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VANISHING CARLESON MEASURES AND POWER COMPACT WEIGHTED COMPOSITION OPERATORS

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Abstract. In this paper, we characterize Carleson measure and vanishing Carleson measure on Bergman spaces with admissible weights in terms of *t-Berezin transform* and *averaging function* as key tools. As an application of the main results of this paper, we characterize power bounded and power compact weighted composition operators on Bergman spaces with admissible weights.

Надано характеризацію міри Карлесона і міри Карлесона, що прямує до нуля, на просторах Бергмана з допустимими вагами в термінах *t-перетворення Березіна* та *функцією усереднення* в якості ключових інструментів. Як застосування основних результатів цієї роботи надано характеризацію степенево обмежених та степенево компактних зважених операторів композиції на просторах Бергмана з допустимими вагами.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Given a positive integrable function $\sigma \in C^2[0,1]$, we extend it on $\mathbb{D}$ by defining $\sigma(z) = \sigma(|z|), z \in \mathbb{D}$, and call such σ a weight function.

For $0 < p < \infty$ and a positive Borel measure Ω, the space $L^p(\Omega)$ consists of all measurable functions f on $\mathbb{D}$ for which

$$\|f\|^p_{L^p(\Omega)} = \int_{\mathbb{D}} |f(z)|^p d\Omega(z) < \infty.$$

In the case $p = \infty$, the space of all complex-valued measurable functions f on $\mathbb{D}$ is defined as

$$L^\infty(\Omega) = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_\infty = \text{ess sup}_{z\in\mathbb{D}}|f(z)| < \infty\},$$

where the essential supremum is taken with respect to the measure Ω. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is norm bounded in $L^\infty(\Omega)$ if $\sup_{n\in\mathbb{N}}\|f_n\|_\infty$ is finite. Let $dA(z) = \frac{dxdy}{\pi}$ be the normalized Lebesgue area measure on $\mathbb{D}$, we define the weighted Bergman space as

$$A^p_\sigma = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{A^p_\sigma} = \int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) < \infty\}.$$

Note that A^2_σ is a closed subspace of $L^2(\sigma dA)$ and hence is a Hilbert space endowed with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\sigma(z)dA(z) \quad f, g \in A^2_\sigma.$$

Throughout this paper, we will consider σ as admissible weight function. Recall that if a weight function σ is non-increasing on $[0,1)$ and $\sigma(r)(1-r)^{-(1+\delta)}$ is non decreasing on $[0,1)$ for some $\delta > 0$, then σ is called admissible weight.

We refer the readers [12] for useful fact over pseudohyperbolic metric. The pseudohyperbolic metric is defined as $\rho(a,z) = |\phi_a(z)|$, where $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ is Möbius transformation. For r in $(0,1)$ and a in $\mathbb{D}$, $E(a,r) = \{z \in \mathbb{D} : \rho(z,a) < r\} = \phi_a(E(0,r)) = \phi_a(\{z : |z| < r\})$

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denote the pseudohyperbolic disk center at a and radius $r \in (0,1)$. It turns out $E(a,r)$ is a Euclidean disk with center $\dfrac{1-r^2}{1-|a|^2r^2}a$ and radius $\dfrac{(1-|a|^2)r}{1-|a|^2r^2}$.

For every $z \in E(a,r)$, $(1-|a|^2)^2 \asymp (1-|z|^2)^2 \asymp |1-\bar{a}z|^2$ and area of $E(a,r)$ is denoted by $|E(a,r)|$ and $|E(a,r)| \asymp (1-|a|^2)^2$ are well known facts. Here the symbol " $\asymp$ " denotes that the left hand side is bounded above and below by constant multiples of the right hand side, where the constants are positive and independent of variables. Given r in $(0,1)$, a sequence $\{z_k\}_{k=1}^{\infty} \subset \mathbb{D}$ is said to be an *r-lattice* if the disk $\{E(z_k,r)\}_{k=1}^{\infty}$ cover $\mathbb{D}$ and there is some integer $M>0$ such that each z in $\mathbb{D}$ belongs to at most M of the disks $\{E(z_k,\frac{1+r}{2})\}_{k=1}^{\infty}$. Equivalently,

$$1 \leq \sum_{k=1}^{\infty} \chi_{E(z_k,\frac{1+r}{2})}(z) \leq M. \tag{1.1}$$

Recall that,

$$K^{\alpha}(z,w) = \frac{1}{(1-\bar{w}z)^{\alpha+2}}, \quad z \in \mathbb{D}$$

is the reproducing kernel in *standard weighted Bergman space* $A^p_{\sigma} = A^p_{\alpha}$, where standard weight $\sigma(z) = (1-|z|^2)^{\alpha+2}$, $\alpha > -1$. The normalized reproducing kernel of A^p_{α} is defined as

$$k^{\alpha}_z(.) = \frac{K^{\alpha}(z,.)}{\sqrt{K^{\alpha}(z,z)}}$$

where $K^{\alpha}(z,z) = \dfrac{1}{(1-|z|^2)^{\alpha+2}}$.

For a finite positive Borel measure Ω on $\mathbb{D}$, the *t-Berezin transform* is defined to be

$$\widetilde{\Omega}_t(z) = \int_{\mathbb{D}} (|k^{\alpha}_z(w)|)^t \, d\Omega(w), \quad z \in \mathbb{D}. \tag{1.2}$$

Note that for $t=2$, the classical Berezin transform is denoted by $\widetilde{\Omega}_2$. Given r in $(0,1)$, the averaging function of Ω is defined to be

$$\widehat{\Omega}_r(z) = \frac{\Omega(E(z,r))}{|E(z,r)|}, \quad z \in \mathbb{D}. \tag{1.3}$$

If we set $d\Omega = f dA$, for a Lebesgue measurable function f, then we can write $\widetilde{f}_t = \widetilde{\Omega}_t$ and $\widehat{f}_r = \widehat{\Omega}_r$ for simplicity.

Motivated by [21] and [9], in this article we characterize Carleson measure and vanishing Carleson measure on Bergman spaces with admissible weights in terms of *t-Berezin transform* and *averaging function* as key tools. An operator T on a normed linear space $(X, \|.\|_X)$ is called *power bounded* if $\{T^n\}$ is a bounded sequence in the space of all bounded operators from X to itself. Also, recall that an operator T on Banach space $(X, \|.\|_X)$ is said to be *power compact* if there exist some integer $m>0$ such that T^m is compact from X to itself, see [2]. Denote by $\Lambda^2(\mathbb{C})$, the linear space of all double sequences with complex entries. A double sequence $\{\gamma_{j,k}\}_{j,k\in\mathbb{N}}$ of complex numbers is bounded if there exists some $M>0$ such that $\sup_{j,k}|\gamma_{j,k}| \leq M$. The space Λ^2_{∞} of all bounded double sequences is defined as

$$\Lambda^2_{\infty} = \{\gamma_{jk} = \{\gamma_{j,k}\}_{j,k\in\mathbb{N}} \in \Lambda^2(\mathbb{C}) : \|\gamma_{jk}\|_{\Lambda^2_{\infty}} = \sup_{j,k}|\gamma_{j,k}| < \infty\}.$$

Let $C_{\psi,\phi}$ denoted the well known *weighted composition operator* on the space $\mathcal{H}(\mathbb{D})$ is defined as

$$C_{\psi,\phi}(f) = \psi(f \circ \phi)$$

where $\psi \in \mathcal{H}(\mathbb{D})$ and ϕ is an analytic self map of $\mathbb{D}$. If $\phi(z) = z$ and $\psi = 1$, then $C_{\psi,\phi}$ becomes the multiplication operator M_ψ and the composition operator C_ϕ respectively. Denote by ϕ_n the nth iteration of ϕ, that is,

$$\phi_n = \underbrace{\phi \circ \phi \cdots \phi}_{n-\text{times}}.$$

Note that any power of $C_{\psi,\phi}$ on $\mathcal{H}(\mathbb{D})$ is a weighted composition operators which is defined as

$$C^n_{\psi,\phi} f = \prod_{j=0}^{n-1} (\psi \circ \phi_j) f \circ \phi_n.$$

For the sake of simplicity, we set

$$\langle \psi, \phi, n \rangle = \prod_{j=0}^{n-1} \psi \circ \phi_j.$$

Thus,

$$\|C^n_{\psi,\phi} f\|^p_{L^p(\Omega)} = \int_{\mathbb{D}} |\langle \psi, \phi, n \rangle(z) f \circ \phi_n(z)|^p d\Omega(z).$$

We define $d\Omega_n = |\langle \psi, \phi, n \rangle| d\Omega \circ \phi_n^{-1}$. One can easily see that Ω_n is a measure and therefore

$$\|C^n_{\psi,\phi} f\|^p_{L^p(\Omega)} = \int_{\mathbb{D}} |f(z)|^p d\Omega_n$$

.For $t > 0$, the t-Berezin transform and for $0 < r < 1$, the averaging function of Ω_n are defined as

$$\widetilde{\Omega}_{n,t}(z) = \int_{\mathbb{D}} \left(|k^\alpha_z(w)|\right)^t d\Omega_n(z), \quad z \in \mathbb{D}$$

and

$$\widehat{\Omega}_{n,r}(z) = \frac{\Omega_n(E(z,r))}{|E(z,r)|}, \quad z \in \mathbb{D},$$

respectively. For more about weighted composition operators, Carleson measures and vanishing Carleson measures, we refer to [12]–[16]. Throughout the paper, the expression $E \lesssim F$ means that there exists a constant C such that $E \le CF$.

2. Preliminaries

In this section, we prove and collect some useful facts and lemmas that are required for the proof of our main results. The next lemma gives a growth estimate for functions in A^p_σ and an asymptotic estimate for norm of $K^\alpha(z,\cdot)$ already proved in [1].

Lemma 1. *Let $1 \le p < \infty$, $\alpha > -1$, $r \in (0,1)$, $t > 0$, σ be an admissible weight, and Ω be a finite positive Borel measure. Then*

(1) *for each $z \in \mathbb{D}$, we have*

$$|f(z)|^p \lesssim \frac{\|f\|^p_{A^p_\sigma}}{\sigma(z)(1-|z|^2)^2} \quad \text{for all } f \in A^p_\sigma. \tag{2.4}$$

(2) *for each $z \in \mathbb{D}$, we have*

$$\|K^\alpha(z,\cdot)\|_{A^p_\sigma} \asymp \frac{1}{(\sigma(z))^{\frac{1}{p}}(1-|z|^2)^{(\alpha+2)-\frac{2}{p}}} \tag{2.5}$$

(3) *the operator $f \longmapsto \widehat{f}_r$ is bounded from $L^p(\mathbb{D})$ to $L^p(\mathbb{D})$.*

(4) *there exists some constant C such that*

$$\int_{\mathbb{D}} h(z)d\Omega(z) \leq C \int_{\mathbb{D}} h(z)\widehat{\Omega}_r(z)dA(z)$$

for all non-negative subharmonic functions $h : \mathbb{D} \longrightarrow [0, \infty)$.

(5) *there exists a constant C such that*

$$\Omega(E(a, R)) \leq \frac{C}{|E(a, R)|} \int_{E(a,R)} \Omega(E(z, r))dA(z). \tag{2.6}$$

(6) *the integral operator*

$$T_{t,s}f(z) = \sigma^s(z)\big(1 - |z|^2\big)^{(\alpha+2)t-2s} \int_{\mathbb{D}} \frac{|K^\alpha(z, \xi)|^t}{\sigma^s(\xi)\,(1 - |\xi|^2)^{2(s-1)}} f(\xi)dA(\xi)$$

is bounded on $L^p(\mathbb{D})$, whenever $\frac{1-p}{p} < s < (\alpha + 2)\frac{t}{2} + \frac{1}{p}$.

Proof. The proof of the lemma is arranged as follows.

(1) and **(2)** are proved in [1].

(3) Since $\chi_{E(z,r)}(\xi) = \chi_{E(\xi,r)}(z),\ \ z, \xi \in \mathbb{D}$. By Fubini's theorem for all $f \in L^1(\mathbb{D})$, we have

$$\begin{aligned}
\|\widehat{f}_r\|_1 &\leq \int_{\mathbb{D}} \left| \frac{1}{|E(z, r)|} \int_{E(z,r)} |f(\xi)|dA(\xi) \right| dA(z) \\
&\leq \int_{\mathbb{D}} \frac{dA(z)}{|E(z, r)|} \int_{E(z,r)} |f(\xi)|dA(\xi) \\
&= \int_{\mathbb{D}} |f(\xi)|dA(\xi) \int_{E(\xi,r)} \frac{dA(z)}{|E(z, r)|} \\
&\leq C\|f\|_1
\end{aligned}$$

The boundedness of the operator $f \longmapsto \widehat{f}_r$ is trivially holds for $p = \infty$, that is, $\|\widehat{f}_r\|_\infty \leq \|f\|_\infty$ and also holds for $1 < p < \infty$, by complex interpolation.

(4) holds, since h is a non negative subharmonic function $h : \mathbb{D} \longrightarrow [0, \infty)$. Then, we have

$$h(z) \leq \frac{C}{|E(z, r)|} \int_{E(z,r)} h(\xi)dA(\xi)$$

for all $z \in \mathbb{D}$, see [12, page 125]. Using the above inequality, Fubini's theorem and (1.3), we have that

$$\begin{aligned}
\int_{\mathbb{D}} h(z)d\Omega(z) &\leq C \int_{\mathbb{D}} \frac{d\Omega(z)}{|E(z, r)|} \int_{E(z,r)} h(\xi)dA(\xi) \\
&= C \int_{\mathbb{D}} h(\xi)dA(\xi) \int_{E(\xi,r)} \frac{d\Omega(z)}{|E(z, r)|} \\
&\leq C \int_{\mathbb{D}} h(z)\widehat{\Omega}_r(z)dA(z)
\end{aligned}$$

This accomplished the result.

(5) For $r, R > 0$, we have

$$\begin{aligned}
\int_{E(a,R)} \Omega(E(z, r))dA(z) &= \int_{\mathbb{D}} \chi_{E(a,R)}(z)dA(z) \int_{\mathbb{D}} \chi_{E(z,r)}(w)d\Omega(w) \\
&= \int_{\mathbb{D}} d\Omega(w) \int_{E(a,R)} \chi_{(E(z,r)}(w)dA(w)
\end{aligned}$$

Since $\chi_{E(z,r)}(w) = \chi_{E(w,r)}(z)$ for all $z, w \in \mathbb{D}$,

$$\begin{aligned}\int_{E(a,R)} \Omega(E(z,r))dA(z) &= \int_{\mathbb{D}} d\Omega(w) \int_{E(a,R)} \chi_{E(w,r)}(z)dA(z) \\ &\geq \int_{E(a,R)} |(E(a,R) \cap E(w,r))| d\Omega(w) \\ &\geq \Omega(E(a,R)) \inf_{w \in E(a,R)} \{|(E(a,R) \cap E(w,r))|\}\end{aligned}$$

For $w \in (E(a,R)$, then there exists a Euclidean disk with diameter $\frac{1}{2}\min\{r,R\}$ contained in $(E(a,R) \cap E(w,r))$. Therefore (2.6) holds.

(6) Let $P = \frac{1-p}{p}(s+1)$, $Q = \frac{p-1}{p}\{(\alpha+2)t - 2 - s\}$, $U = \frac{s+1-(\alpha+2)t}{p}$, and $V = \frac{s}{p}$. Then the intervals (P,Q) and (U,V) are non-empty. By using the hypothesis one can easily find that

$$Q - P = (\alpha+2)(1-\frac{1}{p})(t - \frac{1}{\alpha+2}) > 0$$

and

$$V - U = \frac{(\alpha+2)}{p}(t - \frac{1}{\alpha+2}) > 0$$

implies that the intervals (P,Q) and (U,V) are non-empty. Also,

$$V - P = s - \frac{1-p}{p} > 0$$

and

$$Q - U = (\alpha+2)t - 2 + \frac{1}{p} - s > 0$$

implies that $P < V$ and $U < Q$. Thus, $(P,Q) \cap (U,V)$ is non-empty. For some $m \in (P,Q) \cap (U,V)$ and take $h(\xi) = \big(\sigma(\xi)(1-|\xi|^2)\big)^m$ and $\frac{1}{p} + \frac{1}{p'} = 1$. From part (1) of Lemma 1, there is a positive constant C such that

$$\sigma^s(z)\big(1-|z|^2\big)^{(\alpha+2)t-2s} \int_{\mathbb{D}} \frac{|K^\alpha(z,\xi)|^t}{\sigma^s(\xi)(1-|\xi|^2)^{2(s-1)}} h^{p'}(\xi)dA(\xi) \leq Ch^{p'}(\xi), \quad z \in \mathbb{D},$$

and

$$\sigma^{-s}(z)\left(1-|z|^2\right)^{2(1-s)} \int_{\mathbb{D}} |K^\alpha(z,\xi)|^t \sigma^s(\xi)\big(1-|\xi|^2\big)^{(\alpha+2)t-2s} h^p(\xi)dA(\xi) \leq Ch^p(\xi),$$

$z \in \mathbb{D}$. By Schur's test, boundedness of the operator $T_{t,s}$ on $L^p(\mathbb{D})$ holds. □

Lemma 2. *Let $\{z_k\}_{k=1}^\infty$ be an r- lattice. For $1 < p < \infty$ and $\{\lambda_k\}_{k=1}^\infty \in l^p$, let*

$$f(z) = \sum_{k=1}^\infty \lambda_k \frac{K^\alpha(z_k, z)}{\sigma^{\frac{1}{p}}(z_k)(1-|z_k|^2)^{\frac{2}{p}-(\alpha+2)p}}, \tag{2.7}$$

where $\alpha > -1$. Then $f \in A^p_\sigma(\mathbb{D})$ and $\|f\|_{p,\sigma} \leq C\|\{\lambda_k\}_{k=1}^\infty\|_{l^p}$.

The proof is an easy modification of arguments in [1, Theorem 4.1]. We omit the details.

Lemma 3. *Suppose $\Omega \geq 0$, $1 \leq p \leq \infty$, $t > 0$ and $s \in \mathbb{R}$ satisfies $t < s + \frac{1}{p} < 1$. Then the following are equivalent:*

(a) $\widetilde{M}_{t,s}(z) = \dfrac{\widetilde{\Omega}_t(z)}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha+2}{2})t}} \in L^p(\mathbb{D})$.

(b) $\widehat{M}_{R,s}(z) = \dfrac{\widehat{\Omega}_R(z)}{\sigma^s(z)(1-|z|^2)^{2(s-1)}} \in L^p(\mathbb{D})$ *for some R, $0 < R < 1$.*

(c) *The sequence* $\left\{\frac{\widehat{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2(s-1-\frac{1}{p})}}\right\}_{k=1}^{\infty}$ *belongs to* l^p *for any* r*-lattice* $\{z_k\}_{k=1}^{\infty}$ *with* $0<r<1$.

Moreover, we have

$$\|\widetilde{M}_{t,s}\|_p \asymp \|\widehat{M}_{R,s}\|_p \asymp \left\|\left\{\frac{\widehat{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2(s-1-\frac{1}{p})}}\right\}_{k=1}^{\infty}\right\|_{l^p}.$$

Proof. We will prove the result in the order: $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (c)$.

$(a) \Rightarrow (b)$. For any $R \in (0,1)$, there exists a positive constant C_R such that for any $z \in \mathbb{D}$ the kernel estimate holds. Thus for $s \in \mathbb{R}$, we have

$$\begin{aligned}
\widehat{M}_{R,s}(z) &= \frac{\Omega(E(z,R))}{\sigma^s(z)(1-|z|^2)^{2s}} \\
&\leq C_R \frac{1}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha+2}{2})t}} \int_{E(z,R)} |k_z^\alpha(\xi)|^t d\Omega(\xi) \\
&\leq C_R \frac{1}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha+2}{2})t}} \int_{\mathbb{D}} |k_z^\alpha(\xi)|^t d\Omega(\xi) \\
&\leq C_R \widetilde{M}_{t,s}(z)
\end{aligned}$$

Above implies that $\|\widehat{M}_{R,s}\|_p \leq C\|\widetilde{M}_{t,s}\|_p$.

$(b) \Rightarrow (a)$. By part (3) and part (4) of Lemma 1, there is a positive constant C such that for any $z \in \mathbb{D}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned}
\widetilde{M}_{t,s}(z) &= \frac{\widetilde{\Omega}_t(z)}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha+2}{2})t}} \\
&= \frac{1}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha+2}{2})t}} \int_{\mathbb{D}} |k_z^\alpha(\xi)|^t d\Omega(\xi) \\
&\leq \frac{C}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha+2}{2})t}} \int_{\mathbb{D}} |k_z^\alpha(\xi)|^t \widehat{\Omega}_R(\xi) dA(\xi) \\
&\leq \frac{C}{\sigma^s(z)(1-|z|^2)^{2s-(\alpha+2)t}} \int_{\mathbb{D}} |K^\alpha(z,\xi)|^t \sigma^s(\xi)(1-|\xi|^2)^{2(s-1)} \widehat{M}_{R,s}(\xi) dA(\xi) \\
&\leq C T_{t,-s}(\widehat{M}_{R,s})(z).
\end{aligned}$$

Since $t < s + \frac{1}{p} < 1$, part (6) of Lemma 1 implies that

$$\|\widetilde{M}_{t,s}\|_p \leq \|T_{t,-s}(\widehat{M}_{R,s})\|_p \leq C\|\widehat{M}_{R,s}\|_p.$$

$(b) \Rightarrow (c)$. Assume that $\widehat{M}_{R,s} \in L^p(\mathbb{D})$ for some R, $0<R<1$. Let $\{z_k\}_{k=1}^{\infty}$ be any r-lattice. By part 5 of Lemma 1, we may assume $R < r$. By triangle inequality, we have $E(z_k, r) \subset E(z, 2r)$ for $z \in E(z_k, r)$ and for all k. Thus, we have

$$\begin{aligned}
\frac{\widetilde{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2(s-1)}} &\leq C \frac{\Omega(E(z,2r))}{\sigma^s(z)(1-|z|^2)^{2s}} \\
&\asymp \frac{\widehat{\Omega}_{2r}(z)}{\sigma^s(z)(1-|z|^2)^{2(s-1)}} \qquad (2.8)
\end{aligned}$$

whenever $z \in E(z_k, r)$. Therefore, we have

$$\sum_{k=1}^{\infty} \frac{\widehat{\Omega}_r^p(z_k)}{\sigma^{sp}(z_k)(1-|z_k|^2)^{2(s-1)p-2}} \leq C \sum_{k=1}^{\infty} \int_{E(z_k,r)} \frac{\widehat{\Omega}_{2r}^p(z)}{\sigma^{sp}(z)(1-|z|^2)^{2(s-1)p}} dA(z)$$
$$\leq CM \int_{\mathbb{D}} \frac{\widehat{\Omega}_{2r}^p(z)}{\sigma^{sp}(z)(1-|z|^2)^{2(s-1)p}} dA(z).$$

Thus by part (5) of Lemma 1, we have

$$\left\| \left\{ \frac{\widehat{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2(s-1-\frac{1}{p})}} \right\}_k \right\|_{l^p} \leq C \|\widehat{M}_{2r,s}\|_p \asymp \|\widehat{M}_{R,s}\|_p.$$

$(c) \Rightarrow (b)$. Finally, suppose $\left\{ \frac{\widehat{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2(s-1-\frac{1}{p})}} \right\}_k \in l^p$ for some r-lattice $\{z_k\}_{k=1}^{\infty}$. Similar to (2.8), $\widehat{\Omega}_r(z) \leq C\widehat{\Omega}_{2r}(z_k)$ for $z \in E(z_k, r)$. Therefore, we have

$$\int_{\mathbb{D}} \left(\frac{\widehat{\Omega}_r(\xi)}{\sigma^s(\xi)(1-|\xi|^2)^{2(s-1)}} \right)^p dA(\xi) \leq \sum_{k=1}^{\infty} \int_{E(z_k,r)} \frac{\widehat{\Omega}_r^p(\xi)}{\sigma^{sp}(\xi)(1-|\xi|^2)^{2(s-1)p}} dA(\xi)$$
$$\leq C \sum_{k=1}^{\infty} \frac{\widehat{\Omega}_{2r}^p(z_k)}{\sigma^{sp}(z_k)(1-|z_k|^2)^{2(s-1)p-2}}$$
$$\leq C \sum_{k=1}^{\infty} \frac{\widehat{\Omega}_r^p(z_k)}{\sigma^{sp}(z_k)(1-|z_k|^2)^{2(s-1)p-2}}.$$

Above inequality and part (5) of Lemma 1 implies that

$$\|\widehat{M}_{R,s}\|_p \asymp \|\widehat{M}_{r,s}\|_p \leq C \left\| \left\{ \frac{\widehat{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2(s-1-\frac{1}{p})}} \right\}_k \right\|_{l^p}$$

for any $R \in (0,1)$. □

3. Carleson measure characterizations

In this section, we are using *averaging function* and *t-Berezin transform* as our main tools to characterize the $(p,q,\sigma)-$Bergman Carleson measure for $0 < p, q < \infty$ and $t > 0$. Let Ω be a finite positive Borel measure. Recall that

(i) Ω is a $(p,q,\sigma)-$*Bergman Carleson measure* if the embedding $i : A^p_\sigma \longrightarrow L^q(\Omega)$ is bounded. In other words, we can say Ω is a (p,q,σ)-Bergman-Carleson measure if there exists a finite constant $C > 0$ such that

$$\int_{\mathbb{D}} |f|^q d\Omega \leq C \|f\|^q_{A^p_\sigma}$$

for all $f \in A^p_\sigma$.

(ii) Ω is a *vanishing* (p,q,σ)*-Bergman-Carleson measure* if $\int_{\mathbb{D}} |f_n|^q d\Omega \longrightarrow 0$ as $n \longrightarrow \infty$ whenever $\{f_n\}$ is a bounded sequence in A^p_σ which converges to 0 uniformly on any compact subset of $\mathbb{D}$.

Note that, by taking $p = q$ and $\sigma(z) = 1$, Ω becomes a Bergman-Carleson measure and a vanishing Carleson measure. We divide our result into two cases: $0 < p \leq q < \infty$ and $0 < q < p < \infty$.

Theorem 3.1. *Let Ω be a finite positive Borel measure and $0 < p \leq q < \infty$. Then the following statements are equivalent:*

(a) *Ω is a (p,q,σ)-Bergman-Carleson measure.*

(b) *The function* $\dfrac{\widetilde{\Omega}_t(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})t}}$ *is bounded on* $\mathbb{D}$ *for* $t > \frac{2q}{p(\alpha+2)}$.

(c) *The function* $\dfrac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}}$ *is bounded on* $\mathbb{D}$ *for any* $R \in (0,1)$.

(d) *The sequence* $\left\{\dfrac{\widehat{\Omega}_r(a_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\right\}_{k=1}^{\infty}$ *is bounded for any* r*-lattice* $\{z_k\}_{k=1}^{\infty}$ *with* $0<r<1$.

Furthermore,

$$\|i\|_{A^p_\sigma \to L^p(\Omega)} \asymp \sup_{z\in\mathbb{D}} \frac{\widetilde{\Omega}_t(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})t}} \asymp \sup_{z\in\mathbb{D}} \frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \asymp \sup_{k} \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}} \tag{3.9}$$

Proof. $(d) \Rightarrow (a)$ We assume that $\{z_k\}_{k=1}^{\infty}$ is an r-lattice. We use the elementary inequality $\sum_{k=1}^{\infty} u_k^l \le (\sum_{k=1}^{\infty} u_k)^l$, $u_k \ge 0$, $k=1,2,\cdots$, by taking $l=\frac{p}{q}\ge 1$, using parts (1) and (2) of Lemma 1 and (1.1), we obtain

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q d\Omega &= \sum_{k=1}^{\infty}\int_{E(z_k,r)} |f(z)|^q d\Omega \\ &\le \sum_{k=1}^{\infty} \widehat{\Omega}_r(z_k)|E(z_k,r)|\left(\sup_{z\in E(z_k,r)} |f(z)|^p\right)^{\frac{q}{p}} \\ &\le C\sum_{k=1}^{\infty} \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\left(\int_{E(z_k,\frac{1+r}{2})} |f(\xi)|^p\sigma(\xi)dA(\xi)\right)^{\frac{q}{p}} \\ &\le C\sup_k \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\left(\sum_{k=1}^{\infty}\int_{E(z_k,\frac{1+r}{2})} |f(\xi)|^p\sigma(\xi)dA(\xi)\right)^{\frac{q}{p}} \\ &\le CM^{\frac{q}{p}}\sup_k \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\|f\|^q_{p,\sigma}. \end{aligned}\tag{3.10}$$

The above inequality reveals that

$$\|i\|_{A^p_\sigma \to L^q(\Omega)} \le C\sup_k \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}.$$

$(a) \Rightarrow (c)$. Set $f_z(w) = K^\alpha(z,w)$, $w \in \mathbb{D}$. By part (1), (2) of Lemma 1 and statement(a), we have

$$\begin{aligned}\frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} &= \frac{\Omega(E(z,R))}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}}} \\ &\le C\frac{(1-|z|^2)^{q(\alpha+2-\frac{2}{p})}}{\sigma^{\frac{q}{p}}(z)}\int_{E(z,R)}|f_z(\xi)|^q d\Omega(\xi) \\ &\le C\frac{(1-|z|^2)^{q(\alpha+2-\frac{2}{p})}}{\sigma^{\frac{q}{p}}(z)}\int_{\mathbb{D}}|f_z(\xi)|^q d\Omega(\xi) \qquad (3.11) \\ &\le C\frac{(1-|z|^2)^{q(\alpha+2-\frac{2}{p})}}{\sigma^{\frac{q}{p}}(z)}\|i\|^q_{A^p_\sigma \to L^q(\Omega)}\|f_z\|^q_{A^p_\sigma} \\ &\le C\|i\|^q_{A^p_\sigma\to L^q(\Omega)}.\end{aligned}$$

The above inequality reveals that

$$\sup_{z\in\mathbb{D}}\frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \le \|i\|_{A^p_\sigma\to L^q(\Omega)}. \qquad (3.12)$$

The equivalence of (a), (c), and (d) follows from the above proof of implications. Moreover,

$$\begin{aligned}\|i\|_{A^p_\sigma\to L^p(\Omega)} &\asymp \sup_{z\in\mathbb{D}}\frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \\ &\asymp \sup_k \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}. \qquad (3.13)\end{aligned}$$

$(b) \Rightarrow (c)$. For any R, $0<R<1$, Lemma 3 yields

$$\frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \le C\frac{\widetilde{\Omega}_t(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})t}}. \qquad (3.14)$$

$(a) \Rightarrow (b)$. The estimate (3.13) reveals that the embedding operator $i : A^p_\sigma \longrightarrow L^q(\Omega)$ is bounded for some $0<p\le q<\infty$ if and only if $i : A^{p_1}_\sigma \longrightarrow L^{q_1}(\Omega)$ is bounded for some $0<p_1\le q_1<\infty$ with $\frac{q_1}{p_1}=\frac{q}{p}$. Since Ω is a (p,q,σ)-Carleson measure, $i : A^{Np}_\sigma \longrightarrow L^{Nq}(\Omega)$ where N is some integer with $Np > \frac{4}{(\alpha+2)}$. Let $f_z(.) = K^\alpha_z(.)$, $z\in\mathbb{D}$, and (3.13) tells us that

$$\begin{aligned}\frac{\widetilde{\Omega}_{Nq}(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}} &\asymp \frac{1}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}}\int_{\mathbb{D}}|f_z(\xi)|^{Nq}d\Omega(\xi) \qquad (3.15) \\ &\lesssim \frac{\|i\|^{Nq}_{A^{Np}_\sigma\to L^{Nq}(\Omega)}\|f_z\|^{Nq}_{Np,\sigma}}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}} \\ &\le C\sup_k\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}.\end{aligned}$$

Hence, for any $z\in\mathbb{D}$

$$\begin{aligned}\frac{\widetilde{\Omega}_{Nq}(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}} &\le C\|i\|^{Nq}_{A^{Np}_\sigma\to L^{Nq}(\Omega)} \\ &\asymp C\sup_k\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}.\end{aligned}$$

Statement (a) shows that the operator $i : A_\sigma^{\frac{tp}{Nq}} \to L^{\frac{t}{N}}(\Omega)$ is bounded. Since $t > \frac{2q}{p(\alpha+2)}$, we have $\frac{tp(\alpha+2)}{q} > 2$. The above calculations show that

$$\sup_{z\in\mathbb{D}} \frac{\widetilde{\Omega}_t(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})t}} \leq C\|i\|^{Nq}_{A_\sigma^{Np}\to L^{Nq}(\Omega)}$$
$$\asymp C\sup_k \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}.$$

This completes the proof. □

Corollary 1. *Let $t > 0$, $0 < p < \infty$, $d\Omega = \sigma dA$ be a positive measure, and $C_{\psi,\varphi} : A^p_\sigma \to L^p(\Omega)$ be a bounded operator. Then the following statements are equivalent.*

(a) *$C_{\psi,\varphi}$ is power bounded, that is,*

$$Q_1 = \sup_{n\in\mathbb{N}} \|C^n_{\psi,\varphi}\|^p < \infty.$$

(b) *The sequence of functions $\{f_n\}_{n=1}^\infty$ defined on $\mathbb{D}$ as*

$$f_n(z) = \frac{\widetilde{\Omega}_{n,t}(z)}{\sigma(z)\delta^{2-(\alpha+2)t/2}(z)},\ z\in\mathbb{D},$$

is a norm bounded family in $L^\infty(\mathbb{D})$ for $t > 2/(\alpha+2)$, that is,

$$Q_2 = \sup_{n\in\mathbb{N}} \|f_n\|_\infty = \sup_{n\in\mathbb{N}} \operatorname{ess\,sup}_{z\in\mathbb{D}} \frac{\widetilde{\Omega}_{n,t}(z)}{\sigma(z)\delta^{2-(\alpha+2)t/2}(z)} < \infty.$$

(c) *The sequence of functions $\{g_n\}_{n=1}^\infty$ defined on $\mathbb{D}$ as*

$$g_n(z) = \frac{\widehat{\Omega}_{n,R}(z)}{\sigma(z)},\ z\in\mathbb{D},$$

is a norm bounded family in $L^\infty(\mathbb{D})$ for any $R\in(0,1)$, that is,

$$Q_3 = \sup_{n\in\mathbb{N}} \|g_n\|_\infty = \sup_{n\in\mathbb{N}} \operatorname{ess\,sup}_{z\in\mathbb{D}} \frac{\widehat{\Omega}_{n,R}(z)}{\sigma(z)} < \infty.$$

(d) *The double sequence $\gamma_{nk} = \{\gamma_{n,k}\}_{n,k}$, where*

$$\gamma_{n,k} = \frac{\widehat{\Omega}_{n,r}(z_k)}{\sigma(z_k)}$$

is bounded for any r-lattice $\{z_k\}_{k=1}^\infty$ with fixed r, $0 < r < 1$, that is,

$$Q_4 = \|\gamma_{nk}\|_{\Lambda^2_\infty} = \sup_{n,k\in\mathbb{N}} \frac{\widehat{\Omega}_{n,r}(z_k)}{\sigma(z_k)} < \infty.$$

Moreover, $Q_1 \asymp Q_2 \asymp Q_3 \asymp Q_4$.

Theorem 3.2. *Let $\Omega \geq 0$ and $0 < p \leq q < \infty$. Then the following statements are equivalent:*

(a) *Ω is a vanishing (p,q,σ)-Bergman-Carleson measure.*

(b) *For $t > \frac{2q}{p(\alpha+2)}$, we have*

$$\frac{\widetilde{\Omega}_t(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})t}} \longrightarrow 0$$

as $z \longrightarrow \partial\mathbb{D}$.

(c) *For any* $R \in (0,1)$, *we have*

$$\frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \longrightarrow 0$$

as $z \longrightarrow \partial\mathbb{D}$.

(d) *For any* r*-lattice* $\{z_k\}_{k=1}^{\infty}$ *with* $r \in (0,1)$, *we have*

$$\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}} \longrightarrow 0$$

as $k \longrightarrow 0$.

Proof. The implication $(c) \Rightarrow (d)$ is trivial because $z_k \to \partial\mathbb{D}$ as $k \to \infty$ whenever $\{z_k\}_{k=1}^{\infty}$ is an r-lattice. It follows from (3.14) that $(b) \Rightarrow (c)$.

$(a) \Rightarrow (c)$ Given $0 < R < 1$. For $z \in \mathbb{D}$, we set $f_z(\xi) = \dfrac{K_z^{\alpha}(\xi)}{\sigma^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}-(\alpha+2)}}$, $\xi \in \mathbb{D}$. One can easily find that $f_z \in A^p_\sigma$, $\|f_z\|_{p,\sigma} \le C$ and $f_z \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $z \to \partial\mathbb{D}$. Since Ω is a vanishing (p,q,σ)-Bergman-Carleson measure, it follows from (3.11) that

$$\frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \le C\frac{(1-|z|^2)^{q(\alpha+2-\frac{2}{p})}}{\sigma^{\frac{q}{p}}(z)}\int_{\mathbb{D}} |f_z(\xi)|^q d\Omega(\xi) \longrightarrow 0$$

as $z \to \partial\mathbb{D}$.

$(d) \Rightarrow (a)$. Suppose (d) holds. For any $\epsilon > 0$, there exists a positive integer k_0 such that $\dfrac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}} < \epsilon$, whenever $k > k_0$. Notice that $\cup_{k=1}^{k_0} E(z_k, r)$ is relatively compact in $\mathbb{D}$. Let us consider a bounded sequence $\{f_j\}_{j=1}^{\infty}$ in A^p_σ such that $f_j \longrightarrow 0$ uniformly on any compact subset of $\mathbb{D}$ as $j \to \infty$. Similar to the proof of (3.10) and if j is large enough, we have that

$$\begin{aligned}\int_{\mathbb{D}} |f_j(z)|^q d\Omega(z) &\le \int_{\cup_{k=1}^{k_0} E(z_k,r)} |f_j(z)|^q d\Omega(z) + \sum_{k=k_0+1}^{\infty}\int_{E(z_k,r)} |f_j(z)|^q d\Omega(z)\\ &\le C\epsilon \|f_j\|^q_{p,\sigma}\\ &\le C\epsilon, \end{aligned} \tag{3.16}$$

where C is independent of ϵ.

$(a) \Rightarrow (b)$ The equivalence of (a), (c), and (d) shows that the measure Ω is a vanishing (Np, Nq, σ)-Bergman-Carleson measure if Ω is a vanishing (p,q,σ)-Bergman-Carleson measure. For $z \in \mathbb{D}$, set $f_z(\xi) = \dfrac{K_z^{\alpha}(\xi)}{\sigma^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2}{p}-(\alpha+2)}}$, $\xi \in \mathbb{D}$. One can easily find that $f_z \in A^p_\sigma$, $\|f_z\|_{p,\sigma} \le C$, and $f_z \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $z \to \partial\mathbb{D}$. Since $Np > \frac{4}{(\alpha+2)}$ and it follows from (3.15), we have that

$$\frac{\widetilde{\Omega}_{Nq}(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}} \asymp \frac{1}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}}\int_{\mathbb{D}} |f_z(\xi)|^{Nq} d\Omega(\xi).$$

Statement (a) yields that Ω is a vanishing $(\frac{tp}{Nq}, \frac{t}{N}, \sigma)$-Bergman-Carleson measure. Therefore

$$\lim_{z\to\partial\mathbb{D}} \frac{\widetilde{\Omega}_{Nq}(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})Nq}} = 0.$$

The proof is completed. □

Corollary 2. *Let $t > 0$, $0 < p < \infty$, $d\Omega = \sigma dA$ be a positive measure and $C_{\psi,\varphi} : A^p_\sigma \to L^p(\Omega)$ be a bounded operator. Then the following statements are equivalent.*

(a) *$C_{\psi,\varphi}$ is power compact, that is, $C^m_{\psi,\varphi}$ is compact, for some $m \in \mathbb{N}$.*

(b) *For $t > \frac{2q}{p(\alpha+2)}$, we have*

$$\frac{\widetilde{\Omega}_{m,t}(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}-(\frac{\alpha+2}{2})t}} \longrightarrow 0$$

as $z \longrightarrow \partial\mathbb{D}$.

(c) *For any $R \in (0,1)$, we have*

$$\frac{\widehat{\Omega}_{m,R}(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2(\frac{q-p}{p})}} \longrightarrow 0$$

as $z \longrightarrow \partial\mathbb{D}$.

(d) *For any r-lattice $\{z_k\}_{k=1}^\infty$ with $r \in (0,1)$, we have*

$$\frac{\widehat{\Omega}_{m,r}(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}} \longrightarrow 0$$

as $k \longrightarrow 0$.

Theorem 3.3. *Let Ω be a finite positive Borel measure and $0 < q < p < \infty$. Then the following statements are equivalent:*

(a) *Ω is a (p, q, σ)-Bergman-Carleson measure.*

(b) *Ω is a vanishing $(p, q, \sigma)-$Bergman Carleson measure.*

(c) *For $t > \frac{2(q+p)}{p(\alpha+2)}$, we have*

$$\widetilde{M}_t(z) = \frac{\widetilde{\Omega}_t(z)}{\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{2\frac{q}{p}+1-(\frac{\alpha+2}{2})t}} \in L^{\frac{p}{p-q}}(\mathbb{D}).$$

(d) *For any $R \in (0,1)$, we have*

$$\widehat{M}_R(z) = \frac{\widehat{\Omega}_R(z)}{\sigma^{\frac{q}{p}}(z)} \in L^{\frac{p}{p-q}}(\mathbb{D}).$$

(e) *For any r-lattice $\{z_k\}_{k=1}^p$ with $r \in (0,1)$, we have*

$$\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}} \in l^{\frac{p}{p-q}}(\mathbb{D}).$$

Moreover, we have

$$\|i\|^q_{A^p_\sigma \to L^p(\Omega)} \asymp \|\widetilde{M}_t\|_{\frac{p}{p-q}} \asymp \|\widehat{M}_R\|_{\frac{p}{p-q}} \asymp \left\| \left\{ \frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}} \right\}_{k=1}^\infty \right\|_{l^{\frac{p}{p-q}}}. \quad (3.17)$$

Proof. Since Lemma 3 implies that the statements (c), (d), and (e) are equivalent with the corresponding norm estimate (3.17) and the implication $(b) \Rightarrow (a)$ is trivially true, it is sufficient to prove that $(d) \Rightarrow (a)$, $(a) \Rightarrow (e)$, and $(a) \Rightarrow (b)$.

$(d) \Rightarrow (a)$. Since $0 < q < p < \infty$ implies $\frac{p}{q} > 1$, the conjugate exponent of $\frac{p}{q}$ is $\frac{p}{p-q}$. For $f \in A^p_\sigma$, we have

$$\begin{aligned}\int_{\mathbb{D}} |f(\xi)|^q d\Omega(\xi) &\le C \int_{\mathbb{D}} |f(\xi)|^q \widehat{\Omega}_R(\xi) dA(\xi) \\ &\le C \left\| \widehat{M}_R \right\|_{\frac{p}{p-q}} \|f\|^q_{p,\sigma}.\end{aligned}$$

Above inequality follows from part (4) of Lemma 1 and Holder's inequality shows that Ω is a (p,q,σ)-Bergman-Carleson measure and $\|i\|^q_{A^p_\sigma\to L^p(\Omega)} \le C\|\widehat{M}_R\|_{\frac{p}{p-q}}$.

$(a)\Rightarrow(e)$. Let $\{\lambda_k\}_{k=1}^\infty \in l^p$ and set f as in Lemma 2. Statement (a) and Lemma 2 imply that

$$\int_{\mathbb{D}}\left|\sum_{k=1}^{\infty}\lambda_k\frac{K^\alpha_{z_k}(z)}{\sigma^{\frac{1}{p}}(z_k)(1-|z_k|^2)^{\frac{2}{p}-(\alpha+2)p}}\right|^q d\Omega(z)\le C\|i\|^q_{A^p_\sigma\to L^q}\|\{\lambda_k\}_k\|^q_{l^p}. \tag{3.18}$$

Recall that Rademacher functions ψ_k are defined by

$$\psi_0(t)=\begin{cases}1, & \text{if } 0\le t-[t]<1/2,\\ -1, & \text{if } 1/2\le t-[t]<1,\end{cases}$$

and $\psi_k(t)=\psi_0(2^k t)$ for $k=1,2,\cdots$, where $[t]$ denotes the greatest integer less than or equal to t. For $0<q<\infty$, Khinchine's inequality is given as

$$C_1\left(\sum_{k=1}^m|b_k|^2\right)^{q/2}\le\int_0^1\left|\sum_{k=1}^m b_k\psi_k(t)\right|^q dt\le C_2\left(\sum_{k=1}^m|b_k|^2\right)^{q/2},$$

which holds for all $m\ge1$ and all complex numbers $b_1,b_2,\cdots b_m$. Let $\psi_k(t)$ be the kth Rademacher function on $[0,1]$. Replacing λ_k with $\psi_k(t)\lambda_k$, integrating w.r.t t from 0 to 1 and applying Khinchine's inequality in (3.18), we see that

$$\int_{\mathbb{D}}\left(\sum_{k=1}^{\infty}|\lambda_k|^2\frac{|K^\alpha_{z_k}(z)|^2}{\sigma^{\frac{2}{p}}(z_k)(1-|z_k|^2)^{2(\frac{2}{p}-(\alpha+2))}}\right)^{\frac{q}{2}}d\Omega(z)\le C\|i\|^q_{A^p_\sigma\to L^q}\|\{\lambda_k\}_k\|^q_{l^p}.$$

Thus, we have

$$\begin{aligned}&\sum_{k>k_0}|\lambda_k|^q\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\\
&\asymp\sum_{k>k_0}\int_{E(z_k,r)}|\lambda_k|^q|K^\alpha_{z_k}(z)|^q\sigma^{\frac{q}{p}}(z)(1-|z|^2)^{(\alpha+2)q-\frac{2q}{p}}d\Omega(z)\\
&\le C\sum_{k>k_0}\int_{E(z_k,r)}\left(\sum_{k=1}^{\infty}|\lambda_k|^2\frac{|K^\alpha_{z_k}(z)|^2}{\sigma^{\frac{2}{p}}(z_k)(1-|z_k|^2)^{2(\frac{2}{p}-(\alpha+2))}}\right)^{\frac{q}{2}}d\Omega(z)\\
&\le C\int_{\mathbb{D}}\left(\sum_{k=1}^{\infty}|\lambda_k|^2\frac{|K^\alpha_{z_k}(z)|^2}{\sigma^{\frac{2}{p}}(z_k)(1-|z_k|^2)^{2(\frac{2}{p}-(\alpha+2))}}\right)^{\frac{q}{2}}d\Omega(z)\\
&\le C\|i\|^q_{A^p_\sigma\to L^q}\|\{\lambda_k\}_k\|^q_{l^p}.\end{aligned}\tag{3.19}$$

Since $\cup_{k=1}^{k_0}E(z_k,R)$ is relatively compact in $\mathbb{D}$, we see that

$$\sum_{k=0}^{\infty}|\lambda_k|^q\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\le C\|i\|^q_{A^p_\sigma\to L^q}\|\{\lambda_k\}_k\|^q_{l^p}.$$

Setting $c_k=|\lambda_k|^q$, for each k, then $\{c_k\}_{k=1}^\infty\in l^{\frac{p}{q}}$ because $\{\lambda_k\}_{k=1}^\infty\in l^p$ implies that

$$\sum_{k=0}^{\infty}c_k\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\le C\|i\|^q_{A^p_\sigma\to L^q}\|\{c_k\}\|_{l^{\frac{p}{q}}}.$$

Hence by duality argument,we have

$$\left\|\left\{\frac{\widehat{\Omega}_r(z_k)}{\sigma^{\frac{q}{p}}(z_k)(1-|z_k|^2)^{2(\frac{q-p}{p})}}\right\}_{k=1}^{\infty}\right\|_{l^{\frac{p}{p-q}}}\le C\|i\|^q_{A^p_\sigma\to L^p(\Omega)}.$$

Finally, we will prove the implication $(a) \Rightarrow (b)$. Let us consider a bounded sequence $\{f_n\}_{n=1}^{\infty}$ in A^p_σ such that $f_n \longrightarrow 0$ uniformly on each compact subset of $\mathbb{D}$. Let F be any compact subset of $\mathbb{D}$ and Ω_F be the restriction of Ω to F. Then we have

$$\begin{aligned}\int_{\mathbb{D}} |f_n(z)|^q d\Omega(z) &= \int_F + \int_{\mathbb{D}\setminus F} |f_n(z)|^q d\Omega(z)\\ &= I_1 + I_2. \end{aligned} \tag{3.20}$$

Since $f_n \to 0$ uniformly on F as $n \to \infty$, we have

$$I_1 = \int_F |f_n(z)|^q d\Omega(z) \leq C \sup_{z\in F} |f_n(z)|^q \longrightarrow 0.$$

Fix $R \in (0,1)$ and $z \in \mathbb{D}$, and we obtain that

$$\frac{\widehat{(\Omega_F)}_R(z)}{\sigma^{\frac{q}{p}}(z)} \longrightarrow 0$$

as F extended to $\mathbb{D}$. By the equivalence of (a) and (d), we have

$$\frac{\widehat{(\Omega_F)}_R^{\frac{p}{p-q}}(z)}{\sigma^{\frac{q}{p-q}}(z)} \leq \frac{\widehat{(\Omega_F)}_R^{\frac{p}{p-q}}(z)}{\sigma^{\frac{q}{p-q}}(z)} \in L^1(\mathbb{D}).$$

Therefore,

$$\begin{aligned} I_2 &= \int_{\mathbb{D}} |f_n(z)|^q d\Omega_F(z)\\ &\leq C \sup_n \|f_n\|^q_{p,\sigma} \left\| \frac{\widehat{(\Omega_F)}_R(z)}{\sigma^{\frac{q}{p}}(z)} \right\|_{\frac{p}{p-q}}\\ &\longrightarrow 0 \quad \text{as } F \text{ extended to } \mathbb{D}, \end{aligned} \tag{3.21}$$

which follows from (1.1) and the dominated convergence theorem. Hence

$$\lim_{n\to\infty} \int_{\mathbb{D}} |f_n(z)|^q d\Omega(z) = 0$$

implies that Ω is a vanishing (p, q, σ)-Bergman-Carleson measure. This completes the proof. □

4. Further applications of and future plan

The study of Toeplitz operators on the Bergman spaces with measures as symbols was initiated by Luecking in [4]. Zhu in [14] and [15] characterized boundedness, compactness and the Schatten class of Toeplitz operators using Berezin transform and averaging functions. A detailed study of Toeplitz operators on Bergman spaces is found in Zhu's book [13]. For $\tau \in L^\infty(\mathbb{D})$ and $\alpha > -1$, the Toeplitz type integral operators T_τ on A^2_σ are defined by

$$T_\tau f = \int_{\mathbb{D}} \frac{\tau(w)f(w)}{(1 - z\bar{w})^{2+\alpha}} \sigma(w) dA(w), \quad f \in A^2_\sigma.$$

As an application of Carleson measure and vanishing Carleson measure in the next paper our focus will be to characterize the bounded, compact and schatten class Toeplitz type integral operators on Bergman spaces with admissible weights in terms of Berezin transform and averaging functions.

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