists any weakly saturated complex "based" on K.

For example, take the cube ABCDEFGH and take a complex made of its 12 (oriented) segments and θ a $SU(N)$-connection. With this choice, $Z(G) = e$ so that $c_{2,K}$ is uniquely determined. Then $c_{2,K}$ is non abelian on the 6 faces of the cube, and by choosing an order on the faces, i.e., by choosing complex A made of its 6 faces, $c_{2,K}$ extends to a cosurface $c_{3,A,K}$ which is trivial except on the cube viewed as an element of $\Sigma_{3,*}$. If one wants to get into another dimension (e.g. on the hypercube), the cosurfaces obtained are trivial except for the chosen manifolds along the cube.

3. Cosurfaces and Cobordisms

By a manifold M we always mean a smooth finite dimensional manifold, possibly with boundary; if the boundary, denoted by ∂M, is void, the manifold is said to be *closed.* If N is a an oriented manifold we denote by N^- the manifold N with the opposite orientation.

Let X_1 and X_2 be oriented closed submanifolds, both of dimension $d-1$, where d is a positive integer. By a *pre-cobordism* $(Y, \phi_1, \phi_2) : X_1 \to X_2$ we mean an oriented manifold Y along with an orientation preserving diffeomorphism,

$$\phi : X_1^- \sqcup X_2 \to \partial Y.$$

An isomorphism from a pre-cobordism $Y : X_1 \to X_2$ to a pre-cobordism $(Y', \phi_1', \phi_2') : X_1 \to X_2$ is an orientation-preserving diffeomorphism $f : Y \to Y'$ such that $f \circ \phi_1 = \phi_1'$ and $f \circ \phi_2 = \phi_2'$. A *cobordism* is a pre-cobordism up to isomorphisms.

Next, if $Y : X_1 \to X_2$ is a cobordism, and Y is equipped with a top-dimensional volume form (a measure of volume) we say that Y is a *volume* pre-cobordism. A *volume cobordism* is then an equivalence class of such pre-cobordisms, where the equivalence relation is obtained by using only orientation preserving and volume-preserving diffeomorphisms. A theorem of Morse guarantees that any two diffeomorphic compact oriented manifolds of equal volume are diffeomorphic by means of a volume-preserving diffeomorphism, and so to restrict the considerations to volume cobordism is not a huge restriction).

To keep notation under control, we will simply think of a cobordism from X_1 to X_2 as an oriented manifold Y, of dimension d, running "from" X_1 "to" X_2. Composition of cobordisms is defined in a natural way. The "identity" cobordism $X \to X$ is given by the oriented manifold $X \times [0, 1]$ along with a mapping

$$X^- \sqcup X \to X \times [0, 1]$$

that takes $p \in X^-$ to $(p, 0)$ and $p \in X$ to $(p, 1)$. Let $\mathbf{VCob}_d$ be the category whose objects are $d-1$ dimensional closed oriented manifolds and whose morphisms are volume cobordisms.

We may also work within a fixed oriented d-dimensional manifold M equipped with a volume form, and operate only with cobordisms which are (full dimensional) submanifolds of M. Let VCob_M be the set of all such cobordisms.

3.1. **Adapted saturated complexes.**

Consider Y and Y' two morphisms in the category $VCob_d$, seen as two d-dimensional manifolds equipped with their borders and volume form. Assume also that $\alpha(Y) = \beta(Y')$ so that $Y \circ Y'$ exists in $VCob_d$. Fix now a regular saturated complex K'' in $Y \circ Y'$ and for the $*$-construction so that the set $K'' \cap \alpha(Y)$ is made of complexes on each connected component of $\alpha(Y')$. What we want to construct is a composition rule for saturated complexes adapted to the composition of morphisms in $VCob_d$. Namely, we want to build two complexes $K \subset Y$ and $K' \subset Y'$ and a "composition rule" based on the composition $*$ for which $K * K' = K''$.

Definition 3.1. Let $Y \in Mor(VCob_d)$ and let K be a regular saturated complex of Y. Then K is *adapted* if for each $x \in \partial Y \cap K$ and for each$s \in K$ such that $x \in s$,

$$x \in \partial s$$

and

$$x \in \alpha(s) \Leftrightarrow x \in \alpha(Y).$$

Intuitively, the complex K is adapted if it satisfies a property of transversality on the border of Y, and if the initial and final parts of Y coincide with the corresponding initial and final parts of the surfaces of K reaching the border. Now, let

$$Y'' = Y \circ Y'.$$

We want to build a regular saturated complex of Y'' that splits into Y and Y'. For this, we need 3 parts:

$$\begin{cases} K_1 & = & \{\sigma \in K'' | \sigma \subset \alpha(Y) = \beta(Y')\}, \\ K_0 & = & \{\sigma \in K'' | \sigma \subset Y \text{ and } \sigma \notin K_1\}, \\ K_0' & = & \{\sigma \in K'' | \sigma \subset Y' \text{ and } \sigma \notin K_1\} \end{cases}$$

such that K_1 is a covering of $\alpha(Y)$, and K_0 and K_0' are adapted saturated complexes of Y and Y', respectively. Notice that under this condition, $\alpha(Y)$ is a $m-1$-manifold, and the borders ∂s, with $s \in K_1$, can define a complex on $\alpha(Y)$ by their smooth components up to re-ordering. This is what we specify first, and then we give a precise construction from cutting and pasting.

3.2. **Border reduction.** Let K b an adapted complex on $Y \in Mor(VCob_d)$. Let A be a covering of Y with respect to K and let us consider $A_k \in A$ such that $\partial A_k \cap \alpha(Y)$ has a non empty interior in ∂Y (one can replace here $\alpha(Y)$ by $\beta(Y)$). Let $\tilde{A}_k$ be the closure in ∂Y of the interior of a connected component of $\partial A_k \cap \partial Y$. This is a connected subset of ∂Y, not necessarily simply connected.

• Orientation $\tilde{A}_k$ is a $(d-1)$ manifold with boundary, with the orientation induced by the orientation of the border of Y.

• Initial and final parts. Now, let us consider $\partial\tilde{A}_k$. This is a $(d-2)$ piecewise smooth manifold, since it is a subset of $\bigcup_{s \in K} \partial s$. Let $s \in K \cap \partial A_k$ such that $s \cap \tilde{A}_k \subset \partial\tilde{A}_k$. Then the orientation on $s \cap \partial\tilde{A}_k$ is the one induced by the orientation of s, which defines whether it is an initial or a final part.

Notice that we have here no induced order from the adapted complex K to the border reduction. The *border reduction* is a non ordered regular complex on ∂Y, which is not necessarily saturated because it defines a partition of ∂Y into subsets which are non necessarily simply connected, with orientations induced by Y and K.

3.3. **Complexes for cobordism, cosurfaces and measures part I: cutting.** We now give a more restricted class of complexes.

Definition 3.2. We say that K'' is an *n-complex for cobordism* if $K'' = K''_a \cup K''_\alpha \cup K''_\beta$ if

- K''_α is a covering of $\alpha(Y'') = \alpha(Y')$,
- K''_β is a covering of $\beta(Y) = \beta(Y'')$,
- K''_a is a saturated complex of Y'',

We now need to say how we "cut" $Y'' \in Mor(VCob_d)$. Let $Y, Y' \in Mor(VCob_d)$ be such that $Y'' = Y \circ Y'$ exists. We say that we can cut (Y'', K'') if there exists $\theta \in \mathcal{P}(\mathbb{N}_n)$ such that

$$K''_a = \{\sigma_i \in K'' | i \in \theta\}$$

is an adapted complex in Y' that splits as $K''_a = K_a \cup K'_a \cup K''_b$, where K_a and K'_a are adapted complexes of Y and Y', respectively, and

$$K''_b = \{\sigma_i \in K'' | i \notin \theta\}$$

which defines a covering of $\alpha(Y)$.

The sets $K_a, K'_a, K''_\alpha, K''_\beta$ and K_b are equipped with the order induced by K'', and gathering the corresponding parts, we get two complexes for cobordism:

- $K = K_a \cup K_b \cup K''_\beta$ on Y,
- $K' = K'_a \cup K_\alpha \cup K_b$ on Y'.

Remark 3.1. K_b, as a subcomplex of K'', splits M.

Now let us turn to measures. For this, we now take a stochastic cosurface C''_N on Y'' adapted to the cobordism, that is one that can be divided into two stochastic cosurfaces C_N and C'_N on Y and Y' which coincide on

$$\alpha(Y) \cap \Sigma_* = \{\sigma \in \Sigma_* | \sigma \subset \alpha(Y)\}.$$

Since the order on K'' determines orders on subcomplexes, for each domain A''_i we define $\phi_{A_i} \circ C''$ that equals to $\phi_{A_i} \circ C$ or $\phi_{A_i} \circ C'$ (we recall that we have $A_i \subset Y$ or $A_i \subset Y'$ since K'' is a complex for cobordism) and each domain is connected and simply connected. If G is non abelian, we assume that the indexation of the family $(A_k)_k$ is such that the indexes of the domains in Y' are at the beginning of the list, and that the indexes of the domains in Y are at the end of the list. If G is abelian, this assumption is not necessary.

Theorem 3.1.

$$\mu_K . \mu_{K'} = \mu_{K''}.$$

Proof. Let us build two groups in the formula of Definition 1.10, namely, with the notations of Theorem 3.1,

$$\mu^Q_{K''}(c'') = \prod_{i=1}^{k} q_{|A_i|}\left(\phi_{A_i} \circ c''(K)\right).$$

This formula does not depend on the order among the indexes $\mathbb{N}_k$, so that we can define a twofold partition I, J of $\mathbb{N}_k$ defined as follows: I (resp. J) is the set of indexes i such that $A_i \subset Y$ (resp. $A_i \subset Y'$). Then

$$\begin{aligned} \mu^Q_{K''}(c'') &= \left(\prod_{i\in I} q_{|A_i|}\left(\phi_{A_i} \circ c''(K)\right)\right) . \left(\prod_{j\in J} q_{|A_j|}\left(\phi_{A_j} \circ c''(K)\right)\right) \\ &= \left(\prod_{i\in I} q_{|A_i|}\left(\phi_{A_i} \circ c(K)\right)\right) . \left(\prod_{j\in J} q_{|A_j|}\left(\phi_{A_j} \circ c'(K)\right)\right) \\ &= \mu^Q_K(c) . \mu^Q_{K'}(c') \end{aligned}$$

□

Remark 3.2. In order to get saturated complexes we had to add a complex on the border of the manifold Y. This assumption was not explicitly present in the papers [1, 2, 4, 5] where open manifolds were also considered. For volume cobordism, only compact manifolds with boundary are considered. A link with finite volume open manifolds can be done in a particular case of cosurfaces c such that, for any complex for cobordism K, we have the property $c|_{K_b} = e$.

3.4. **Complexes for cobordism, cosurfaces and measures part II: pasting.** Now let us consider the inverse problem, and let us only point out extra facts that give "anomalies" to the pasting procedure. Let us consider $Y, Y' \in VCob_d$, equipped with two complexes for cobordisms K and K' and two cosurfaces c and c'. Here are conditions to be able to build up a complex for cobordism K''on $Y'' = Y \circ Y'$:

(A) $K_b \cap \alpha(Y) = K'_b \cap \beta(Y')$ with corresponding orientations, initial and final parts on each hypersurface and on each border.

(B) $c|_{K_b \cap \alpha(Y)} = c'|_{K'_b \cap \beta(Y')}$.

With this, one can build up c'', but one cannot build up K'' in a unified way. This depends on a choice of reindexation, compatible with the orders of K_a, K_b, $K'b$ and $K'a$ that we have recovered by "extraction" from K''.

Proposition 3.1. *There exists such a cobordism K'' with the properties (A) and (B).*

Proof. Let us start with

$$K_0'' = K_b \cap \alpha(Y) = K_b' \cap \beta(Y') = (s_1, \ldots, s_l).$$

We build up by induction a complex K'' which satisfies (A) and (B). In a complex K_e, for $s \in K_e$, we denote by $bef_{K_e}(s)$ the subcomplex of elements of K_e before s in the list, and by $aft_{K_e}(s)$ the subcomplex of elements that are after s in the list.

- First step. Let

$$K_1'' = bef_K(s_1) \cup bef_{K'}(s_1) \cup$$

(this union is an ordered union, made first of the ordered set $bef_K(s_1)$, secondly of $bef_{K'}(s_1)$ and finally of K_0).

- Intermediate steps. Let $i \in \mathbb{N}_{l-1}$. Assume that we know K_i''. Set

$$\begin{aligned} K_{i+1}'' &= bef_{K_i''}(s_{i+1}) \cup (aft_K(s_i) \cap bef_K(s_{i+1})) \cup \\ & \quad (aft_{K'}(s_i) \cap bef_{K'}(s_{i+1})) \cup aft_{K_i''}(s_i). \end{aligned}$$

(with the ordered union)

- Final step. We have obtained K_l'' the last element of which is s_l. Then

$$K'' = K_l'' \cup aft_K(s_l) \cup aft_{K'}(s_l).$$

Then one can extract K and K' from K'' with the desired order. □

Since the corresponding coverings $A = (A_i)_{i \in I}$ of Y and $A' = (A_j')_{j \in J}$ of Y' are well defined and since all the quantities depend only on the indexation of the hypersurfaces on the borders of each domain, with an order already defined by K and K' and that *will not be changed* while passing to K'', the quantity

$$\left(\prod_{i \in I} q_{|A_i|} \left(\phi_{A_i} \circ c(K) \right) \right) \cdot \left(\prod_{j \in J} q_{|A_j|} \left(\phi_{A_j} \circ c'(K) \right) \right) = \mu_K^Q(c) \mu_{K'}^Q(c')$$

corresponds to the (classical) definition of $\mu_{K''}^Q(c'')$ for any possible choice of indexation for K''.

4. Series indexed by stochastic cosurfaces and cobordisms

We describe here a class of formal series where the indexes which remain in cobordisms, based on the setting of stochastic cosurfaces. We do not wish to consider homotopy invariant properties, and describe some kind of "pseudo-cobordism". We now consider the set

$$Gr = \coprod_{m \in \mathbb{N}^*} Gr_m,$$

where Gr_m is the set of m-dimensional connected oriented manifolds M, possibly with boundary, where the boundaries ∂M are separated into two disconnected parts: the initial part $\alpha(M)$ and the final part $\beta(M)$. Then, we have a composition law $*$, called cobordism composition in the rest of the text, defined by the following relation.

Definition 4.1. Let $m \in \mathbb{N}^*$. Let $M, M' \in Gr_m$. Then $M'' = M * M' \in Gr_m$ exists if

(1) $\alpha(M) = \beta(M') \neq \emptyset$, up to diffeomorphism,
(2) $\alpha(M'') = \alpha(M')$,
(3) $\beta(M'') = \beta(M)$,
(4) M'' cuts into two pieces $M'' = M \cup M'$ with $M \cap M' = \alpha(M) = \beta(M')$.

This composition, that we call *cobordism composition*, extends naturally to embedded manifolds.

Definition 4.2. Let N be a smooth (finite dimensional) manifold,

$$Gr(N) = \coprod_{m\in\mathbb{N}^*} \coprod_{M\in Gr_m} Emb(M,N),$$

where the notation $Emb(M,N)$ denotes the smooth manifold of smooth embeddings of M into N.

Notice that, since $dim(N) < \infty$, we have $m \leq dim(N)$. We recall that that $Gr(N)$ is naturally a smooth manifold, since $Emb(M,N)$ is a smooth manifold [9], and that $*$ is obviously smooth because it is smooth in the sense of the underlying diffeologies. When we only consider manifolds without boundary (in this case, cobordism composition is not defined), these spaces are called non linear grassmanians in the literature, which explains the notations.

Definition 4.3.

- Let $I = (Gr \times \mathbb{N}^*)\coprod(\emptyset, 0)$, graded by the second component. Assuming $\emptyset$ as a neutral element for $*$, we extend the cobordism composition into a composition, also denoted $*$, defined as:
$$(M,p)*(M',p') = (M*M', p+p')$$
when $M*M'$ is defined. We call *length* of (M,p) the number $len(M,p) = p$.
- Let $I(N) = (Gr(N)\times\mathbb{N}^*)\coprod(\emptyset,0)$, be graded by the second component. Assuming $\emptyset$ as a neutral element for $*$, we extend the cobordism composition into a composition, also denoted $*$, defined as:
$$(M,p)*(M',p') = (M*M', p+p')$$
whenever $M*M'$ is defined.
- Let $m\in\mathbb{N}^*$. We denote by I_m and $I_m(N)$ the set of indexes based on Gr_m and on $Gr_m(N)$, respectively

Let us now gather the framework of cobordism and stochastic cosurfaces into series. First, fix $m > 1$, the dimension of the theory of cobordism. The set of indexes is the one described in Theorem 3.1, namely, $\Gamma \subset \coprod_{p\in\mathbb{N}^*} I_p$, resp. $\Gamma(N) \subset \coprod_{p\in\mathbb{N}^*} I_p(N)$, be a family of indexes, stable under $*$, such that $\forall p \in \mathbb{N}^*$,

(1) $\forall p\in\mathbb{N}^*$, $\Gamma\cap I_p$ is finite or, more general;
(2) $\forall\gamma\in\Gamma$, the set of pairs $(\gamma',\gamma'')\in\Gamma^2$ such that $\gamma = \gamma' *$ 'γ'' is finite.

We fix a family

$$\mathcal{A}_\Gamma = \prod_{\gamma\in\Gamma}\mathcal{A}_\gamma$$

of Fréchet vector spaces $\mathcal{A}_\gamma$ which is equipped with a multiplication $*$ which is defined component-wise,

$$\mathcal{A}_\gamma \times \mathcal{A}_{\gamma'} \to \mathcal{A}_{\gamma*\gamma'}$$

smooth (in the Gâteaux sense), and such that $(\mathcal{A}_\Gamma, +, *)$ is an Fréchet algebra for the product Fréchet structure.

Let us now turn to our motivating example. Let Γ be a family of piecewise smooth manifolds, made along the infinite lattice $\mathbb{Z}^\infty$, of fixed dimension m. Let us also normalize the volume of a m-cube to 1. The family Γ is $\mathbb{N}$-graded by the volume, and assume that we have a stochastic cosurface on o $(m-1)$-cubes, either defined directly, or by dimension extension. Assume now that the complex K, supporting the family Γ, is a (maybe infinite) complex for cobordism. Then, following Theorem 3.1, we have the map

$$\mu_K : M\in\Gamma \mapsto \mu_K(M) \in \mathcal{A} \subset M(\Omega),$$

where $M(\Omega)$ is the space of measures on Ω, and $\mathcal{A}$ is a complete vector space of measures such that convolution is associative. Then, we are in the context of application of the main theorems of this paper.

Theorem 4.1. *For a fixed choice of stochastic cosurface, the mapping $\gamma \in \Gamma \mapsto \mu_K(\gamma)$ defines an element of $\mathcal{A}_\Gamma$*

The proof is obvious.

Acknowledgments

I would like to thank Professor Ambar Sengupta for stimulating discussions on the topics of cobordism, that influenced the corresponding section of this paper. These discussions mostly occurred during two stays (2012 and 2013) at the Hausdorff Center für Mathematik at Bonn, Germany, invited by Sergio Albeverio and Matthias Lesch who are warmly acknowledged.

References

[1] Albeverio, S.; Hoegh-Krohn, R.; *Brownian motion, Markov cosurfaces and Higgs fields*, Fundamental aspects of quantum field theory (Corno, 1985), NATO Adv. Sci. Inst. Ser. B. Phys. **144** (1986), 95–104

[2] Albeverio, S.; Hoegh-Krohn, R.; Holden, H.; *Markov cosurfaces and gauge fields*, Acta Phys. Austr., suppl. **26** (1984), 211–231

[3] Albeverio, S.; Hoegh-Krohn, R.; Holden, H.; Stochastic multiplicative measures, generalized Markov semi-groups and group-valued stochastic processes and fields; *J. Funct. Anal.* **78**, 154-184 (1988)

[4] Albeverio,S.; Hoegh-Krohn, R.; Holden, H.; *Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces*, in Stochastic processes-mathematical physics (Bielefeld, 1984) Lect. Notes in Math. **1158** (1986), 1–24

[5] Albeverio, S.; Hoegh-Krohn, R.; Holden, H.; Random fields with values in Lie groups and Higgs fields; in stochastic Processes in Classical and Quantum System, Proceedings. Ascona 1985. Lect. Notes in Physics **262** (1986), l–13

[6] Baez, J. C.; Dolan, J.; *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995) no. 11, 6073–6105

[7] Bergner, J.N.; *Models of (∞, n) categories and the cobordism hypothesis*, Proc. Symp. Pure Math. **53** (2012) 18-30

[8] Freed, D.S.; *The Cobordism hypothesis* Bull. Amer. Math. Soc. **50** (2011), no. 1 57–92

[9] Kriegl, A.; Michor, P.W.; *The convenient setting for global analysis* Math. surveys and monographs **53**, American Mathematical society, Providence, USA. (2000)

[10] Lévy, Thierry; *The Master field on the plane*; arXiv:1112.2452v2

[11] Lurie, J.; *On the classification of classical quantum field theories* arXiv:0905.0465.

[12] Magnot, J-P.; Ambrose-Singer theorem on diffeological bundles and complete integrability of the KP equation; *Int. J. Geom. Meth. Mod. Phys.* **10**, No. 9, Article ID 1350043, 31 p. (2013). (2013)

[13] Magnot, J-P.; *Remarks on a new discretization scheme for gauge theories*, Int. J. Theoret. Phys. (2018) **57** no . 7, pp.2093–2102

[14] Sengupta, A.N.; Yang-Mills in two dimensions and Chern-Simons in three, In **Chern-Simons Theory: 20 years after**, Editors: Jorgen Ellegaard Anderson, Hans U. Boden, Atle Hahn, and Benjamin Himpel. AMS/IP Studies in Advanced Mathematics (2011), 311–320

Jean-Pierre Magnot: jp.magnot@gmail.com
CNRS, LAREMA, SFR MATHSTIC, F-49000 Angers, France; and Lycé Jeanne d'Arc, Avenue de Grande Bretagne, F-63000 Clermont-Ferrand, France

Received 15/08/2022