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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDERS IN A BANACH SPACE AND THE VANDERMONDE OPERATOR

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Abstract. We study the question of existence of a unique bounded solution to a Cauchy problem for a higher-order differential equation with bounded operator coefficients. The case under consideration is where the corresponding "algebraic" operator equation has separated pairwise commuting roots. Using the Vandermonde operator constructed from such roots, representations for a unique bounded solution and the Cauchy problem are obtained.

Вивчається питання існування єдиного обмеженого рохв'язку задачі Коші для диференціального рівняння вищого порядку з обмеженим оператором коефіцієнтами. Розглядається випадок, в якому відповідне "алгебраїчне" операторне рівняння має відокремлені попарно коммутуючі корені. Використовуючи оператор Вандермонда, який побудований за такими коренями, отримано представлення для єдиного обмеженого розв'язку задачі Коші.

1. Introduction

Let X be a complex Banach space with norm $\|\cdot\|$ and zero element $\bar{0}$, $L(X)$ the Banach algebra of bounded linear operators defined on X, I and O the identity and zero operators in X, correspondingly, $C_b(\mathbb{R}, X)$ the Banach space of functions $x : \mathbb{R} \to X$ continuous and bounded on $\mathbb{R}$ with the norm $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|$, $C_b^{(k)}(\mathbb{R}, X)$ the Banach space of functions $x \in C_b(\mathbb{R}, X)$ that have the k-th derivative, continuous and bounded on $\mathbb{R}$, with the norm $\|x\|_{\infty,k} = \|x\|_\infty + \|x^{(k)}\|_\infty$.

Let us fix a natural number $p \geq 2$ and consider the differential equation

$$x^{(p)}(t) = A_1 x^{(p-1)}(t) + \cdots + A_{p-1} x'(t) + A_p x(t) + y(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $y \in C_b(\mathbb{R}, X)$ and $A_1, A_2, \ldots, A_p$ are fixed operators belonging to $L(X)$. As usual, a bounded solution of equation (1.1) is a function $x \in C_b^{(p)}(\mathbb{R}, X)$ such that identity (1.1) holds for each $t \in \mathbb{R}$.

The purpose of the paper is to find criteria for the operators $A_1, A_2, \ldots, A_p$ such that the differential equation (1.1) has a unique bounded solution x for each function $y \in C_b(\mathbb{R}, X)$.

If $p = 1$, then, according to M.G. Krein's theorem (see [3]) the differential equation

$$x'(t) = A_1 x(t) + y(t), \qquad t \in \mathbb{R}, \tag{1.2}$$

has a unique bounded solution for each function $y \in C_b(\mathbb{R}, X)$ if and only if the spectrum $\sigma(A_1)$ of the operator A_1 does not intersect with the imaginary axis $i\mathbb{R} = \{it \,|\, t \in \mathbb{R}\}$. In this case, the unique bounded solution x of equation (1.2), corresponding to the function $y \in C_b(\mathbb{R}, X)$, is constructed as follows. Let σ_-, σ_+ be the parts of the spectrum $\sigma(A_1)$

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of the operator A_1 lying in the left and right half-planes, respectively (one of them may be empty), $P_\pm$ be the Riesz projections corresponding to $\sigma_\pm$,

$$G_{A_1} = \begin{cases} -e^{A_1 t} P_+, & t < 0, \\ \ \ e^{A_1 t} P_-, & t \geq 0. \end{cases}$$

The bounded solution x is the convolution

$$x(t) = (G_{A_1} * y)(t) = \int_{\mathbb{R}} G_{A_1}(t-s) y(s) ds, \qquad t \in \mathbb{R}. \tag{1.3}$$

Put $X^p = \{\bar{x} = (x_1, x_2, \dots, x_p)^\tau \mid x_k \in X,\ 1 \leq k \leq p\}$. Then X^p is a complex separable Banach space with respect to coordinatewise addition and multiplication by scalar, and the norm $\|\bar{x}\|_* = \sup_{1 \leq k \leq p} \|x_k\|$. If $\{T_{ij},\ 1 \leq i, j \leq p\} \subset L(X)$, then, as in the case of numerical matrices, $T = (T_{ij})_{1 \leq i,j \leq p}$ defines an operator, which belongs to $L(X^p)$, by

$$T\bar{x} = \Big(\sum_{k=1}^{p} T_{1k} x_k, \sum_{k=1}^{p} T_{2k} x_k, \dots, \sum_{k=1}^{p} T_{pk} x_k \Big)^\tau, \qquad \bar{x} \in X^p.$$

Together with equation (1.1), we will consider a first-order differential equation

$$\bar{x}'(t) = T_A \bar{x}(t) + \bar{y}(t),\ t \in \mathbb{R}, \tag{1.4}$$

where $\bar{y} \in C_b(\mathbb{R}, X^p)$ and

$$T_A = \begin{pmatrix} A_1 & A_2 & \dots & A_{p-1} & A_p \\ I & O & \dots & O & O \\ O & I & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots \\ O & O & \dots & I & O \end{pmatrix}.$$

Put $\Delta(\lambda) = I\lambda^p - A_1 \lambda^{p-1} - \dots - A_{p-1}\lambda - A_p,\ \lambda \in \mathbb{C}$. The following theorem holds true.

Theorem 1.1. *The following conditions are equivalent:*

- (i1) *the differential equation* (1.1) *has a unique bounded solution x for each function $y \in C_b(\mathbb{R}, X)$;*
- (i2) *for each $\lambda \in i\mathbb{R}$, the operator $\Delta(\lambda)$ is continuously invertible;*
- (i3) *the differential equation* (1.4) *has a unique bounded solution $\bar{x}$ for each function $\bar{y} \in C_b(\mathbb{R}, X^p)$.*

The proof of Theorem 1.1 is given in Appendix A.

In the general case, verification of the conditions of Theorem 1.1 is not trivial. We will study the case where, for this verification, one can use properties of the "algebraic" operator equation

$$\Lambda^p - A_1 \Lambda^{p-1} - \dots - A_{p-1}\Lambda - A_p = O \tag{1.5}$$

corresponding to equation (1.1). A similar approach was used for the second-order differential equation in [1]. Various existence conditions and properties of bounded solutions of linear differential equations with operator coefficients are studied in [1, 6, 4, 2].

2. Separated roots of equation (1.5) and the Vandermonde operator

Similarly to [1], the roots $\Lambda_1, \Lambda_2, \dots, \Lambda_p$ of the operator equation (1.5) will be called separated if, for any $1 \leq i < j \leq p$, the operator $\Lambda_i - \Lambda_j$ is continuously invertible.

Further, the following assumption is used.

Assumption 2.1. The operator equation (1.5) has separated and pairwise commutable roots $\Lambda_1, \Lambda_2, \dots, \Lambda_p$.

As in [5], the Vandermonde operator corresponding to $\Lambda_1, \Lambda_2, \dots, \Lambda_p$ acts in the space X^p and is defined by the formula

$$W = \begin{pmatrix} \Lambda_1^{p-1} & \Lambda_2^{p-1} & \dots & \Lambda_p^{p-1} \\ \Lambda_1^{p-2} & \Lambda_2^{p-2} & \dots & \Lambda_p^{p-2} \\ \dots & \dots & \dots & \dots \\ \Lambda_1 & \Lambda_2 & \dots & \Lambda_p \\ I & I & \dots & I \end{pmatrix}.$$

For further reasoning, it is important that Assumption 2.1 allows one to generalize properties analogous to the well-known properties of the Vandermonde numerical determinant to the case of the Vandermonde operator. The following theorem contains the properties of the operator W, which will be used in what follows.

Theorem 2.2. *If Assumption 2.1 is true, then:*

(j1) *the operator W has a continuous inverse operator W^{-1};*

(j2) $W^{-1}T_AW = \begin{pmatrix} \Lambda_1 & O & \dots & O \\ O & \Lambda_2 & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & \Lambda_p \end{pmatrix}$;

(j3) $\sigma(T_A) = \bigcup\limits_{k=1}^{p} \sigma(\Lambda_k)$.

Proof. Since the roots $\Lambda_1, \Lambda_2, \dots, \Lambda_p$ are separated, the operator

$$D = \prod_{1 \le i < j \le p} (\Lambda_i - \Lambda_j),$$

which is the same as the Vandermonde determinant in the scalar case, is continuously invertible. Therefore, taking into account the pairwise commutativity of $\Lambda_1, \Lambda_2, \dots, \Lambda_p$, the operator W^{-1} is constructed using algebraic additions, as in the case of scalar matrices. Thus, assertion (j1) is satisfied.

Assertion (j2) follows from (j1) and Corollary 1.1 in [5]. Finally, assertion (j3) is an immediate consequence of (j2). □

Note that if Assumption 2.1 holds, then it follows from assertion (j1) of Theorem 2.2 and Theorems 1.2 and 1.3 in [5] that there is no "algebraic" operator equation of the p-th degree with the identity operator at Λ^p, different from (1.5) , which also has the roots $\Lambda_1, \Lambda_2, \dots, \Lambda_p$.

3. Main results

The properties of the Vandermonde operator are used to prove the following assertions.

Theorem 3.1. *Assume that Assumption 2.1 holds. The differential equation (1.1) has a unique bounded solution x for each function $y \in C_b(\mathbb{R}, X)$ if and only if*

$$\Big(\bigcup_{k=1}^{p} \sigma(\Lambda_k)\Big) \cap i\mathbb{R} = \emptyset. \tag{3.6}$$

In this case, the unique bounded solution x of equation (1.1), corresponding to the function $y \in C_b(\mathbb{R}, X)$, is represented as

$$x = \sum_{k=1}^{p} G_{\Lambda_k} * D_{1k} D^{-1} y, \tag{3.7}$$

where, for each $1 \leq k \leq p$,

$$D_{1k} = (-1)^{k+1} \prod_{\substack{1\leq i<j\leq p \\ i\neq k,\ j\neq k}} (\Lambda_i - \Lambda_j)$$

is an algebraic complement to the element Λ_k^{p-1} *of the first row of the matrix that determines the operator* W.

Proof. Since equation (1.4) is an equation of the form (1.2), the equivalence of condition (3.6) and the condition on the existence of a unique bounded solution to equation (1.1) follows from Theorem 1.1 and assertion (j3) of Theorem 2.2.

Let us check the correctness of representation (3.7). Due to the pairwise commutativity of the roots $\Lambda_1, \Lambda_2, \ldots, \Lambda_p$, as for numerical matrices, the equality

$$\sum_{k=1}^{p} \Lambda_k^j D_{1k} = \delta_{j,p-1} I \tag{3.8}$$

holds for each $0 \leq j \leq p-1$. Here $\delta_{l,m}$ is the Kronecker symbol. Put $x_k = G_{\Lambda_k} * D_{1k} D^{-1} y$, $1 \leq k \leq p$. Taking into account (1.2), (1.4) for each k, $1 \leq k \leq p$, we have

$$x_k'(t) = \Lambda_k x_k(t) + D_{1k} D^{-1} y(t), \qquad t \in \mathbb{R}. \tag{3.9}$$

Therefore, from (3.7), (3.8), (3.9) it follows that for every $t \in \mathbb{R}$

$$\begin{aligned}
x(t) &= \sum_{k=1}^{p} x_k(t), \\
x'(t) &= \sum_{k=1}^{p} \Lambda_k x_k(t) + \Big(\sum_{k=1}^{p} I D_{1k}\Big) D^{-1} y(t) = \sum_{k=1}^{p} \Lambda_k x_k(t), \\
&\ \ \vdots \\
x^{(p-1)}(t) &= \Big(\sum_{k=1}^{p} \Lambda_k^{p-2} x_k(t)\Big)' = \sum_{k=1}^{p} \Lambda_k^{p-1} x_k(t) \\
&\quad + \Big(\sum_{k=1}^{p} \Lambda_k^{p-2} D_{1k}\Big) D^{-1} y(t) = \sum_{k=1}^{p} \Lambda_k^{p-1} x_k(t), \\
x^{(p)}(t) &= \sum_{k=1}^{p} \Lambda_k^{p} x_k(t) + \Big(\sum_{k=1}^{p} \Lambda_k^{p-1} D_{1k}\Big) D^{-1} y(t) = \sum_{k=1}^{p} \Lambda_k^{p} x_k(t) + y(t).
\end{aligned}$$

Hence, taking also into account that $\Lambda_1, \Lambda_2, \ldots, \Lambda_p$ are roots of the operator equation (1.5), we obtain

$$\begin{aligned}
&x^{(p)}(t) - A_1 x^{(p-1)}(t) - \cdots - A_{p-1} x'(t) - A_p x(t) \\
&\qquad = \sum_{k=1}^{p} (\Lambda_k^p - A_1 \Lambda_k^{p-1} - \cdots - A_{p-1}\Lambda_k - A_p) x_k(t) + y(t) = y(t).
\end{aligned}$$

Thus, the function x defined by formula (3.7) is indeed a bounded solution of equation (1.1) corresponding to the function $y \in C_b(\mathbb{R}, X)$. □

Remark 3.2. If $p = 2$, then, by Remark 1.3 in [5], regardless of the commutability of the roots Λ_1, Λ_2, the operator W is continuously invertible if and only if the operator $\Lambda_1 - \Lambda_2$ is continuously invertible. In this case, it is directly verified that instead of Assumption 2.1 in the statement of Theorem 3, it suffices to require that the operator equation $\Lambda^2 - A_1 \Lambda - A_2 = O$ have separated roots Λ_1, Λ_2.

Remark 3.3. It is erroneously stated in [1] that for $p = 2$ the unique bounded solution x of equation (1.1) corresponding to the function $y \in C_b(\mathbb{R}, X)$ has the form

$$x = (\Lambda_1 - \Lambda_2)^{-1}\Lambda_2(G_{\Lambda_2} - G_{\Lambda_1}) * y. \tag{3.10}$$

Indeed, for example, for the numerical differential equation $x''(t) = x'(t)+2x(t)-2,\ t \in \mathbb{R}$, we have $\Lambda_1 = -1,\ \Lambda_2 = 2$ and, due to (3.10),

$$x(t) = -\frac{2}{3}\Big(-\int_t^{\infty} e^{2(t-s)}(-2)ds - \int_{-\infty}^{t} e^{-(t-s)}(-2)ds\Big) = -2, \qquad t \in \mathbb{R}.$$

However, it is easy to verify that this function is not a solution to the differential equation.

Consider now the differential equation

$$x^{(p)}(t) = A^p x(t) + y(t),\ t \in \mathbb{R}, \tag{3.11}$$

where A is a fixed operator belonging to $L(X)$. The corresponding operator equation is written as $\Lambda^p - A^p = O$ and has pairwise commutable roots

$$U_k = u_k A,\ 0 \le k \le p-1, \tag{3.12}$$

where $u_k = \exp\Big(\frac{2\pi k}{p} i\Big),\ 0 \le k \le p-1$, are the p-th roots of 1.

Theorem 3.4. *For equation (3.11) to have a unique bounded solution x for any $y \in C_b(\mathbb{R}, X)$, it is necessary and sufficient that*

$$\Big(\bigcup_{k=0}^{p-1}\{u_k z \mid z \in \sigma(A)\}\Big) \cap i\mathbb{R} = \emptyset. \tag{3.13}$$

In this case, the unique bounded solution x of equation (3.11), corresponding to the function $y \in C_b(\mathbb{R}, X)$, is represented as

$$x = \frac{1}{p}\sum_{k=0}^{p-1} u_k G_{U_k} * A^{-p+1} y. \tag{3.14}$$

Proof. Necessity. Since equation (3.11) has a unique bounded solution x for each function $y \in C_b(\mathbb{R}, X)$, due to Theorem 1.1, the operator $\Delta(\lambda) = I\lambda^p - A^p$ is continuously invertible for each $\lambda \in i\mathbb{R}$. Hence, using the equality $\Delta(0) = -A^p$, we conclude that the operator A has a continuous inverse operator A^{-1} and, therefore, the roots $U_0, U_1, \dots, U_{p-1}$ are separated. Consequently, by Theorem 3.1, condition (3.13) is satisfied.

Sufficiency. It follows from (3.13) that the operator A is continuously invertible and condition (3.6) is satisfied for the operators $U_0, U_1, \dots, U_{p-1}$. Therefore, by Theorem 3.1, equation (3.11) has a unique bounded solution x for each function $y \in C_b(\mathbb{R}, X)$.

Let us check that, for the differential equation (3.11), formula (3.7) is written in the form (3.14). Indeed, taking into account properties of the p-th roots of 1, for each $1 \le k \le p$ we obtain

$$\begin{aligned}
D_{1k}D^{-1} &= (-1)^{1+k}A^{-p+1}\prod_{0\le j\le k-2}(u_j - u_{k-1})^{-1}\prod_{k\le j\le p-1}(u_{k-1} - u_j)^{-1} \\
&= A^{-p+1}u_{k-1}^{-p+1}\prod_{\substack{0\le j\le p-1 \\ j\ne k-1}}\left(u_0 - \frac{u_j}{u_{k-1}}\right)^{-1} \\
&= A^{-p+1}u_{k-1}^{-p+1}\prod_{1\le l\le p-1}(u_0 - u_l)^{-1} = u_{k-1}^{-p+1}D_{11}D^{-1}.
\end{aligned}$$

Consequently,

$$D = \sum_{k=1}^{p} U_{k-1}^{p-1} D_{1k} = \sum_{k=1}^{p} (u_{k-1}A)^{p-1} u_{k-1}^{-p+1} D_{11} = pA^{p-1} D_{11},$$

and hence $D_{11}D^{-1} = \frac{u_{k-1}}{p} A^{-p+1}$. Therefore, representation (3.14) of the unique bounded solution x of equation (3.11) corresponding to the function $y \in C_b(\mathbb{R}, X)$ is correct. $\square$

Theorem 3.5. *If Assumption 2.1 is satisfied, then for any* $x_0, x_0', \ldots, x_0^{(p-1)} \in X$ *and continuous function* $f : [t_0; \infty) \to X$ *the solution of the Cauchy problem*

$$\begin{gathered} x^{(p)}(t) = A_1 x^{(p-1)}(t) + \cdots + A_{p-1} x'(t) + A_p x(t) + f(t), \qquad t \geq t_0, \\ x(t_0) = x_0, \quad x'(t_0) = x_0', \quad \ldots, \quad x^{(p-1)}(t_0) = x_0^{(p-1)}, \end{gathered} \tag{3.15}$$

is represented as

$$\begin{aligned} x(t) = \sum_{k=1}^{p} e^{\Lambda_k (t-t_0)} D^{-1} (D_{pk} x_0 + D_{(p-1)k} x_0' + \cdots + D_{1k} x_0^{(p-1)}) \\ + \sum_{k=1}^{p} \int_{t_0}^{t} e^{\Lambda_k (t-s)} D_{1k} D^{-1} f(s) ds, \qquad t \geq t_0, \end{aligned} \tag{3.16}$$

where D_{ik} *is an algebraic complement to the element* Λ_k^{p-i} *of the* i*-th row of the matrix that determines the operator* W.

Proof. Since $\Lambda_1, \Lambda_2, \ldots, \Lambda_p$ are roots of the operator equation (1.5), taking into account an explicit form of the solution of the Cauchy problem for a first-order differential equation (see [3]), it is directly verified that the function x defined by formula (3.16) is a solution to the Cauchy problem (3.15). $\square$

Appendix A. Proof of Theorem 1.1

Proof. Equivalence of conditions (i2) and (i3) follows from M. G. Krein's theorem and the statement about the spectrum of a polynomial pencil of operators proved in [5].

Let us prove implication (i3)⇒(i1). Fix $y \in C_b(\mathbb{R}, X)$. Let $\bar{x}(t) = (x_1(t), x_2(t), \ldots, x_p(t))^\tau$, $t \in \mathbb{R}$, be the unique bounded solution of equation (1.4) corresponding to $\bar{y}(t) = (y(t), \bar{0}, \ldots, \bar{0})^\tau$, $t \in \mathbb{R}$. Then

$$x_1'(t) = A_1 x_1(t) + \cdots + A_{p-1} x_{p-1}(t) + A_p x_p(t) + y(t), \qquad t \in \mathbb{R},$$

and $x_{k+1}'(t) = x_k(t)$, $t \in \mathbb{R}$, for every $1 \leq k \leq p-1$. Consequently, the differential equation (1.1) has a bounded solution $x_p(t)$, $t \in \mathbb{R}$, corresponding to y.

If, by contradiction, this solution is not unique, then the homogeneous differential equation corresponding to (1.1), has a nonzero bounded solution $u(t)$, $t \in \mathbb{R}$. But then $\bar{u}(t) = (u^{(p-1)}(t), u^{(p-2)}(t), \ldots, u(t))^\tau$, $t \in \mathbb{R}$, is a nonzero bounded solution of the homogeneous equation corresponding to (1.4), which is impossible.

Finally we prove that (i1)⇒(i2). Fix $w \in X$, $\alpha \in \mathbb{R}$. Let $x_\alpha(t)$, $t \in \mathbb{R}$, be the unique bounded solution of equation (1.1) corresponding to $y_\alpha(t) = e^{i\alpha t} w$, $t \in \mathbb{R}$. It is directly verified that then $\bar{x}_\alpha(t) = (x_\alpha^{(p-1)}(t), x_\alpha^{(p-2)}(t), \ldots, x_\alpha(t))^\tau$, $t \in \mathbb{R}$, is a bounded solution of the differential equation (1.4) corresponding to the function $\bar{y}_\alpha(t) = (y_\alpha(t), \bar{0}, \ldots, \bar{0})^\tau$, $t \in \mathbb{R}$.

If, by contradiction, equation (1.4) has several bounded solutions corresponding to $\bar{y}_\alpha$, then the homogeneous differential equation corresponding to (1.4) has some nonzero bounded solution $\bar{u}(t) = (u_1(t), u_2(t), \ldots, u_p(t))^\tau$, $t \in \mathbb{R}$. But then the homogeneous differential equation corresponding to (1.1) also has a nonzero bounded solution $u_p(t),\ t \in \mathbb{R}$. This contradicts condition (i1).

Making the change of function $\bar{x}(t) = e^{i\alpha t}\bar{v}(t)$ in the equation $\bar{x}'(t) = T_A\bar{x}(t)+\bar{y}_\alpha(t),\ t \in \mathbb{R}$, we conclude that the differential equation

$$\bar{v}'(t) = (T_A - i\alpha\boldsymbol{I})\bar{v}(t) + (w, \bar{0}, \dots, \bar{0})^\tau,\ t \in \mathbb{R}, \tag{A.17}$$

where $\boldsymbol{I}$ is the identity operator in X^p, also has a unique bounded solution $\bar{v}(t),\ t \in \mathbb{R}$.

Since, for each fixed $s \in \mathbb{R}$, the function $\bar{v}(t+s),\ t \in \mathbb{R}$, is also a bounded solution of equation (A.17), we see that $\bar{v}(t) = \bar{v}(0),\ t \in \mathbb{R}$. Therefore, it follows from (A.17) that

$$(i\alpha)^p v_p(0) - (i\alpha)^{p-1}A_1 v_p(0) - \dots - A_p v_p(0) = w.$$

Thus, the operator equation $\Delta(i\alpha)u = w$ has a solution $u = v_p(0)$.

If, by contradiction, this solution is not unique, then there is a nonzero element $u \in X$ such that $\Delta(i\alpha)u = \bar{0}$. But then the function $x(t) = e^{i\alpha t}u,\ t \in \mathbb{R}$, is a nonzero bounded solution of the homogeneous differential equation corresponding to (1.1), which contradicts condition (i1).

Thus, the operator $\Delta(i\alpha)$ is continuously invertible by the Banach inverse operator theorem. □

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