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ON INTEGRAL EQUATIONS OF FREDHOLM TYPE FOR A CLASS OF BOUNDED FUNCTIONS ON THE REAL LINE

MAMMAD BAYRAMOGLU, ILGAR JABBAROV, AND SABINA IBRAHIMOVA

Abstract. We consider the problem of extending the notion of a Fredholm integral equation and investigate its solvability in the class of bounded functions on the real line.

Розглядається задача розширення поняття інтегрального рівняння Фредгольма і досліджено його розв'язність у класі обмежених функцій на дійсній прямій.

1. Introduction

Integral equations of Fredholm type,

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,\xi)\varphi(\xi)d\xi, \tag{1.1}$$

were extensively studied in the literature [6, 2]. Our goal is to investigate such equations in classes of functions defined on $\mathbf{R}$ or $\mathbf{R}_+ = [0,+\infty)$. We will consider continuous and bounded functions.

Definition 1. A function $f(x)$, defined on $\mathbf{R}$, is called bounded if there exists a positive constant c such that

$$|f(x)| \le c \qquad \forall x \in \mathbf{R}.$$

By an analogy we define the notion of a bounded function in two or more real variables. If a function $f(x)$ is bounded on $\mathbf{R}^n$ for a natural n, then we use the notation $f \in B(\mathbf{R}^n)$. The class of continuous bounded functions will be denoted by $CB(\mathbf{R}^n)$. It easy to observe that both classes are linear spaces over the field $\mathbf{R}$.

It is well-known that equations (1.1) are closely related to the first order differential equations

$$y' = f(x,y).$$

Many investigations were devoted to study integral equations in different classes of functions [1]. In the Favard theory, for example, special cases of this equations were considered for the class of Bohr almost periodic functions which belong to the class of continuous bounded functions. In some cases, the equation

$$y' + A(x)y = f(x),$$

where the functions $f(x)$ and $A(x)$ are almost periodic, does not have almost periodic solutions, see [7].

There are examples showing existence of unbounded functions that cannot be solutions to any differential equation of the form

$$F(y, y', \dots, y^{(n)}) = 0$$

with a continuous function F, see [5, 4].

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In this paper, we consider equations of the type

$$\varphi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\varphi(\xi)d\xi \tag{1.2}$$

in the class of bounded functions on the real line, where $f(x)$ is bounded and continuous on $\mathbf{R}$. Despite that equation (1.2) differs from a Fredholm integral equation, the method used to solve it is similar to the one used to solve equation (1.1). For this reason, we call it a limit integral equation of Fredholm type. We suppose that the function $K(x,\xi)$, the kernel, is bounded and continuous on $\mathbf{R}\times\mathbf{R}$. Existence of the limit in the right-hand side, in the general case, may demand some constraints on the kernel, and needs to consider more special classes of functions. For this reason, we will at first consider the problem in following general setting: is there a sequence of values of the parameter $1 < T_1 < \ldots < T_k < \ldots$ such that the equation

$$\varphi(x) = f(x) + \lambda \lim_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} K(x,\xi)\varphi(\xi)d\xi$$

has solutions in the class of bounded continuous functions? Below we introduce analogs of some notions of the Fredholm theory, and give an explicit solution to the above equation, assuming some natural conditions.

Below we introduce analogues of Fredholm functions and show that they are integral functions. Defining eigenvalues of the kernel, we shall give a criterion of solvability of the non-homogenous equation. We give then an explicit solution of equation (1.2).

2. Auxliary result.

The main auxiliary result of the present paper is the following.

Theorem 1. *Let $K(x,\xi)\in CB(\mathbf{R}\times\mathbf{R})$ and the equation*

$$\varphi(x) = f(x) + \frac{\lambda}{T} \int_0^T K(x,\xi)\varphi(\xi)d\xi \tag{2.3}$$

have uniformly bounded solutions $\varphi_k(x)$ for some unbounded sequence of values of the parameter T, $1 < T_1 < T_2 < \cdots < T_k < \cdots$ Then the sequence $\big(\varphi_k(x)\big)$ has a subsequence (T_{k_l}), uniformly converging to some uniformly continuous function $\varphi(x)$, such that

$$\varphi(x) = f(x) + \lambda \lim_{l\to\infty} \frac{1}{T_{k_l}} \int_0^{T_{k_l}} K(x,\xi)\varphi(\xi)d\xi. \tag{2.4}$$

Proof. Take any segment $[0,\, M]$. Denote by

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$$

a sequence of solutions of equation (2.3) corresponding to the values of $T = T_1, T_2, \ldots$ Let us prove that the sequence $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$ is an equicontinuous family of functions on $[0,\, M]$. These functions can be extended to the whole real line by means of equation (2.3). Since, a change of a finite number of functions in this sequence does not change its property of being equicontinuous, we can suppose that $M < T_1$. Let ε be any positive number. The function $K(x,y)$ is continuous on the compact set $[0,T_k]\times[0,T_k]$ for each natural k. Thus it is uniformly continuous. So, for a given ε there exists $\delta > 0$ such that following inequalities are satisfied:

$$|K(x,y) - K(x',y')| < \varepsilon, \qquad |x-x'| < \delta, |y-y'| < \delta$$

for every two pair of real numbers (x, y) and (x', y') from the set $[0, T_k] \times [0, T_k]$ satisfying the conditions above. Then for each natural k we have

$$\begin{aligned}|\varphi_k(x) - \varphi_k(x')| &= \left| f(x) - f(x') + \frac{1}{T_k} \int_0^{T_k} (K(x, \xi) - K(x', \xi))\, \varphi_k(\xi) d\xi \right| \\ &\le |f(x) - f(x')| + \frac{C}{T_k} \int_0^{T_k} |K(x, \xi) - K(x', \xi)|\, d\xi \\ &= |f(x) - f(x')| + \frac{C}{T_k} \sum_{m=1}^{N} \int_{(m-1)\delta}^{m\delta} |K(x, \xi) - K(x', \xi)|\, d\xi,\end{aligned}$$

where N is a least natural number satisfying the condition $N\delta \ge T_k$ and C is a bound for the sequence of solutions $(\varphi_n(x))$, $|\varphi_n(x)| \le C$. From uniform continuity of the function $K(x, y)$ it follows that

$$|K(x, \xi) - K(x', \xi)| < \varepsilon$$

for points ξ in each interval $[(m-1)\delta, m\delta]$. Therefore,

$$|\varphi_k(x) - \varphi_k(x')| \le |f(x) - f(x')| + \frac{C}{T_k} N\delta\varepsilon \le |f(x) - f(x')| + C(1 + \delta T_k^{-1})\varepsilon.$$

Since $f(x)$ is also continuous, the right-hand side of the last inequality does not exceed $(2C + 1)\varepsilon$. This shows that the considered sequence of solutions is equicontinuous on $[0, M]$. Consider now the values $M = T_1, T_2, \ldots$ By [3, Theorems 7.24, 7.25, pp. 157-158] this sequence contains a subsequence uniformly convergent on $[0, T_1]$ to some continuous function $\phi_1(x)$,

$$\lim_{k\to\infty} \varphi_{n_k}(x) = \phi_1(x).$$

Taking $M = T_2$ we repeat the same procedure. Then it is possible to choose a new subsequence $(n_{k_m})_{m\ge1}$ from the sequence $(n_k)_{k\ge1}$ such that

$$\lim_{m\to\infty} \varphi_{n_{k_m}}(x) = \phi_2(x).$$

It can be seen that the function $\phi_2(x)$ is an extension of the function $\phi_1(x)$ to the segment $[0, T_2]$. Continuing in this manner, we get a sequence of functions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$ and, for each natural k,

$$\phi_{k+1}(x) = \phi_k(x), \qquad x \in [0, T_k].$$

Note that $\mathbf{R}_+ = [0, \infty) = \bigcup_{k=1}^{\infty} [0, T_k]$. Now we define a function on $\mathbf{R}_+$ by

$$\varphi(x) = \phi_n(x), \qquad x \in [0, T_n],$$

which is uniformly continuous on $\mathbf{R}_+$. □

3. Main result

Theorem 1 does not answer the question about existence of solutions of equation (1.2). Let us investigate the equation (1.2) and establish conditions for which (1.2) has solutions. To apply Theorem 1, we have to show that equations (2.3), for a sequence of real numbers $T_1, T_2, \ldots$, have uniformly bounded solutions.

Let us introduce analogues of notions known in the theory of Fredholm equations [6, 2]. First define analogues of the functions $D(\lambda)$ and $D(x, y; \lambda)$ as follows. Take some unbounded sequence of values of the parameter T, $1 < T_1 < T_2 < \cdots < T_k < \ldots$ .We put

$$D_k(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{b_{n,k}\lambda^n}{n!},$$

where

$$b_{n,k} = (-1)^n \frac{1}{T_k^n} \int_0^{T_k} \cdots \int_0^{T_k} \begin{vmatrix} K(x_1,\xi_1) & K(x_1,\xi_2) & \cdots & K(x_1,\xi_n) \\ K(x_2,\xi_1) & K(x_2,\xi_2) & \cdots & K(x_2,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n,\xi_1) & K(x_n,\xi_2) & \cdots & K(x_n,\xi_n) \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_n.$$

The function $D(x, y; \lambda)$ is defined by the relation

$$D_k(x, y; \lambda) = \lambda D_k(\lambda) K(x, y) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{Q_{n,k}(x, y)\lambda^{n+1}}{n!}, \qquad x, y \in \mathbf{R},$$

where

$$Q_{n,k}(x, y) = -\frac{n}{T_k^n} \int_0^{T_k} \cdots \int_0^{T_k} P_{n,k}(x, \xi, \xi_1, \ldots, \xi_{n-1}) K(\xi, y) d\xi d\xi_1 \cdots d\xi_{n-1},$$

and

$$P_{n,k}(x, \xi, \xi_1, \ldots, \xi_{n-1}) = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_{n-1}) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_{n-1},\xi) & K(\xi_{n-1},\xi_1) & \cdots & K(\xi_{n-1},\xi_{n-1}) \end{vmatrix}.$$

These functions are defined in analogy with the corresponding quantities in [6, 2].

Since the function $K(x, y)$ is bounded, there exists a positive constant L such that $|K(x, y)| \leq L$. By Hadamard's inequalitywe have

$$|b_{n,k}| \leq L^n n^{n/2}, \qquad k = 1, 2, \ldots \tag{3.5}$$

Therefore, the sequence $(b_{1,k})$ is bounded. We can find a subsequence (b_{1,k_m}) that tends to some real b_1 as $m \to \infty$,

$$\lim_{m \to \infty} b_{1,k_m} = b_1.$$

It is clear that $|b_1| \leq L$.

Now consider the sequence $(b_{2,k})$. All of the said above remains true for this sequence, and we can choose, from the sequence (k_m), a new subsequence, say (k_{m_p}) (which for simplicity we denote by (k_p)) such that

$$\lim_{p \to \infty} b_{2,k_p} = b_2.$$

Moreover, from inequalities (3.5) it follows that $|b_2| \leq 2L^2$.

Inductively continuing the above procedure of choosing a subsequence and applying the diagonal procedure, we arrive at some final subsequence (k_μ) such that, for every natural n, we have

$$\lim_{\mu \to \infty} b_{n,k_\mu} = b_n, |b_n| \leq L^n n^{n/2}.$$

Let us prove that, on every closed domain of the complex plane, the following limit relations are uniformly satisfied for some integral functions $D(z)$ and $D(x, y, z)$:

$$D(z) = \lim_{\mu \to \infty} D_{k_\mu}(z), \qquad D(x, y, z) = \lim_{\mu \to \infty} D_{k_\mu}(x, y, z).$$

We can suppose that $|z| \leq M$ for some large M. Take some positive $\varepsilon > 0$ and some natural N that will be specified below. We have

$$\Big| \sum_{n=N+1}^{\infty} \frac{b_n \lambda^n}{n!} \Big| \leq \sum_{n=N+1}^{\infty} \frac{(|\lambda| \, L\sqrt{n})^n}{n!} \leq \sum_{n=N+1}^{\infty} \Big(\frac{eLM}{\sqrt{N}} \Big)^n.$$

Taking now N large enough so that

$$\left| \sum_{n=N+1}^{\infty} \frac{b_n \lambda^n}{n!} \right| \leq 0.25\varepsilon$$

and noting that the Nth reminder of the series $D_k(z)$ has the same bound, we get the needed result,

$$\left|D(\lambda) - D_{k_\mu}(\lambda)\right| \leq 0.5\varepsilon + 0.5\varepsilon + \sum_{n=1}^{N} \frac{\left|b_n - b_{n,k_\mu}\right| M^n}{n!} \leq \varepsilon + eM^N \max_{n \leq N} \left|b_n - b_{n,k_\mu}\right|,$$

because taking sufficiently large μ we can make the last expression less than 2ε. Indeed, if we write the expansion of the determinant for b_{k_μ} with term-by-term integration of the obtained sum, then one can note that, for $n \leq N$, the relation $\lim_{p\to\infty} b_{n,k_\mu} \to b_n$ is satisfied as $\mu \to \infty$. Similarly, one can establish the relation

$$D(x, y, z) = \lim_{\mu\to\infty} D_{k_\mu}(x, y, z).$$

Uniformly in the disc $|z| \leq M$ follows From the bounds obtained above for the coefficients of the series, and we deduce that the all analytic functions above given by the power series are integral functions.

As in the theory of ordinary Fredholm integral equations, the roots of the function $D(\lambda)$ play an essential role. Here we at first consider the case where λ is a real number does not satisfying the equation $D(\lambda) = 0$.

Theorem 2. *Let λ be a real number such that $D(\lambda) \neq 0$ and $f(x) \in CB(\mathbf{R})$. Then equation (1.2) has a solution defined by*

$$\varphi(x) = f(x) + \lim_{\mu\to\infty} \frac{1}{T_{k_\mu}} \int_0^{T_{k_\mu}} f(\xi) \frac{D(x, \xi; \lambda)}{D(\lambda)} d\xi. \tag{3.6}$$

Proof. Let us consider equation (2.3) for some fixed k, and take $T = T_\mu = T_{k_\mu}$ for simplicity of notations. Then

$$\varphi(x) = f(x) + \frac{\lambda}{T_\mu} \int_0^{T_\mu} K(x, \xi)\varphi(\xi) d\xi. \tag{3.7}$$

Let λ be a real number such that $D(\lambda) \neq 0$. Let λ_0 be the nearest to the number λ zero of the function $D(\lambda)$, and set $|\lambda_0 - \lambda| = r$. By a theorem of Hurwitz [5, p.128], there can exist not more than a finite number of zeroes of the functions $D_{k_\mu}(\lambda)$ in the neighborhood $|\lambda - z| \leq r/2$ of the number λ. For this reason, for sufficiently large values of k, we have $D_k(\lambda) \neq 0$ (and, moreover, the number $D_k(\lambda)$ is sufficiently close to $D(\lambda)$).

For such values of k, due to [6, p.305], equation (3.7) has a unique solution given by the formula

$$\varphi_{k_\mu}(x) = f(x) + \int_0^{T_\mu} f(\xi) \frac{\bar{D}_\mu(x, \xi; \lambda/T_\mu)}{\bar{D}_\mu(\lambda/T_\mu)} d\xi,$$

where

$$\bar{D}_\mu(\lambda/T_\mu) = 1 + \sum_{n=1}^{\infty} \frac{a_{n,\mu} \left(\lambda/T_\mu\right)^n}{n!}$$

and

$$a_{n,\mu} = (-1)^n \int_0^{T_\mu} \cdots \int_0^{T_\mu} \begin{vmatrix} K(x_1, \xi_1) & K(x_1, \xi_2) & \cdots & K(x_1, \xi_n) \\ K(x_2, \xi_1) & K(x_2, \xi_2) & \cdots & K(x_2, \xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, \xi_1) & K(x_n, \xi_2) & \cdots & K(x_n, \xi_n) \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_n.$$

The function $\bar{D}_\mu(x,y;\lambda/T_\mu)$ is defined by

$$\bar{D}_\mu(x,y;\lambda/T_\mu) = (\lambda/T_\mu)D_\mu(\lambda/T_\mu)K(x,y) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{Q_{n,\mu}(x,y)(\lambda/T_\mu)^{n+1}}{n!}, \qquad x,y\in\mathbf{R},$$

where

$$Q_{n,\mu}(x,y) = -n\int_0^{T_\mu}\cdots\int_0^{T_\mu} P_{n,\mu}(x,\xi,\xi_1,...,\xi_{n-1})K(\xi,y)d\xi d\xi_1\cdots d\xi_{n-1}$$

and

$$P_{n,\mu}(x,\xi,\xi_1,...,\xi_{n-1}) = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_{n-1}) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_{n-1},\xi) & K(\xi_{n-1},\xi_1) & \cdots & K(\xi_{n-1},\xi_{n-1}) \end{vmatrix}.$$

It is not difficult to note that the expressions for $\bar{D}_\mu(\lambda/T_\mu)$ and $D_{k_\mu}(\lambda)$ are identical. The same is true for the functions $\bar{D}_\mu(x,y;\lambda/T_\mu)$ and $D_{k_\mu}(x,y;\lambda)/T_\mu$. So, we have the unique solution

$$\varphi_{k_\mu}(x) = f(x) + \frac{1}{T_\mu}\int_0^{T_\mu} f(\xi)\frac{D_{k_\mu}(x,\xi;\lambda)}{D_{k_\mu}(\lambda)}d\xi.$$

It follows from the expressions for the functions $D_{k_\mu}(\lambda)$ and $D_{k_\mu}(x,y;\lambda)$ and the assumption $f(x)\in CB(\mathbf{R})$ that the conditions of the Theorem 1. By this theorem the sequence $\varphi_{k_\mu}(x)$ of solutions has a subsequence uniformly converging to $\varphi_{k_\mu}(x)$, a solution of equation (1.2) corresponding to the chosen subsequence. Without loss of generality, we can assume that the sequence of the obtained solutions is itself converges to a solution of equation (1.2). So, the equation (1.2) has a solution defined by the relation

$$\varphi_{k_\mu}(x) = f(x) + \frac{1}{T_\mu}\int_0^{T_\mu} f(\xi)\frac{D_{k_\mu}(x,\xi;\lambda)}{D_{k_\mu}(\lambda)}d\xi.$$

It is obvious, due to uniform continuity and boundedness, that we can pass to the limit under the integral and write the solution of the equation as follows:

$$\varphi(x) = f(x) + \lim_{\mu\to\infty}\frac{1}{T_\mu}\int_0^{T_\mu} f(\xi)\frac{D(x,\xi;\lambda)}{D(\lambda)}d\xi. \qquad \square$$

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Mammad Bayramoglu: mamed.bayramoglu@yahoo.com
Institute of Mathematics and Mechanics of NAS Azerbaijan

Ilgar Jabbarov: ilgar_js@rambler.ru
Departmrnt of Geometry and Algebra of Ganja State University, AZ2000, Ganja city, Azerbaijan

Sabina Ibrahimova: ilgar_js@rambler.ru
Departmrnt of Geometry and Algebra of Ganja State University, AZ2000, Ganja city, Azerbaijan

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