Methods of Functional Analysis and Topology
Vol. 28 (2022), no. 4, pp. 302–308
https://doi.org/10.31392/MFAT-npu26_4.2022.03

REAL ALGEBRAIC FUNCTIONS ON CLOSED MANIFOLDS WHOSE REEB GRAPHS ARE GIVEN GRAPHS

NAOKI KITAZAWA

Abstract. In this paper, we construct a real-algebraic function on some closed manifold whose Reeb (Kronrod-Reeb) graph is a graph respecting some algebraic domain: a graph for this is called a Poincaré-Reeb graph.

The Reeb graph of a smooth function is defined as a natural graph which is the quotient space of the manifold of the domain under a natural equivalence relation for some wide and nice class of smooth functions. The vertex set is defined as the set of all connected components containing some singular points of the function: a singular point of a smooth function is a point where the differential vanishes. Morse-Bott functions give very specific cases. The relation is to contract each connected component of each preimage to a point.

Sharko has posed a natural and important problem: can we construct a nice smooth function whose Reeb graph is a given graph? Explicit answers have been given first by Masumoto-Saeki in a generalized manner for closed surfaces. After that various answers have been presented by various researchers and most of them are essentially for functions on closed surfaces and Morse functions such that connected components of preimages that contain no singular points are spheres. Recently the author has also considered questions and answered them in the cases where the preimages are general manifolds.

У статті побудовано дійсну алгебраїчну функцію на деякому замкнутому многовиді, графом Реба (Кронрода-Реба) для якого є граф, який зберігає деяку алгебраїчну область: його графік називається графіком Пуанкаре-Реба.

Граф Реба гладкої функції визначається як природний граф, який є фактор-простором многовида, що відповідає області, відносно природньому відношенню еквівалентності для деякого широкого класу гладких функцій. Множина вершин визначається як множина всіх зв'язаних компонентів, що містять деякі особливі точки функції: особливою точкою гладкою функції є точка, в якій диференціал дорінює нулю. Функції Морсе-Ботта є конкретними випадками таких функцій. Відношення еквівалентності полягає в тому, щоб звести кожен зв'язаний компонент кожного прообразу до точки.

Шарко поставив природну і важливу проблему: чи можемо ми побудувати хорошу гладку функцію, граф Реба якої є заданим графом? Чіткі відповіді були дані спочатку Масумото-Саекі в узагальненому вигляді для замкнутих поверхонь. Після цього були дані відповіді різними дослідниками, і більшість з них були для функцій на замкнутих поверхнях і функцій Морса для випадку, коли зв'язані компоненти прообразів, що не містять особливих точок, є сферами. Нещодавно автор також розглянув і відповів на ці питання в випадках, де прообрази є загальними многовидами.

2020 *Mathematics Subject Classification.* Primary 57R45, 58C05. Secondary 57R19.

Keywords. Kronrod-Reeb graphs. Real algebraic manifolds. Real algebraic domains. Poincaré-Reeb graph.

The author is a member of the project JSPS KAKENHI Grant Number JP22K18267 "Visualizing twists in data through monodromy" (Principal Investigator: Osamu Saeki). Our present study is supported by the project. This work was also supported by the Research Institute of Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University: more precisely, the conference "Singularity theory and its applications (RIMS-Sing 1)" (http://www.math.kobe-u.ac.jp/HOME/saji/math/conf2022/spmon/index.html). The data essentially supporting our present study are all contained in our present paper.

1. Introduction

The *Reeb graph* (*Kronrod-Reeb graph*) of a differentiable function $c : X \to Y$ is a graph whose underlying topology is the quotient space of the manifold of the domain with respect to an equivalence relation defined in the following way: two points $x_1, x_2 \in X$ are equivalent if and only if they are in a same connected component of some preimage $c^{-1}(y)$.

Let W_c denote this quotient space. We define the quotient map $q_c : X \to W_c$ and we can define a map $\bar{c}$ enjoying the relation $c = \bar{c} \circ q_c$ uniquely.

This is regarded as a graph for some wide class of smooth functions. We explain the structure of our Reeb graph for W_c.

Remark 1. We call such graphs simply "Reeb graphs" in situations we consider where "Kronrod-Reeb graphs" are also well-known of course.

Such topological and combinatorial objects have been fundamental and powerful tools in understanding the manifolds; [19] is a related pioneering paper.

We define fundamental terminology, notions, and notation.

For a topological space X having the structure of some cell complex whose dimension is finite, its dimension is uniquely defined as the dimension of the cell complex. We denote it by $\dim X$.

A polyhedron and a CW complex is of course of such a class and a topological manifold is of such a class, being known to have the structure of a CW complex of a finite dimension.

A graph is a 1-dimensional CW complex where the vertex set is the set of all 0-dimensional cells and the edge set is the set of all 1-dimensional cells. If the closure of an edge is homeomorphic to a circle, then it is called a *loop*. Hereafter, we do not consider such graphs and a graph is always a 1-dimensional polyhedron. An *isomorphism* from a graph K_1 onto K_2 means a piecewise smooth homeomorphism mapping the edge set and the vertex set of K_1 onto those of K_2.

For a differentiable map $c : X \to Y$, a *singular* point $x \in X$ is a point where the differential is smaller than both $\dim X$ and $\dim Y$. The value at some singular point of c is a *singular value* of c. If a smooth function on a closed manifold has finitely many singular values, then the Reeb space of it is a graph where the vertex set consists of all points p whose preimages ${q_c}^{-1}(p)$ contain some singular points of c. Morse-Bott functions and smooth functions of some considerably wide classes satisfy this. This is due to [20].

Problem 1. For a graph, can we construct a smooth function on some closed manifold whose Reeb graph is isomorphic to a given one, and which enjoys some nice (differential) topological properties and properties on singularity? We do not need to fix the manifold of the domain.

In [23], this question has been posed for the first time. Smooth functions on closed surfaces have been explicitly constructed for graphs satisfying some nice conditions there. In [14], the author generalizes this for arbitrary graphs. Later, answers were given in [15, 16]. These works study cases of smooth functions on closed surfaces and Morse functions for which each connected component of each preimage containing no singular points is always a sphere, for example. In [10, 11, 3, 4], the author has studied cases where such connected components are general manifolds satisfying mild conditions on singularities of the functions, for example. Paper [21] appears as a paper motivated by [10] and through related informal discussions by us.

Problem 2. Can we construct a smooth map in Problem 1 as a morphism from a nicer or finer category. In other words, can we construct this as a real analytic one, and as a nicer one, real algebraic one, for example?

The category we discuss is always the smooth one. In this paper, we consider Problem 2 and obtain related answers in some specific or explicit situations.

To present our main result, we need some terminology, notions, and notation from the theory of smooth manifolds and (real) algebraic manifolds, for example.

By $\mathbb{R}^k$ we denote the k-dimensional Euclidean space, and this is the simplest smooth manifold of dimension k for an arbitrary integer $k > 0$. This is also the simplest real algebraic manifold, which is also called the (k-dimensional) *real affine space*. It is also a Riemannian manifold equipped with the so-called Euclidean metric. For a point $x \in \mathbb{R}^k$, $||x|| \geq 0$ is the distance between x and the origin $0 \in \mathbb{R}^k$. We denote $\mathbb{R} := \mathbb{R}^1$, and let $S^k := \{x \in \mathbb{R}^{k+1} \mid ||x|| = 1\}$ be the k-dimensional unit sphere. This is a k-dimensional algebraic submanifold of $\mathbb{R}^{k+1}$ which is compact and has no boundary. It is also a smooth submanifold. It is connected for $k \geq 1$, and it is a discrete set with exactly two points for $k = 0$.

An *algebraic domain* D of $\mathbb{R}^k$ is some open subset there such that the boundary of the closure $\overline{D}$ consists of finitely many smooth algebraic submanifolds of dimension $k-1$ or smooth algebraic hypersurfaces with no boundaries.

To simplify our arguments, let us assume the following with $l \geq 0$ being an integer:

- For each hypersurface S_j in the family $\{S_j\}_{j=1}^{l}$, a real polynomial f_{P,S_j} is given so that the zero set and S_j coincide and that the polynomial function $f_{\mathrm{P},S_j} : S_j \to \mathbb{R}$ defined canonically has no singular points on S_j.
- D is assumed to be the intersection $\bigcap_{j=1}^{l} \{x \in \mathbb{R}^k \mid f_{\mathrm{P},S_j}(x) > 0\}$.

For example, the interior Int D^k of $D^k := \{x \in \mathbb{R}^k \mid ||x|| \leq 1\}$ is the simplest example and D^k is the k-dimensional unit disk. This is also a k-dimensional smooth, compact, and connected submanifold. Note that $||x|| = \Sigma_{j=1}^{k} {x_j}^2$, where $x := (x_1, \cdots, x_k)$.

A *Poincaré-Reeb graph* is defined for a pair of an algebraic domain D of the real affine space of dimension $k > 1$ and a canonical projection $\pi_{k,1}$ mapping $(x_1, x_2) \in \mathbb{R}^k$ to $x_1 \in \mathbb{R}$. This can be presented in a more general manner. Hereafter, we mainly respect the preprint [18] and there such cases are discussed. Note that terminologies and situations are different in considerable cases and that here we can argue in a self-contained way.

Definition 1. A *Poincaré-Reeb graph for the pair* $(D, \pi_{k,1})$ is a graph in the real affine space embedded by a piecewise smooth embedding satisfying the following conditions.

- Each edge e intersects each preimage of the projection $\pi_{k,1}$ in a so-called *generic* way or satisfying the "transversality". In other words, each edge is embedded smoothly and for each point p_e in each edge e, the image of the differential at the point and the tangent vector space at the value $v(p_e)$ in the preimage ${\pi_{k,1}}^{-1}(p)$ of a suitable (unique) point p by the projection $\pi_{k,1}$ generate the tangent vector space at the point $v(p_e) \in \mathbb{R}^k$.
- Two points in the closure $\overline{D}$ of D can be defined to be equivalent if and only if they are in a same connected component of the preimage $\overline{D} \bigcap {\pi_{k,1}}^{-1}(p)$ for some point $p \in \mathbb{R}$ and the map obtained by the restriction of the projection to the closure $\overline{D}$. Let π_D denote the restriction to the closure $\overline{D}$. Our Poincaré-Reeb graph for the pair can also be defined as the quotient space with respect to this equivalence relation. This is isomorphic to the Reeb graph of π_D. Furthermore, an isomorphism is defined as the canonically obtained correspondence.
- The vertex set of our Poincaré-Reeb graph for the pair is the union of the set of all singular points of the restrictions of the projection $\pi_{k,1}$ or π_D to all connected components of the boundary $\partial\overline{D} \subset \overline{D}$. This set is also finite.

See also [24, 25] for the related theory, for example. We present our main result. In the following section we prove this result and present related comments as our main content.

Main Theorem 1. *Consider a Poincaré-Reeb graph K for the pair in Definition 1 such that the closure $\overline{D}$ is compact. Take an arbitrary integer $k_0 > k+1$. Then we can construct a real algebraic function on some (k_0-1)-dimensional smooth closed manifold regarded as a smooth real algebraic manifold whose Reeb graph is isomorphic to the graph K as a graph.*

2. On Main Theorem 1.

Proof of Main Theorem 1. Let k_0 be an arbitrary integer satisfying $k_0 > k$ as assumed.

We use $x := (x_1, \cdots, x_k)$ for (local) coordinates for $\mathbb{R}^k$, $y := (y_1, \cdots, y_{k'})$ for (local) coordinates for $\mathbb{R}^{k'}$, where $k' := k_0 - k$.

We take two steps to complete the proof.

STEP 1. Define a set in $M_D \subset \mathbb{R}^{k_0}$, which is a real algebraic hypersurface and a smooth regular compact submanifold of dimension $k_0 - 1$ with no boundary.

First we define $M_{D_0} := \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{k'} = \mathbb{R}^{k_0} \mid \prod_{j=1}^{l}(f_{\mathrm{P},S_j}(x)) - ||y||^2 = 0\}$, where $||y|| = \Sigma_{j=1}^{l} {y_j}^2$.

We show this is also a smooth regular submanifold in $\mathbb{R}^{k_0}$. We consider the partial derivative of the function $\prod_{j=1}^{l}(f_{\mathrm{P},S_j}(x)) - \Sigma_{j=1}^{l} {y_j}^2$ with respect to the variables x_j and y_j. First we take a point $(x_0, y_0) \in M_{D_0}$ such that $\prod_{j=1}^{l}(f_{\mathrm{P},S_j}(x_0)) > 0$. We use $x_0 := (x_{0,1}, \cdots, x_{0,k})$ and $y_0 := (y_{0,1}, \cdots, y_{0,k})$. Here we consider the partial derivative of the function for some y_j and we have the value $2y_j = 2y_{0,j} \neq 0$. The differential of the restriction of the function $\prod_{j=1}^{l}(f_{\mathrm{P},S_j}(x)) - \Sigma_{j=1}^{l} {y_j}^2$ at $(x_0, y_0) \in M_{D_0}$ is not of rank 0 and this is not a singular point of the function.

Second we take a point $(x_{S_a}, y_{S_a}) \in M_{D_0}$ such that $f_{\mathrm{P},S_a}(x_{S_a}) = 0$. By the assumption on the hypersurfaces S_b and the polynomials $f_{\mathrm{P},S_j}(x_{S_b})$, $f_{\mathrm{P},S_{a'}}(x_{S_a}) > 0$ for $a' \neq a$. The polynomial function defined canonically from the polynomial f_{P,S_a} is assumed to have no singular points on S_a. We use $x_{S_a} := (x_{S_a,1}, \cdots, x_{S_a,k})$ and $y_{S_a} := (y_{S_a,1}, \cdots, y_{S_a,k})$. Here we consider the partial derivative of the function with respect to some x_j, and we have a non-zero value represented as the product of the partial derivatives of the function $f_{\mathrm{P},S_a}(x)$ with respect to x_j at (x_{S_a}, y_{S_a}) and the product of $l-1$ numbers defined as the values of polynomials (or the canonically defined polynomial functions) in the family $\{f_{\mathrm{P},S_j}\}_{j=1}^{l}$ at x_{S_a} except for the number $j \neq a$. The differential of the restriction of the function at $(x_{S_a}, y_{S_a}) \in M_{D_0}$ is not of rank 0 and this is not a singular point of the function.

We have shown that M_{D_0} is a smooth regular submanifold by the implicit function theorem.

We define M_D as the set of all points in M_{D_0} such that $x \in \overline{D} \supset D$. We investigate a small neighborhood of each point in M_D.

First we consider a point $p_1 \in D$ and a point $(p_1, q_1) \in M_D$ and take its sufficiently small open neighborhood U_{p_1,q_1} in $\mathbb{R}^{k_0}$. For any point in $M_{D_0} \bigcap U_{p_1,q_1}$, by the definition, it is also a point in M_D. Second we consider a point $p_2 \in \partial\overline{D}$ in the boundary $\partial\overline{D} \subset \overline{D}$ and a point $(p_2, q_2) \in M_D$ and take its sufficiently small open neighborhood U_{p_2,q_2} in $\mathbb{R}^{k_0}$. Take an arbitrary point (p', q') in $M_{D_0} \bigcap U_{p_2,q_2}$. By the definition and the assumption on the hypersurfaces S_b and the polynomials $f_{\mathrm{P},S_j}(x_{S_b})$, we can assert that $f_{\mathrm{P},S_{b'}}(p') > 0$ for $1 \leq b' \leq l$ except for one $b' := {b_0}'$. It follows that $f_{\mathrm{P},S_{{b_0}'}}(p') < 0$ cannot occur due to the form of the function $\prod_{j=1}^{l}(f_{\mathrm{P},S_j}(x)) - \Sigma_{j=1}^{l} {y_j}^2$. We have that (p', q') is also a point in M_D.

We have shown that M_D is also a smooth regular submanifold with $M_D = M_{D_0}$. By the form of the function $\prod_{j=1}^{l}(f_{\mathrm{P},S_j}(x)) - \Sigma_{j=1}^{l} {y_j}^2$ and the compactness of the closure $\overline{D}$, it is also a smooth compact manifold with no boundary.

STEP 2. Define the composition of the restriction of the canonical projection mapping $(x_1, x_2) \in \mathbb{R}^{k_0}$ to $x_1 \in \mathbb{R}^k$ to the submanifold M_D with the restriction of the given projection $\pi_{k,1}$ or π_D in Definition 1.

First restrict the canonical projection mapping $(x_1, x_2) \in \mathbb{R}^{k_0}$ to $x_1 \in \mathbb{R}^k$ to the submanifold M_D. We thus have a surjection onto $\overline{D}$. We restrict this to the preimage of D. By the form of the function, this is regarded as a projection and a submersion. If we restrict it to the preimage of the boundary $\partial\overline{D}$, then, by the form of the function, we have a smooth and real algebraic embedding onto $\partial\overline{D}$. We compose the surjection onto $\overline{D}$ with π_D to have a new real algebraic function.

By our definitions and constructions, we can see that the composition obtained before can be regarded as the desired function for $k_0 > k + 1$, where we need to respect connectedness of the preimages. □

Remark 2. Let us make some comments that could help to understand our arguments more rigorously. Firstly, our resulting function is a function for [21], having finitely many singular values. Secondly, our map on M_D onto $\overline{D}$ can be topologically regarded as a so-called *special generic* map. The class of special generic maps contains Morse functions with exactly two singular points on spheres, or Morse functions in the so-called Reeb's theorem, and canonical projections of unit spheres. See [20] for fundamental theory on special generic maps and some advanced results on manifolds admitting such maps. For construction of special generic maps related to our construction of the map on M_D onto $\overline{D}$, consult also the preprints [2, 1] of the author for example.

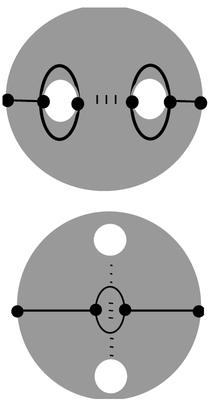

FIGURE 1. Some simplest Poincaré-Reeb graphs for Main Theorem 1. Small dots are used to denote circles, edges, and vertices.

Example 1. Figure 1 shows two simplest explicit cases.

The upper figure shows a Poincaré-Reeb graph for a pair of the algebraic domain surrounded by $l \geq 1$ circles centered at points and having fixed radii and a canonical projection into (a copy of) the 1-dimensional real affine space where $l \geq 1$ is an arbitrary positive integer. It shows a graph with exactly 2 vertices of degree 1, exactly $2(l-1)$ vertices of degree 3, and exactly $2(l-1) + l = 3l - 2$ edges.

The lower figure shows a Poincaré-Reeb graph for a pair of the algebraic domain surrounded by $l \geq 1$ circles centered at points and having fixed radii and a canonical projection into (a copy of) the 1-dimensional real affine space where $l \geq 2$ is an arbitrary integer greater than 1. It shows a graph with exactly 2 vertices of degree 1, exactly 2 vertices of degree $l + 1$, and exactly $l + 2$ edges.

Let us make some remarks related to our result.

Remark 3. In the proof of Main Theorem 1, $||y|| = \Sigma_{j=1}^{l} {y_j}^2$ can be replaced with a polynomial of the form $\Sigma_{j=1}^{l} k_{1,j} {y_j}^{2k_{2,j}}$ with arbitrary positive real numbers $k_{1,j}$ and arbitrary positive integers $k_{2,j}$, for example.

Remark 4. According to the preprint [18], for any graph of some certain wide class, we can obtain some algebraic domain respecting the situation that the underlying 2-dimensional real affine space and a more general projection are given. More precisely, we also have a Poincaré-Reeb graph for the pair of the real affine space and the general projection and the graph is isomorphic to a given graph as a graph. It tries to obtain domains arguing in the topological category or in the class C^r that may not be the class C^{∞} with $r \geq 1$ regarding differentiability. After some arguments, it applies a so-called Weierstrass-type theorem and approximations.

This can give various examples if the algebraic domains satisfy our conditions. However, it is in general difficult to investigate such conditions. See Example 2.2 and Figure 2 of the preprint for an explicit example.

Another remark, which is not directly related to our study in the present paper, is closely related to our future study.

Remark 5. Let $\mathbb{C}^k$ denote the k-dimensional complex space, whose underlying Euclidean space is $2k$-dimensional real affine space. It is also a simplest complex algebraic manifold. Let $\mathbb{C} := \mathbb{C}^1$. It has been difficult to construct very explicit examples of real algebraic functions into $\mathbb{R}$ or maps into higher dimensional real affine spaces for explicitly given closed and connected real algebraic manifolds via explicit polynomial maps. In [22], Sakurai gives an explicit example via celebrating theory of Milnor on links of complex polynomials [17]. He first considers a polynomial function on the 3-dimensional complex space $\mathbb{C}^3$ mapping $(z_1, z_2, z_3) \in \mathbb{C}^3$ to ${z_1}^2 + {z_2}^2 + {z_3}^2 \in \mathbb{C}$ and the link associated with this link is represented as the intersection of the unit sphere S^5 in $\mathbb{R}^6$ and the zero set of the polynomial. He restricts a very explicit complex linear function on the outer complex space $\mathbb{C}^3 = \mathbb{R}^6$ to the link, which is diffeomorphic to the 3-dimensional real projective space, and obtains a smooth map into $\mathbb{R}^2$. This map enjoys nice properties. The image of the set of all singular points of the map is two smoothly and disjointly embedded circles. This is conjectured to be essentially the same as a so-called *round* fold map in [7, 9, 8, 5, 6, 13, 12], by Osamu Saeki and the author.

3. Acknowledgment

The author would like to thank Osamu Saeki and Shuntaro Sakurai for interesting discussions on [22] and construction of explicit smooth maps which are also real algebraic. This and an interesting talk in the conference "Singularity theory and its applications (RIMS-Sing 1)"[1] by M. S. Sorea on [18] and her interesting answers to questions on construction of nice domains and extensions of the presented study and results to graphs of wider classes by the author has motivated us to start studies on our present paper. The author would like to thank anonymous referees.

References

[1] N. Kitazawa, *A note on cohomological structures of special generic maps, a revised version is submitted based on positive comments by referees and editors after the third submission to a refereed journal.*

[2] N. Kitazawa, *Notes on explicit special generic maps into Euclidean spaces whose dimensions are greater than* 4*, a revised version is submitted based on positive comments (major revision) by referees and editors after the first submission to a refereed journal, arXiv:2010.10078.*

[1] http://www.math.kobe-u.ac.jp/home/saji/math/conf2022/spmon/index.html

[3] N. Kitazawa, *On reeb graphs induced from smooth functions on* 3*-dimensional closed manifolds which may not be orientable, a revised version is submitted to a refereed journal after based on positive comments by editors and referees after the second submission to a refreed journal, arXiv:2108.01300*.

[4] N. Kitazawa, *Realization problems of graphs as Reeb graphs of Morse functions with prescribed preimages, arXiv:2108.06913*.

[5] N. Kitazawa, *Round fold maps and the topologies and the differentiable structures of manifolds admitting explicit ones, submitted to a refereed journal, arXiv:1304.0618* (*the title has changed*).

[6] N. Kitazawa, *Round fold maps on* 3*-dimensional manifolds and their integral and rational cohomology rings, submitted to a refereed journal, arXiv:2301.07008*.

[7] N. Kitazawa, *On round fold maps* (*in Japanese*), RIMS Kokyuroku Bessatsu **B38** (2013), 45–59.

[8] N. Kitazawa, *Fold maps with singular value sets of concentric spheres*, Hokkaido Mathematical Journal **43** (2014), no. 3, 327–359.

[9] N. Kitazawa, *On manifolds admitting fold maps with singular value sets of concentric spheres*, Ph.D. thesis, Tokyo Institute of Technology, 2014.

[10] N. Kitazawa, *On Reeb graphs induced from smooth functions on* 3*-dimensional closed orientable manifolds with finitely many singular values*, Topol. Methods in Nonlinear Anal. **57** (2022), no. 2B, 897–912.

[11] N. Kitazawa, *On Reeb graphs induced from smooth functions on closed or open manifolds*, Methods of Functional Analysis and Topology **28** (2022), no. 2, 127–143.

[12] N. Kitazawa and O. Saeki, *Round fold maps on* 3*-manifolds, accepted for publication after a refereeing process and to appear in Algebraic & Geometric Topology, arXiv:2105.00974*.

[13] N. Kitazawa and O. Saeki, *Round fold maps of* n*-dimensional manifolds into* R^{n-1}*, arXiv:2111.13510*, Journal of Singularities **26** (2023), 1–12.

[14] Y. Masumoto and O. Saeki, *A smooth function on a manifold with given Reeb graph*, Kyushu J. Math. **65** (2011), 75–84.

[15] J. M. A. I. S. Meza-Sarmiento and R. Oliveira, *Topological classification of simple Morse Bott functions on surfaces*, Contemp. Math. **675** (2016), no. 4, 165–179.

[16] L. P. Michalak, *Realization of a graph as the Reeb graph of a Morse function on a manifold*, Topol. Methods Nonlinear Anal. **52** (2018), no. 2, 749–762, arXiv:1805.06727.

[17] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, vol. 61, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1968.

[18] A. B. P. Popescu-Pampu and M. S. Sorea, *Poincaré-Reeb graphs of real algebraic domains, https://link.springer.com/article/10.1007/s13163-023-00469-y, arxiv:2207.06871v2*, Revista Matemática Complutense (2023).

[19] G. Reeb, *Sur les points singuliers d´une forme de Pfaff complétement intègrable ou d´une fonction numérique*, Comptes Rendus Hebdomadaires des Séances de I´Académie des Sciences **22** (1946), 847–849.

[20] O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. **49** (1993), 265–293.

[21] O. Saeki, *Reeb spaces of smooth functions on manifolds*, International Mathematics Research Notices **maa301, https://doi.org/10.1093/imrn/maa301** (2021), arXiv:2006.01689.

[22] S. Sakurai, *Master Thesis, Kyushu Univ.*

[23] V. Sharko, *About Kronrod-Reeb graph of a function on a manifold*, Methods of Functional Analysis and Topology **12** (2006), 389–396.

[24] M. S. Sorea, *The shapes of level curves of real polynomials near strict local maxima*, Ph.D. thesis, Université de Lille, Laboratoire Paul Painlevé, 2018.

[25] M. S. Sorea, *Measuring the local non-convexity of real algebraic curves*, J. Symbolic Compute **109** (2022), 482–509.

Naoki kitazawa: n-kitazawa@imi.kyushu-u.ac.jp
Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku Fukuoka 819-0395, Japan

Received 13/02/2023; Revised 16/02/2023