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EXISTENCE OF SOLUTIONS FOR SOLITONS TYPE EQUATIONS IN SEVERAL SPACE DIMENSIONS: DERRICK'S PROBLEM WITH (r, p)-LAPLACIAN

N.E. TAIBI, A. DELLAL, J. HENDERSON, AND A. OUAHAB

Abstract. In this paper we study a class of Lorentz invariant nonlinear field equations in several space dimensions. The main purpose is to obtain soliton-like solutions with twice (r, p)-Laplacian. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge.

У статті вивчається клас нелінійних рівнянь, інваріантних відносно лоренцевих перетворень, для поля з декількома просторовими змінними. Основною метою є отримання солітоноподібних розв'язків з подвійним (r, p)-лапласіаном. Поля характеризуються топологічним інваріантом, який ми називаємо зарядом. Доведено існування статичного розв'язку, який мінімізує енергію в конфігураціях з нетривіальним зарядом.

1. Introduction

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect solitons have a particle-like behavior and they occur in many areas of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, and plasma physics; see [9]. Probably, the simplest equation which has soliton solutions is the sine-Gordon equation,

$$-\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial t^2}+\sin\psi=0, \tag{1.1}$$

where $\psi=\psi(x,t)$ is a scalar field, x, t are real numbers representing, respectively, the space and the time variable. Derrick, in a celebrated paper [8], considers the more realistic three space dimension model,

$$-\Delta\psi+\frac{\partial^2\psi}{\partial t^2}+V'(\psi)=0, \tag{1.2}$$

Δ being the 3-dimensional Laplace operator and V' is the gradient of a nonnegative C^1 real function V. In [8] it is proved by a simple rescaling argument that (1.2) does not possess any nontrivial finite-energy static solution. This fact leads the author to say, "We are thus faced with the disconcerting fact that no equation of type (1.2) has any time-independent solutions which could reasonably be interpreted as elementary particles." Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler-Lagrange equations of the action functional relative to the functional

$$S=\iint \mathcal{L}\,dx\,dt,$$

where the Lorentz invariant Lagrangian density proposed in [8] has the form

$$\mathcal{L}(\psi)=-\left(|\nabla\psi|^2-|\psi_t|^2\right)^{\frac{p}{2}}-V(\psi),\quad p>3. \tag{1.3}$$

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However, Derrick does not continue his analysis and he concludes that a Lagrangian density of type (1.3) leads to a very complicated differential equation. He has been unable to demonstrate either the existence or nonexistence of stable solutions. In this spirit, a considerable amount of work has been done by Benci and collaborators, and a model equation proposed in [2]. The Lorentz invariant Lagrangian density proposed in [2] has the form

$$\rho = |\nabla\psi|^2 - |\psi_t|^2;\ \alpha(\rho) = a\rho + b|\rho|^{\frac{p}{2}},\ p > n,$$
$$\mathcal{L}(\psi,\rho) = -\frac{1}{2}\alpha(\rho) - V(\psi). \tag{1.4}$$

In the case where p is constant, various mathematical results (existence, multiplicity results, asymptotic behavior, etc.), have been obtained for different classes of solution models (see [2, 3, 4, 1, 9, 5, 12, 7] and the references therein).

The aim of this paper is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension for a class of Lagrangian densities $\mathcal{L}$ which include (1.4) with (r,p)-Laplacian.

2. Statement of the Problem

The class of Lagrangian densities we consider generalizes the problem studied in [2], Lagrangian density with variable exponent, in such a way as to include the Lorentz invariant Lagrangian density proposed in [2]. First we introduce some notation. If n, m are positive integers, and will denote, respectively, the physical space-time (typically $n = 3$) and the internal parameters space. We are interested in the multi-dimensional case, so we assume that $n \geq 2$. A point in $\mathbb{R}^{n+1}$ will be denoted by $X = (x,t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The fields we are interested in are maps $\psi : {}^{n+1} \to \mathbb{R}^m,\ \psi = (\psi_1,\dots,\psi_m)$. We set

$$\rho = |\nabla\psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t.

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi,\rho) = -\frac{1}{2}\alpha(\rho) - V(\psi), \tag{2.5}$$

where the function V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho) = a\rho|\rho|^{\frac{r}{2}-1} + b|\rho|^{\frac{p}{2}}, \qquad a \geq 0,\ b > 0,\ 1 < r \leq 2 \leq n < p. \tag{2.6}$$

The results of [2] were concerned with the case: $r \equiv 2$. The action functional related to (2.5) is

$$S(\psi) = \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi,\rho)dxdt = \int_{\mathbb{R}^{n+1}} -\frac{1}{2}\alpha(\rho) - V(\psi)dxdt.$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha'\psi_t) - \nabla\big(\alpha'\nabla\psi\big) + V'(\psi) = 0, \tag{2.7}$$

where $\nabla\big(\alpha'\nabla\psi\big)$ denotes the vector whose j-th component is given by $div\big(\alpha'\nabla\psi^j\big)$, and V' denotes the gradient of V. The equation (2.7) is Lorentz invariant. Static solutions $\psi(x,t) = u(x)$ of (2.7) solve the equation

$$-\nabla\big(\alpha'\nabla u\big) + V'(u) = 0. \tag{2.8}$$

Using (2.6) and (2.8) we obtain

$$-a\frac{r}{2}\Delta_r u - b\frac{p}{2}\Delta_p u + V'(u) = 0, \tag{2.9}$$

where

$$\Delta_r u = \nabla\big(|\nabla u|^{r-2}\nabla u\big),$$

and

$$\Delta_p u = \nabla\big(|\nabla u|^{p-2}\nabla u\big).$$

Recall that the results of [2] were concerned with the case $r \equiv 2$.

It is easy to verify that, if $u = u(x)$ is a solution of the (2.7) and $v = (\nu, 0, \ldots, 0)$ with $|\nu| < 1$, the field

$$\psi_\nu(x,t) = u\left(\frac{x_1 - \nu t}{\sqrt{1-\nu^2}}, x_2, \ldots, x_n\right) \tag{2.10}$$

is solution of (2.7). Notice that the function undergoes a contraction by a factor,

$$\gamma = \frac{1}{\sqrt{1-\nu^2}},$$

in the direction of the motion; this is a consequence of the fact that (2.7) is Lorentz invariant. Clearly (2.9) are the Euler-Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^r + \frac{b}{2}|\nabla u|^p + V(u)\right) dx, \tag{2.11}$$

where $m = n+1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \to \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$. $V : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1,0)$, and V is positive and singular at η. More precisely we assume:

(V_1) $V \in C^1(\Omega, \mathbb{R})$.
(V_2) $V(\xi) \geq V(0) = 0$.
(V_3) V is twice differentiable at 0 and the Hessian matrix $V''(0)$ is nondegenerate.
(V_4) There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c|\xi|^{-q}$$

where

$$\frac{1}{q} = \frac{1}{n} - \frac{1}{p}.$$

(V_5) For every $\xi \in \Omega \backslash \{0\}$ we have

$$V(\xi) > 0, \text{ and } \lim_{|\xi|\to\infty} \inf V(\xi) = \nu > 0.$$

(V_6) There exist $R > 0$, $|\xi| < R \Longrightarrow V(\xi) \geq \omega_R |\xi|^r$, $\omega_R > 0$.

Example 2.1. A potential satisfying the assumptions $(V_1) - (V_6)$ is

$$V(\xi) = \omega_0^2\left(|\xi|^r + \frac{|\xi|^4}{|\xi - \eta|^q}\right).$$

Definition 2.2. We call soliton a solution of equation (2.7) having the form of equation (2.10), where u is a local minimum of the energy functional.

3. Functional Setting

Let $p > n \geq 2$ and, with no loss of generality, we can consider the functional (2.11) with $b = 1$. It will be convenient to introduce the following notation:

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^r + \frac{1}{2} |\nabla u|^p + V(u) \right) dx,$$

and we define the space E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a\|\nabla u\|_{L^r} + \|\nabla u\|_{L^p} + \|u\|_{L^r}, \quad a > 0,$$

$$p > n \geq 2 \geq r > 1,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a};$$

$$\|u\|_{L^r} = \left(\sum_{j=1}^{n+1} \|u_j\|_{L^r}^r \right)^{\frac{1}{r}},$$

$$\|\nabla u\|_{L^r} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^r}^r \right)^{\frac{1}{r}},$$

and

$$\|\nabla u\|_{L^p} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^p}^p \right)^{\frac{1}{p}}.$$

For every $a > 0$, the norms $\|\cdot\|_a$ are equivalent, so we have to study only two cases: $a = 0,\ a > 0$.

Proposition 3.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$, for every $s \in [r, \infty], 1 < r \leq 2$.*

Proof. The space E_0 is continuously embedded in $L^r(\mathbb{R}^n, \mathbb{R}^{n+1})$, therefore it is sufficient to show that E_0 is embedded also in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, and also in L^s, so it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_{L^\infty} \leq c\|u\|_0.$$

We fix $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$mes(Q_k) = 1, \qquad \mathbb{U}_{k\in\mathbb{N}} Q_k = \mathbb{R}^n.$$

Then, by a well-known inequality (see [6, page 283]), for every $k \in \mathbb{N}$ and $Q_k \subset \mathbb{R}^n$,

$$|u(x)| \leq \left| \int_{Q_k} u dy \right| + M\|\nabla u\|_{L^p(Q_k)}, \tag{3.12}$$

where $M \geq 0$ being independent of u. Thus

$$|u(x)| \leq \|u\|_{L^r(Q_k)} + M\|\nabla u\|_{L^p(Q_k)} \leq \|u\|_{L^r(\mathbb{R}^n)} + M\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq (1 + M)\|u\|_0.$$

Hence

$$\|u\|_{L^\infty} \leq c\|u\|_0, \quad c = 1 + M.$$

□

Corollary 3.2. *The Banach space E_0 is continuously embedded in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. By definition of the space E_0, we have for every $u \in E_0$

$$\|u\|_0 > \|\nabla u\|_{L^p}.$$

From Proposition 3.1 there exists $c_1 > 0$ such that

$$c_1\|u\|_0 > \|u\|_{L^p}$$

and so

$$\|u\|_0 > c\|u\|_{W^{1,p}}.$$

□

Corollary 3.3. *For every $a > 0$, the space E_a can be identified with the Banach space*

$$W = W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,r}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,r}} + \|u\|_{W^{1,p}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$ and also in $W^{1,r}(\mathbb{R}^n, \mathbb{R}^{n+1})$. For any $u \in E_a$ we have

$$\|u\|_a \leq \sup(1, a)\|u\|_W.$$

From Corollary 3.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_a \geq c(\|u\|_{W^{1,r}} + \|u\|_{W^{1,p}}).$$

□

By Proposition 3.1 and well-known Sobolev embeddings, we now make a Remark.

Remark 3.4. (see [6, Theorem 9.12 (Morrey), page 282]).
Since $p > n$, by the preceding Corollaries and well-known Sobolev embeddings, we get easily some useful properties of the Banach space E_a :

(1) We have

$$E_a \subset W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}), \tag{3.13}$$

if $\{u_k\}$ converges weakly in E_a to u , then it converges uniformly on every compact set contained in $\mathbb{R}^n$.

(2) Furthermore the E_a functions are Holder continuous of order $(p-n)/p$,

$$|u(x) - u(y)| = C^{(p-n)/p}|x-y|\|\nabla u\|_{L^p}, \tag{3.14}$$

i.e.,

$$E_a \subset C^{0,(p-n)/p}(\mathbb{R}^n, \mathbb{R}^{n+1})$$

is a locally compact injection.

(3) For every value $a \geq 0$, the functions in E_a are bounded and decay to zero at infinity,

$$\lim_{|x|\to\infty} u(x) = 0. \tag{3.15}$$

Recall that η is a singular point of the potential V, so it is reasonable to consider in space E_a, the open subset

$$\Lambda_a = \{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\},$$

which is open in E_a. In fact, if $u \in \Lambda_a$, by Remark 3.4, we have

$$\inf_{x\in\mathbb{R}^n} |u(x) - \eta| = d > 0.$$

Then, by using Proposition 3.1 (E_0 is continuously embedded in L^∞), we deduce that there exists a small neighborhood of u contained in Λ_a.

The boundary of Λ_a is given by

$$\partial\Lambda_a = \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}.$$

We can show that Λ_a has a rich topological structure, more precisely it consists of infinitely many connected components. These components are identified by the topological charge we are going to introduce.

4. Topological Charge and Connected Components of Λ_a

For the sake of simplicity, we consider the function space

$$C = \left\{ u : \mathbb{R}^n \to \mathbb{R}^{n+1} \backslash \{\eta\} \text{ is continuous and } \lim_{|x|\to\infty} u(x) = 0 \right\}$$

where $\eta = (1, 0)$. Every function $u \in C$ we write in the form $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$ where $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4.1. For every function $u \in C$ we define the support of u

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

Then we define the topological charge of u

$$ch(u) := \begin{cases} deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset, \end{cases}$$

so that with the Brouwer degree,

$$deg(\tilde{u}, K_u, 0) = \sum_{x \in \tilde{u}^{-1}(0)} sgn J_{\tilde{u}}(x).$$

where $J_{\tilde{u}}$ denotes the determinant of the Jacobian matrix. For more information about this subject, see [11].

We notice that the above definition is well posed. Indeed, since

$$\lim_{|x|\to\infty} u(x) = 0,$$

we have that K_u is an open, bounded set; moreover, for every $x \in \partial K_u$, we have which, together with $u(x) \neq \eta$ implies $\tilde{u}(x) \neq 0$.

We notice that this definition of charge is the same as in [2]. We recall that the topological charge is continuous with respect to the uniform convergence.

Now, for every $q \in \mathbb{Z}$ we set

$$\Lambda_q = \{u \in \Lambda_a : ch(u) = q\}.$$

Since the topological charge is continuous with respect to the uniform convergence and the continuity of the embeddings E_a in L^∞ (see Proposition 3.1) assure that the topological charge is continuous on Λ_a, it follows that Λ_q is open in E_a, since we have also

- $\Lambda_a = \bigcup_{q\in\mathbb{Z}} \Lambda_q$,
- $\Lambda_q \cap \Lambda_p = \emptyset, \quad p \neq q$.

We conclude that every Λ_q is a connected component of Λ_a.

If we assume that the space dimension is odd then we conclude that for every $q \in \mathbb{Z}$ the component Λ_q is isomorphic to the component Λ_{-q}.

So for every $u \in \Lambda_a$ we can define the charge $ch(u) \in \mathbb{Z}$. Now, we consider the set of a minimizer of f_a in the open set

$$\Lambda^*{}_q = \{u \in \Lambda_a : ch(u) \neq 0\}.$$

Remark 4.2. We can easily see that $ch(u) \neq 0$ implies $\|u\|_{L^\infty} > 1$.

5. Properties of the Energy Functional

Lemma 5.1. *The functional f_a takes real values and it is continuous on Λ_a.*

Proof. We have

$$f_a(u) = \underbrace{\int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^r + \frac{b}{2}|\nabla u|^p\right) dx}_{} + \underbrace{\int_{\mathbb{R}^n} V(u)dx}_{}.$$

First we show that

$$f_a(u) < \infty.$$

The first term on the left-hand side of energy f_a is finite and continuous. Let us prove that the second term is finite and continuous.

From the assumption (V_2) we have $V(\xi) = V''(0)\xi\ \xi + o(\xi^2)$.

By (V_3) there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every $\xi \in \mathbb{R}^{n+1}$, we have

$$V(\xi) \leq M|\xi|^2. \tag{5.16}$$

Since every $u \in E_a$ decays to zero at infinity (see (3.15)), there exists a ball B_u such that, for every $x \in \mathbb{R}^n \setminus B_u$, $|u(x)| < \epsilon$.

By (5.16) and for ϵ sufficiently small

$$V(u(x)) \leq M|u(x)|^2. \tag{5.17}$$

Since $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Proposition 3.1), we deduce

$$\int_{\mathbb{R}^n \setminus B_u} V(u)dx < \infty.$$

On the other hand, since u is continuous (see (3.14)), we also have

$$\int_{B_u} V(u)dx < \infty.$$

Let $\{u_k\} \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u\ \ in\ \ E_a$.
We show that

$$\int_{\mathbb{R}^n} V(u_k) \longrightarrow \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$ and with Lemma 5.4, u belongs to Λ_a.

We have $u_k \to u$ *on* $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see (3.13)), and we deduce that $V(u_k) \to V(u)$ uniformly on $\mathbb{R}$. Then

$$\int_{B_u} V(u_k)dx \to \int_{B_u} V(u)dx. \tag{5.18}$$

By (5.17)

$$\int_{\mathbb{R}^n \setminus B_u} V(u(x))dx \leq \int_{\mathbb{R}^n \setminus B_u} |u(x)|^2 dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, by using the dominated convergence theorem,

$$\int_{\mathbb{R}^n \setminus B_u} V(u_k)dx \to \int_{\mathbb{R}^n \setminus B_u} V(u)dx. \tag{5.19}$$

□

Lemma 5.2. *The map $f' : E_a \to E'_a$ defined by*

$$\begin{aligned} < f'_a(u), v > &= < -a\frac{r}{2}\Delta_r u - b\frac{p}{2}\Delta_p u + V'(u), v > \\ &= \int_{\mathbb{R}^n} \Big(a\frac{r}{2}|\nabla u|^{r-2}(\nabla u|\nabla v) + b\frac{p}{2}|\nabla u|^{p-2}(\nabla u|\nabla v) + V'(u).v \Big)\, dx \end{aligned}$$

is continuous.

Proof. We have

$$f'_a(u) = \underbrace{-a\frac{r}{2}\Delta_r u - b\frac{p}{2}\Delta_p u}+\underbrace{V'(u)}.$$

The proof for the first term on the left-hand side of f'_a is given in the Appendix A.

Let us prove that the second term is continuous. Let $\{u_k\} \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \longrightarrow u$. We show that

$$V'(u_k) \longrightarrow V'(u) \ \ in \ \ E'_a.$$

Since $f_a(u_k) < \infty$ and with Lemma 5.4, u belongs to Λ_a. Recall that E_a is continuously embedded in L^∞; see (3.13). We have

$$\|V'(u_k) - V'(u)\|_{E'_a} = \sup_{\|h\|_{E_a}\le 1} < V'(u_k) - V'(u), h >_{E'_a \times E_a},$$

with

$$\begin{aligned} < V'(u_k) - V'(u), h >_{E'_a\times E_a} &= \int_{\mathbb{R}^n} (V'(u_k) - V'(u))h\ dx \\ &= \underbrace{\int_{B_u} (V'(u_k) - V'(u))h\ dx}_{1} + \underbrace{\int_{\mathbb{R}^n/B_u} (V'(u_k) - V'(u))h\ dx}_{2}. \end{aligned}$$

In the term 1: since $\|h\|_{L^\infty} \le \|h\|_{E_a} \le 1$, with the same reasoning as in (5.18), we have

$$\int_{B_u} (V'(u_k) - V'(u))h\ dx < \frac{\epsilon}{2},$$

with the same choice of B_u as in proof of Lemma 5.1.

In the term 2: we have $V'(\xi) = V''(0)\xi + o(\xi)$, and then by V_3

$$\begin{aligned} \int_{\mathbb{R}^n\setminus B_u} (V'(u_k))h\ dx &= M\int_{\mathbb{R}^n\setminus B_u} |u_k||h|\ dx \\ &\le \|u_k\|_{L^2}\|h\|_{L^2} \\ &\le \|u_k\|_{L^2}. \end{aligned} \tag{5.20}$$

From (5.20), with the same reasoning as in (5.19), we have

$$\int_{\mathbb{R}^n\setminus B_u} (V'(u_k) - V'(u))h\ dx < \frac{\epsilon}{2}.$$

□

Lemma 5.3. *The functional f_a is coercive in Λ_a; that is, for every sequence $u_k \subset \Lambda_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. In the case $a > 0,\ n > r$, we have

$$\|u\|_a = a\|\nabla u\|_{L^r} + \|\nabla u\|_{L^p} + \|u\|_{L^2}.$$

Let $u_k \in \Lambda_a$ such that

$$\|u_k\|_a \to \infty \ \text{ as } \ k \to \infty.$$

It is clear that, if

$$a\|\nabla u_k\|_{L^r} + \|\nabla u_k\|_{L^p} \to \infty \quad \text{as} \quad k \to \infty, \tag{5.21}$$

we have

$$f_a(u_k) \to \infty \quad \text{as} \quad k \to \infty.$$

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^r} + \|\nabla u_k\|_{L^p} < c_* \tag{5.22}$$

and

$$\|u_k\|_{L^r} \to \infty \quad \text{as} \quad k \to \infty. \tag{5.23}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)dx \to \infty \text{ as } \quad k \to \infty.$$

From (V_6), there exist $R > 0$, $\omega_R > 0$ such that

$$|\xi| < R \Longrightarrow V(\xi) \geq \omega_R|\xi|^r. \tag{5.24}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n : \ |u_k(x)| \leq R\},$$

where $u_k \in W^{1,r}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the Sobolev inequality (see [6, Theorem 9.9, page 278]),

$$\|u_k\|_{L^{r^*}} \leq c\|\nabla u_k\|_{L^r}, \ r^* = \frac{rn}{n-r}, \ n > r > 1. \tag{5.25}$$

From (5.22), we obtain

$$\|u_k\|_{L^{r^*}} < c_*. \tag{5.26}$$

Moreover, from (3.12), there exists $M \geq 0$ independent of u_k, such that, for $mes(Q_k) = 1$,

$$|u_k(x)| \leq \left|\int_{Q_k} udy\right| + M\|\nabla u_k\|_{L^p(Q_k)} \leq \|u\|_{L^{r^*}(Q_k)} + M\|\nabla u_k\|_{L^p(Q_k)}.$$

By (5.21) and (5.26), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + Mc_*. \tag{5.27}$$

Then, there exists $c > 0$ such that

$$mes(\mathbb{R}^n \backslash A_k) < c. \tag{5.28}$$

From (5.27) and (5.28), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \backslash A_k} |u_k|^r dx < c_1. \tag{5.29}$$

By (5.24), we obtain

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \int_{A_k} V(u_k)dx \geq \omega_r \int_{A_k} \|u_k\|^r dx \geq \omega_r \left(\|u_k\|_{L^r}^r - \int_{\mathbb{R}^n \backslash A_k} |u_k|^r dx\right).$$

From (5.29) and (5.23), we have

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \omega_r(\|u_k\|_{L^r}^r - c_1) \to \infty \text{ as } k \to \infty.$$

In the case, $a = 0$ or $n = 2$, by (V_5), there exists $R_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq R_*$, we have

$$V(\xi) \geq \frac{\nu}{2}. \tag{5.30}$$

Let $u_k \in \Lambda_a$ be a sequence such that

$$\|u_k\|_0 \to \infty \text{ as } k \to \infty.$$

Since the functional f_a is invariant with respect to translation in $\mathbb{R}^n$, we can assume

$$\|u_k\|_{L^\infty} = |u_k(0)|. \tag{5.31}$$

Now, we consider the case

$$\|\nabla u_k\|_{L^p} \le M_* \quad \text{and} \quad \|u_k\|_{L^r} \to \quad \text{as} \quad k \to \infty.$$

Here we have two subcases:

(a)
$$\|u_k\|_{L^\infty} \to \infty \quad \text{as} \quad k \to \infty \quad , \tag{5.32}$$

or

(b)
$$\|u_k\|_{L^\infty} \quad \text{is bounded.} \quad , \tag{5.33}$$

In the subcase (a), by (5.32), we can choose a sequence $(R_k) \subset (0, \infty)$ such that

$$r_* \le \|u_k\|_{L^\infty} - K(R_k^{\frac{p-n}{p}}) \text{ and } R_k \to \infty, \tag{5.34}$$

where $K = cM_*$ and c is the same constant as in (3.14). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \le |u_k(0) - u_k(y)|.$$

Hence by (3.14), we obtain

$$|u_k(0)| - |u_k(y)| \le K(|y|^{\frac{p-n}{p}}).$$

From (5.31), we get

$$|u_k(y)| \ge \|u_k\|_{L^\infty} - K(|y|^{\frac{p-n}{p}}).$$

For $|y| \le R_k$ and (5.34), we have

$$|u_k(y)| \ge \|u_k\|_{L^\infty} - K(R_k^{\frac{p-n}{p}}) \ge R_*. \tag{5.35}$$

From (5.30) and (5.35), we get

$$\int_{\mathbb{R}^n} V(u_k)dx \ge \int_{B(0,R_k)} V(u_k)dx \ge \frac{\nu}{2} mes(B(0,R_k)).$$

This implies that

$$\int_{\mathbb{R}^n} V(u_k)dx \to \infty \text{ as } \ R_k \to \infty.$$

In the last subcase (b), we assume there exists $\bar{M} > 0$ such that

$$\|u_k\|_{L^\infty} \le \bar{M}.$$

From (5.24), we obtain

$$\int_{\mathbb{R}^n} V(u_k)dx \ge \omega_{\bar{M}} \|u_k\|_{L^r} \to \infty \text{ as } \ k \to \infty.$$

□

We are going to study the behaviour of energy f_a when u approaches the boundary of Λ_a; we remark that $\partial\Lambda_a = E_a \setminus \Lambda_a$.

Lemma 5.4. *Let $(u_k) \subset \Lambda_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Lambda_a$, then*

$$f_a(u_k) \to \infty \text{ as } k \to \infty.$$

Proof. The proof is the same as in [4, Lemma 3.7].

□

Corollary 5.5. *For every $b > 0$, there exists $d = d(b)$ such that, for every $u \in \Lambda_a$ we have*

$$f_a(u) \le b \Rightarrow \min_{x \in \mathbb{R}^n} |u(x) - \eta| \ge d.$$

Proof. The proof is the same as in [4, Proposition 3.9]. □

Lemma 5.6. *The functional f_a is weakly lower semicontinuous in Γ_a.*

Proof. The proof is the same as in [4, Proposition 3.10]. Let $u \in \Lambda_a$ and let a sequence $(u_k) \subset \Lambda_a$ weakly converge to u.

We show that

$$\liminf_{k\to\infty} f_a(u_k) \geq f_a(u).$$

The result is obvious when

$$\liminf_{k\to\infty} f_a(u_k) = +\infty.$$

We have

$$f_a(u_k) = \underbrace{\int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u_k|^r + \frac{b}{2}|\nabla u_k|^p\right) dx}_{A} + \underbrace{\int_{\mathbb{R}^n} V(u_k)dx}_{B}$$

The part A is convex and strongly continuous, and so is weakly lower semicontinuous (see[6, Remark 6, page 61]).

Now we have to study the part B. Since $\{u_k\}$ converges to u uniformly on every compact set, we fix a sphere $B_R(0)$ and we have

$$\lim_{k\to\infty} \int_{B_R(0)} V(u_k)dx = \int_{B_R(0)} V(u)dx.$$

On the other hand, since V is nonnegative, we have

$$\liminf_{k\to\infty} \int_{\mathbb{R}^n} V(u_k)dx \geq \liminf_{k\to\infty} \int_{B_R(0)} V(u_k)dx = \int_{B_R(0)} V(u)dx,$$

and taking the limit as $R \to \infty$, we obtain

$$\liminf_{k\to\infty} \int_{\mathbb{R}^n} V(u_k)dx \geq \int_{\mathbb{R}^n} V(u)dx.$$

The proof is complete. □

Proposition 5.7. *There exists $\Delta_a > 0$ such that, for every $u \in \Lambda_a$ satisfying $\|u\|_{L^\infty} \geq 1$, we have*

$$f_a(u) \geq \Delta_a.$$

Proof. By the continuous injection in Proposition 3.1,

$$\|u\|_a \geq \|u\|_{L^\infty} \geq 1,$$

and by the coercivity of f_a, we get

$$\|u\|_a \geq 1 \Rightarrow \exists\ \Delta_a > 0 \text{ such that } \ f_a(u) \geq \Delta_a.$$

□

6. Existence Result

Theorem 6.1. *The minimum points $u \in \Lambda_a$ for the functional f_a are weak solutions of the system (2.9).*

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the j^{th}-vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Lambda_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, then

$$0 = \left.\frac{df(u+\epsilon e_j h)}{d\epsilon}\right|_{\epsilon=0} = \int_{\mathbb{R}^n} \left(a\frac{r}{2}(|\nabla u|^{r-2}\nabla u_j \nabla h) + b\frac{p}{2}(|\nabla u|^{p-2}\nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h \right) dx,$$

$$1 \leq j \leq n+1.$$

By Green's formula,

$$\int_{\mathbb{R}^n} b\frac{p}{2}(|\nabla u|^{p-2}\nabla u_j \nabla h)dx = \int_{\mathbb{R}^n} -b\frac{p}{2} div(|\nabla .u|^{p-2}\nabla u_j)hdx.$$

So

$$\int_{\mathbb{R}^n} \left(-a\frac{r}{2} div(|\nabla .u|^{r-2}\nabla u_j) - b\frac{p}{2} div(|\nabla .u|^{p-2}\nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j} \right) .hdx = 0,$$

for $1 \leq j \leq n+1$, and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n} \left[-a\frac{r}{2}\Delta_r u - b\frac{p}{2}\Delta_p u + V'(u) \right] \phi \; dx = 0, \text{ for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

By Lemma 5.2 and by density we have

$$-\frac{a}{2}\Delta_r u - \frac{b}{2}\Delta_p u + V'(u) = 0.$$

□

Proposition 6.2. *(Splitting lemma) Let $(u_k) \in \Lambda_a^*$ be a sequence and M be a positive real number such that*

$$f_a(u_k) \leq M.$$

Then there exists $l \in \mathbb{N}$ such that

$$1 \leq l \leq \frac{M}{\Delta_a}$$

where Δ_a was introduced in Proposition 5.7 and there exist $\bar{u}_1, \ldots, \bar{u}_l \in \Lambda_a$, $(x_k^1), \ldots, (x_k^l) \subset \mathbb{R}^n$ such that, up to a subsequence,

$$u_k(\cdot + x_k^i) \to \bar{u}_i,$$

$$|x_k^i - x_k^j| \to \infty, \quad i \neq j,$$

$$\sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k),$$

and

$$ch(u_k) = \sum_{i=1}^{l} ch(\bar{u}_i).$$

Proof. From Lemmas 5.3, 5.4 and 5.6, and by the same method as used in [4, Lemma 4.1], we can conclude the result of this proposition. □

The minimum is attained on the set Λ_a, and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But, of course, we are interested in nontrivial solutions. We consider the following problem

$$I_* = \inf_{u \in \Lambda_a^*} f_a(u), \quad \Lambda_a^* = \{u \in E_a : ch(u) \neq 0\}.$$

The functional is bounded below and the set E_a is not empty. We consider fields u having the form

$$u(x) = \left(\frac{2}{1+|x|^m}, \frac{1}{1+|x|^m} x \right). \tag{6.36}$$

Lemma 6.3. *There exists a suitable $m \geq 1$, such that the field u defined in (6.36) belongs to Λ_a^* .*

Proof. Clearly, if m is sufficiently large, then the field u defined in (6.36) belongs to E_a. For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = \eta = (1, 0)$. We deduce that

$$\frac{2}{1 + |\bar{x}|^m} = 1,$$

$$\frac{1}{1 + |\bar{x}|^m}\bar{x} = 0.$$

We get the contradiction: $|\bar{x}| = 1$ and $\bar{x} = 0$. So, $u \in \Lambda_a$.

We show that $ch(u) \neq 0$.

We set $g(x) = \frac{1}{2}x$. We have

$$K_u = \{x \in \mathbb{R}^n : \frac{2}{1 + |x|^m} > 1\} = B(0, 1),$$

$$\text{if} \quad |x| = 1 \quad \text{then} \quad g(x) = \frac{1}{1 + |x|^m}x,$$

and then by the properties of the topological degree (see [11]) we get,

$$deg\left(\frac{1}{1 + |x|^m}x, B(0, 1), 0\right) = deg(g(x), B(0, 1), 0) \neq 0.$$

□

And moreover the set Λ_a^* is open in the space E_a; indeed,

- $\Lambda_a^* = \bigcup_{q \in \mathbb{N}^*} \Lambda_a^q$,
- $\Lambda_a^q \cap \Lambda_a^p = \emptyset, \quad p \neq q$.

where Γ_q is a connected component.

Theorem 6.4. *Let $a, b > 0$, and $p > n \geq 2 \geq r > 1$. If V satisfies $(V_1) - (V_6)$, then there exists a weak solution of (2.9) (i.e., a static solution of (2.7)), which is a minimizer of the energy functional (2.11) in the class of maps whose topological charge is different from 0.*

Proof. By the Splitting lemma (Proposition 6.2) and the same technique used in [2], we can conclude that there exists a weak solution of (2.9). And with suitable change of variable (2.10) we deduce a solution of equation (2.7) □

Remark 6.5. The functional exhibits an invariance for the symmetry group of rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $u_g(x) = u(gx)$, we have immediately

$$f_a(u_g) = f_a(u).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

Appendix A. Continuity of (Δ_p, $p > 2$) and (Δ_r, $1 < r \leq 2$)

Lemma 6.6. *The maps* ($\Delta_r : E \to E'$, $p > 2$) *and* ($\Delta_r : E \to E'$, $1 < r \leq 2$) *defined respectively by*

$$\langle -\Delta_r u, v\rangle_{E'_a \times E_a} = \int_{\mathbb{R}^n} |\nabla u|^{r-2}(\nabla u|\nabla v)dx, \quad 1 < r \leq 2,$$

and

$$\langle -\Delta_p u, v\rangle_{E'_a \times E_a} = \int_{\mathbb{R}^n} |\nabla u|^{p-2}(\nabla u|\nabla v)dx, \quad p > 2$$

are continuous.

The proof of the Lemma 6.6 follows from the following lemma.

Lemma 6.7. (*see R. Glowinski and A. Marroco* [10].)

(*i*) *If* $p \in [2;\infty)$ *then it holds that*

$$\left|z|z|^{p-2} - y|y|^{p-2}\right| \leq \beta|z-y|(|z|+|y|)^{p-2} \quad \textit{for all} \quad z, y \in \mathbb{R}^n$$

with β *independent of* y *and* z;

(*ii*) *If* $p \in (1;2]$, *then it holds that*

$$\left|z|z|^{p-2} - y|y|^{p-2}\right| \leq \beta(|z|+|y|)^{p-1} \quad \textit{for all} \quad z, y \in \mathbb{R}^n$$

with β *independent of* y *and* z.

Proof. Recall E to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, let $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$.

The map ($\Delta_r : E \to E'$, $1 < r \leq 2$) is continuous.

$$\begin{aligned}
\langle \Delta_r u - \Delta_r v, h\rangle &= \int_{\mathbb{R}^n} \left(|\nabla v|^{r-2}(\nabla u|\nabla h) - |\nabla u|^{r-2}(\nabla v|\nabla h)\right) dx \\
&= \int_{\mathbb{R}^n} \left(|\nabla v|^{r-2}\nabla u - |\nabla u|^{r-2}\nabla v \quad | \quad \nabla h\right) dx \\
&\leq \int_{\mathbb{R}^n} \left||\nabla v|^{r-2}\nabla u - |\nabla u|^{r-2}\nabla v\right| . |\nabla h|\, dx \\
\text{(from Lemma 6.7)} \quad &\leq \beta \int_{\mathbb{R}^n} |\nabla v - \nabla u|^{r-1} . |\nabla h|\, dx \\
\text{(from Hölder's inequality)} \quad &\leq \beta . \|\nabla u - \nabla v\|_{L^r}^{r-1} . \|\nabla h\|_{L^r}.
\end{aligned}$$

The map ($\Delta_r : E \to E'$, $p > 2$) is continuous.

$$\begin{aligned}
\langle \Delta_p u - \Delta_p v, h\rangle &= \int_{\mathbb{R}^n} \left(|\nabla v|^{p-2}(\nabla u|\nabla h) - |\nabla u|^{p-2}(\nabla v|\nabla h)\right) dx \\
&= \int_{\mathbb{R}^n} \left(|\nabla v|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v \quad | \quad \nabla h\right) dx \\
&\leq \int_{\mathbb{R}^n} \left||\nabla v|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v\right| . |\nabla h|\, dx \\
\text{(from Lemma 6.7)} \quad &\leq \beta \int_{\mathbb{R}^n} \left||\nabla v|^{p-2} + |\nabla u|^{p-2}\right| . |\nabla u - \nabla v| . |\nabla h|\, dx \\
\text{(from Hölder's inequality)} \quad &\leq \beta \left(\|\nabla v\|_{L^p}^{p-2} + \|\nabla u\|_{L^p}^{p-2}\right) . \|\nabla u - \nabla v\|_{L^p} . \|\nabla h\|_{L^p}.
\end{aligned}$$

□

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N.E.Taibi: taibi.nour.el.houda@gmail.com
Laboratory of Mathematics, University of Sidi Bel Abbès, P.O. Box 89, 22000 Sidi-Bel-Abbès, Algeria

A.Dellal: adellal@yahoo.fr
Department of Mathematics and Computer Science, Ahmed Ben Yahia El Wancharissi University, Tissemsilt, Algeria

J. Henderson: Johnny_Henderson@baylor.edu
Department of Mathematics, Baylor University Waco, Texas 76798-7328 USA

A. Ouahab: agh_ouahab@yahoo.fr
Laboratory of Mathematics, University of Sidi Bel Abbès, P.O. Box 89, 22000 Sidi-Bel-Abbès, Algeria

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