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REGULARIZED SOLUTIONS FOR ABSTRACT VOLTERRA EQUATIONS

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Abstract. The aim of this work is to introduce the domain and the Favard spaces of order α where $\alpha \in]0,1]$ for k-regularized resolvent family, extending some of the well-known theorems for semigroup and resolvent family. Furthermore, we show some relationship between the Favard temporal spaces and the Favard frequential spaces for scalar Volterra linear systems in Banach spaces, extending some results in [8, 3].

Метою цієї роботи є ввести область та простори Фавара порядку α, де $\alpha \in]0,1]$ для k — регуляризованої сім'ї резольвент, та розширити деякі з добре відомих теорем для напівгруп і сімей резольвент. Крім того, ми показуємо деякий взаємозв'язок між часовими просторами Фавара та просторовими просторами Фавара для скалярних лінійних систем Вольтерра в банахових просторах, розширюючи деякі результати в [8, 3].

1. Introduction

A Favard class for semigroups was developed as early as 1967 by B. L. Butzer and H. Berens, which is presented in the monograph [4]. In semigroup theory, the Favard class plays an important role particularly in perturbation theory. The body of knowledge has been increasing steadily since then, and the recent monograph [8] gives a good account of modern developments. Applications appear, in particular, in [7, 16, 19], but are certainly not restricted to this. However, these concepts have been slightly introduced to Volterra integral equations in [11, 14, 3], although they are closely related to perturbation theory, which plays an important role in various fields and has been treated in terms ofo Favard spaces. The aim of this paper is to give an extension to Favard classes for a k-regularized resolvent family of scalar Volterra integral equations similar to the one for semigroups and resolvent families. In fact, we recover several well-known results for semigroups if we consider $k(t) = a(t) = 1$ and for the resolvent family, if we consider $k(t) = 1$ and $a(t)$ is arbitrary.

In Section 2, we give some preliminaries about the concept of a k-regularized resolvent family, and discuss a relationship between linear integral equations of Volterra type with scalar kernel. It is well-known that for a Cauchy problem there are strong relations connecting its semigroup solution and its associated generator. Likewise, for a Volterra scalar problem, there are some results connecting its k-regularized resolvent family and the domain of the associated generator, which will be reviewed in Section 3. In Section 4, we define the Favard spaces "temporal and frequency" for a k-regularized resolvent family of scalar Volterra integral equations, and for these spaces we account for some results which are similar to those of semigroups and resolvent families (see e.g. [8, 3]).

2. Preliminaries

In this section we collect some elementary facts about scalar Volterra equations and a regularized resolvent family. These topics have been covered in detail in [13, 15]. We refer to these works for reference to the literature and further information.

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Let $(X, \|\cdot\|)$ be a Banach space, A be a linear closed densely defined operator in X and $a \in L^1_{loc}(\mathbb{R}^+)$ be a scalar kernel. We consider the linear Volterra equation

$$x(t) = \int_0^t a(t-s)\, Ax(s)ds + f(t), \qquad t \geq 0, \tag{2.1}$$

where $f \in \mathcal{C}(\mathbb{R}^+, X)$.

Definition 2.1. [13, Definition 2.1] Let $k \in \mathcal{C}(\mathbb{R}^+)$. A strongly continuous family $(R(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a k-regularized resolvent family for equation (2.1), if the following three conditions are satisfied:

(R1) $R(0) = k(0)I$.
(R2) $R(t)$ commutes with A, which means $R(t)(D(A)) \subset D(A)$ for all $t \geq 0$, and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.
(R3) For each $x \in D(A)$ and all $t \geq 0$ the resolvent equations hold:

$$R(t)x = k(t)\,x + \int_0^t a(t-s)\, AR(s)xds.$$

If $k(t) = 1$, $(R(t))_{t\geq 0}$ is called a resolvent family.

Definition 2.2. [18, Definition 1.3] Let $a \in L^1_{loc}(\mathbb{R}^+)$. A strongly continuous family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a resolvent family for equation (2.1), if the following three conditions are satisfied:

(S1) $S(0) = I$.
(S2) $S(t)$ commutes with A, which means $S(t)(\mathcal{D}(A)) \subset \mathcal{D}(A)$ for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in \mathcal{D}(A)$ and $t \geq 0$.
(S3) For each $x \in \mathcal{D}(A)$ and all $t \geq 0$ the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s)\, S(s)Axds.$$

Note that the k-regularized resolvent family for (2.1) is uniquely determined and further information on the k-regularized resolvent family can be found by C. Lizama in [13]. We also notice that the choice of the kernel a classifies different families of strongly continuous solution operators in $\mathcal{L}(X)$. For instance if $k(t) = a(t) = 1$, then $R(t)$ corresponds to a C_0-semigroup and if $k(t) = 1$ and $a(t) = t$, then $R(t)$ corresponds to the cosine operator function.

We define the convolution product of a scalar function a with a vector-valued function f by

$$(a * f)(t) := \int_0^t a(t-s)\, f(s)\, ds, \qquad t \geq 0.$$

We start with the following important theorem. In what follows, $\mathcal{R}$ denotes the range of a given operator.

Theorem 2.3. [13, Lemma 2.2] *If (2.1) admits a k-regularized resolvent family $(R(t))_{t\geq 0}$ then $\mathcal{R}(a * R(t)) \subset D(A)$ for all $t \geq 0$ and*

$$R(t)x = k(t)\,x + A\int_0^t a(t-s)\, R(s)xds. \tag{2.2}$$

for each $x \in X$, $t \geq 0$.

From this we obtain that if $(R(t))_{t\geq 0}$ is a k-regularized resolvent family of (2.1), we have $A(a * R)(\cdot)$ is strongly continuous.

Definition 2.4. A k-regularized resolvent family $(R(t))_{t\geq 0}$ is called exponentially bounded, if there exist $M>0$ and $\omega\in\mathbb{R}$ such that $\|R(t)\|\leq Me^{\omega t}$ for all $t\geq 0$ and the pair (M,ω) is called the type of $(R(t))_{t\geq 0}$. The growth bound of $(R(t))_{t\geq 0}$ is

$$\omega_0=\inf\left\{\omega\in\mathbb{R},\ \|R(t)\|\leq Me^{\omega t},\ t\geq 0,\ M>0\right\}.$$

Note that, contrary to the case of C_0-semigroup, a k-regularized resolvent family for (2.1) need not to be exponentially bounded (see [6, 18, 13]). However, there are verifiable conditions guaranteeing that (2.1) possesses an exponentially bounded k-regularized resolvent operator.

Remark 2.5. We note that if $k(t)=1$ and $a\in\mathcal{C}^{\infty}(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$ with $a(0)=1$ and such that $\hat{a}(\lambda)$ admits zeros with arbitrary large real part, problem (2.1) cannot admit an exponentially bounded regularized resolvent (for more details see [18, Page 45-46]).

We will use the Laplace transform at times. Suppose $g:\mathbb{R}^+\to X$ is measurable and there exist $M>0$, $\omega\in\mathbb{R}$, such that $\|g(t)\|\leq Me^{\omega t}$ for almost $t\geq 0$. Then the Laplace transform

$$\widehat{g}(\lambda)=\int_0^{\infty}e^{-\lambda t}g(t)\,dt,$$

exists for all $\lambda\in\mathbb{C}$ with $Re\lambda>\omega$.

A function $a\in L^1_{loc}(\mathbb{R}^+)$ is ω (resp. ω^+)-exponentially bounded if $\int_0^{\infty}e^{-\omega s}|a(s)|\,ds<+\infty$ for some $\omega\in\mathbb{R}$ (resp. $\omega>0$.)

The following proposition stated in [13, Proposition 3.1], establishes a relation between k-regularized resolvent family and Laplace transform.

Proposition 2.6. *Let a, $k\in L^1_{loc}(\mathbb{R}^+)$ be ω-exponentially bounded and let $(R(t))_{t\geq 0}\subset\mathcal{L}(X)$ be a strongly continuous exponentially bounded such that the Laplace transform $\widehat{R}(\lambda)$ exists for $\lambda>\omega$. Then $(R(t))_{t\geq 0}$ is a k-regularized resolvent family of (2.1) if and only if the following conditions hold:*

(1) $\hat{a}(\lambda)\neq 0$ *and* $\frac{1}{\hat{a}(\lambda)}\in\rho(A)$, *for all* $\lambda>\omega$.

(2) $H(\lambda):=\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}$, *called the resolvent associated to* $R(t)$, *satisfies*

$$\left\|H^{(n)}(\lambda)\right\|\leq Mn!(\lambda-\omega)^{-(n+1)}\text{for all }\lambda>\omega\ \text{ and }\ n\in\mathbb{N}$$

Under these assumptions the Laplace-transform of $R(\cdot)$ is well-defined and it is given by $\widehat{R}(\lambda)=H(\lambda)$ for all $\lambda>\omega$.

Remark 2.7. Proposition 2.6 is a well known result.

(1) In the case where $k(t)=1$, if $a(t)=1$, it is the Hille-Yosida theorem and if $a(t)=t$ it is the generation theorem for generators of cosine functions due to Sova and Fattorini [20] and for arbitrary $a(t)$ it is the generation theorem due essentially to Da Prato and Iannelli [17].

(2) In the case where $k(t)=\frac{t^n}{n!}$ and $a(t)=1$, it is the generation theorem for n-times semigroups due to H. Kellermann and M. Hieber [12] and if $k(t)=\frac{t^n}{n!}$ and $a(t)$ is arbitrary, it corresponds to the generation theorem for integrated solutions of Volterra equations due to W. Arendt and H. Kellermann [2].

3. Domain of A

Assuming the existence of a k-regularized resolvent family $(R(t))_{t\geq 0}$ for (2.1), it is natural to ask how to characterize the domain $D(A)$ of the operator A in terms of the k-regularized resolvent family. This is important, for instance in order to study the Favard class in perturbation theory (see [11, 14, 3]). For a very spacial case, the answer to the above question is well-known. For instance, if $k(t) = 1$ and $a(t) = 1$ or $a(t) = t$, A is the generator of a C_0-semi-group $(\mathbb{T}(t))_{t\geq 0}$ or a cosine family $(C(t))_{t\geq 0}$ and we have

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{\mathbb{T}(t)\,x - x}{t} = Ax \right\},$$

and

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} 2 \cdot \frac{C(t)\,x - x}{t^2} = Ax \right\},$$

respectively (see [20, 8]). It was observed in [11] that $D(A)$ has the following characterization in terms of the resolvent family.

Proposition 3.1. *Let (2.1) admit a resolvent family (i.e, a k-regularized resolvent family with $k(t) = 1$) with growth bound ω (such that the Laplace transform of the resolvent exists for $\lambda > \omega$) for ω-exponentially bounded $a \in L^1_{loc}(\mathbb{R}^+)$. Set, for $0 < \theta < \frac{\pi}{2}$ and $\epsilon > 0$,*

$$\Omega_\theta^\epsilon := \left\{ \frac{1}{\widehat{a}(\lambda)} : \ Re\lambda > \omega + \epsilon,\ |arg\lambda| \leq \theta \right\}.$$

Then the following characterization of $D(A)$ holds:

$$D(A) = \left\{ x \in X : \lim_{|\mu|\to\infty,\, \mu\in\Omega_\theta^0} \mu A (\mu I - A)^{-1} x \quad exists \right\}.$$

We give the following characterization of $D(A)$ in terms of a k-regularized resolvent family, which is a consequence of [18, Corollary I.1.6].

Proposition 3.2. *Let (2.1) admit a k-regularized resolvent family with growth bound ω (such that the Laplace transform of the resolvent exists for $\lambda > \omega$) for ω-exponentially bounded $a, k \in L^1_{loc}(\mathbb{R}^+)$. Then the following characterization of $D(A)$ holds:*

$$D(A) = \left\{ x \in X : \lim_{|\lambda|\to\infty} \lambda \widehat{k}(\lambda) \frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \quad exists \right\}.$$

Proof. Under Theorem 2.6 with $n = 0$ and for all $\lambda > \omega$, we have

$$\|H(\lambda)\| \leq \frac{M}{\lambda - \omega},$$

i.e,

$$\left\| \frac{\widehat{k}(\lambda)}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} \right\| \leq \frac{M}{\lambda - \omega},$$

then

$$\lim_{|\lambda|\to\infty} \lambda \widehat{k}(\lambda) \frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \quad \text{exists for all} \quad x \in D(A).$$

□

Without loss of generality we may assume that $\int_0^t |a(t-s)\,k(s)|^p\, ds \neq 0$ for all $t > 0$ and some $1 \leq p < \infty$. Otherwise we would have for some $t_0 > 0$ and $p_0 \geq 1$ that $a(t) = 0$ or $k(t) = 0$ for almost all $t \in [0, t_0]$ and thus, by definition of k-regularized resolvent family, $R(t) = k(t)$ or $R(t) = (a * AR)(t)$ for $t \in [0, t_0]$. This implies that A is bounded, which is the trivial case with $X = D(A)$.

In what follows, we will use, in the forthcoming sections, the following assumption on a and k:

$$\int_0^t |a(t-s)\,k(s)|^p\, ds \neq 0$$

for all $t > 0$ and some $1 \leq p < \infty$. It corresponds to

– [3, Assumption H1] if $k(t) = 1$,
– [15, Assumption Ha] if $p = 1$ and $|k(t)|$ is increasing and satisfies the condition

$$\limsup_{t\mapsto 0^+} \frac{\|R(t)\|}{|k(t)|} < +\infty$$

– [11, Assumption 2.3] if $k(t) = 1$ and $p = 1$.

Assumption A1: There exist $\epsilon_{a,k} > 0$ and $t_{a,k} > 0$ such that, for all $0 < t \le t_{a,k}$, we have

$$\left|\int_0^t a(t-s)\, k(s)\, ds\right| \ge \epsilon_{a,k} \int_0^t |a(t-s)\, k(s)|^p\, ds.$$

This is the case for functions a and k that are positive (resp. $h(I) \subset]0,1]$, with $h(t) = a(t-s)\, k(s),\ s \in [0,t],\ t \in I$) on some interval $I = [0, t_{a,k}[$ for $p = 1$, (resp. $p > 1$).

In fact that, if a and k are positive and $p = 1$, there exist $0 < \epsilon_{a,k} \le 1$ and $t_{a,k} > 0$ such that, for all $0 < t \le t_{a,k}$, we have

$$\int_0^t a(t-s)\, k(s)\, ds \ge \epsilon_{a,k} \int_0^t a(t-s)\, k(s)\, ds.$$

On the other hand for $p > 1$ and $t \in]0,1]$, we have

$$\int_0^t |a(t-s)\, k(s)|^p\, ds = \int_0^t |h(s)|^p\, ds \le \int_0^1 |h(s)|^p\, ds \le 1,$$

so that, for

$$\int_0^t |h(s)|^p\, ds \le \frac{1}{\epsilon_{a,k}} \left|\int_0^t h(s)\, ds\right|,$$

it is necessary and sufficient that

$$1 \le \frac{1}{\epsilon_{a,k}} \left|\int_0^t h(s)\, ds\right|,$$

i.e.,

$$\epsilon_{a,k} \le \left|\int_0^t h(s)\, ds\right|.$$

Then we have

$$\left|\int_0^t h(s)\, ds\right| \ge \epsilon_{a,k},$$

i.e.,

$$\begin{aligned} \frac{1}{\epsilon_{a,k}} \left|\int_0^t h(s)\, ds\right| &\ge 1 \\ &\ge \int_0^t |h(s)|^p\, ds. \end{aligned}$$

Hence, there exist $\epsilon_{a,k} > 0$ and $t_{a,k} > 0$ such that, for all $0 < t \le t_{a,k}$, we have

$$\left|\int_0^t a(t-s)\, k(s)\, ds\right| \ge \epsilon_{a,k} \int_0^t |a(t-s)\, k(s)|^p\, ds.$$

It is easy to see that almost all reasonable functions in applications satisfy this assumption. There are nonetheless examples of functions that do not satisfy this assumption.

Now let us define the set $\widetilde{D}(A)$ as follows:

$$\widetilde{D}(A) := \left\{ x \in X : \ \lim_{t\to 0^+} \frac{R(t)\, x - k(t)\, x}{(k * a)(t)} \ \text{ exists} \right\}.$$

It was proved in [11] under Assumption A1 with $p = 1$ and $k(t) = 1$ and in [15] under Assumption A1 with $p = 1$ and $|k(t)|$ increasing with $\limsup_{t\mapsto 0^+} \frac{\|R(t)\|}{|k(t)|} < +\infty$, respectively, that

$$D(A) = \widetilde{D}(A) = \{x \in X : \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(k*a)(t)} = Ax\}. \tag{3.3}$$

In view of this result we say that the pair (A, a) is a generator of a k-regularized resolvent family $(R(t))_{t\geq 0}$.

Remark 3.3. When $a = 1 + 1 * l$ with $l \in L^1_{loc}(\mathbb{R}^+)$, the Volterra system (2.1) with $f(t) = x_0$ is equivalent to the following integro-differential Volterra system:

$$\dot{x}(t) = Ax(t) + \int_0^t l(t-s)Ax(s)ds, \ \ t \geq 0. \tag{3.4}$$

Furthermore, if (3.4) admits a k-regularized resolvent family $(R(t))_{t\geq 0}$ with $k(0) = 1$, then we have

$$\begin{aligned}\widetilde{D}(A) =& \{x \in X / \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{[k*(1+1*l)](t)} = Ax\}, \\ =& \left\{x \in X / \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{t} = Ax\right\}.\end{aligned}$$

In fact, we have that

$$\lim_{t\to 0^+} \frac{[k*(1+1*l)](t)}{t} = \lim_{t\to 0^+} \frac{1}{t}\left(\int_0^t k(s)\,ds\right) + \lim_{t\to 0^+} \frac{1}{t}\left(\int_0^t (k*l)(s)\,ds\right) = 1 \tag{3.5}$$

and we write

$$\lim_{t\to 0^+} \frac{R(t)x - k(t)x}{[k*(1+1*l)](t)} = \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{t},$$

hence, we obtain

$$\widetilde{D}(A) = \left\{x \in X / \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{t} = Ax\right\}.$$

In the case $k(t) = 1$ and $l \in BV_{loc}(\mathbb{R}^+)$, the space of functions of locally bounded variation, the operator A becomes a generator of a C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$, which is a necessary and sufficient condition for existence of a resolvent family (see [18]). Whence $\widetilde{D}(A)$ is also characterized in terms of $(\mathbb{T}(t))_{t\geq 0}$, and we have

$$\widetilde{D}(A) = \left\{x \in X : \lim_{t\to 0^+} \frac{\mathbb{T}(t)x - x}{t} = Ax\right\} = \left\{x \in X : \lim_{t\to 0^+} \frac{R(t)x - x}{t} = Ax\right\}.$$

(see [18, Corollary 1.4, Page 40] and [3, Remark 3.2]).

4. Favard spaces for k-regularized resolvent family with kernel

The following definition that corresponds to a natural extension, in our context, of the Favard class frequently used in approximation theory for semigroups and resolvent families (see e.g., [16, 8, 14, 3]).

Definition 4.1. Let (2.1) admit a bounded k-regularized resolvent family $(R(t))_{t\geq 0}$ on X, for ω^+-exponentially bounded a, $k \in L^1_{loc}(\mathbb{R}^+)$. For $0 < \alpha \leq 1$ we define the Favard space (frenquency) and the Favard space (temporal) of order α associated to (A,a) as follows:

$$F^{\alpha}(A) := \left\{ x \in X / \sup_{\lambda>\omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\| < \infty \right\},$$
$$= \left\{ x \in X / \sup_{\lambda>\omega} \left\| \lambda^{\alpha} A H(\lambda) x \right\| < \infty \right\}.$$

and

$$\widetilde{F}^{\alpha}(A) := \left\{ x \in X / \sup_{t>0} \frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^{\alpha}} < \infty \right\},$$

Remark 4.2. (i) It is clear that $\widetilde{D}(A) \subset \widetilde{F}^{1}(A)$ and, in virtue of Proposition 3.2, we have $D(A) \subset F^{1}(A)$.

(ii) If $a(t) = k(t) = 1$, we recall that $(R(t))_{t\geq 0}$ corresponds to a bounded C_0-semigroup generated by A. In this situation we obtain

$$F^{\alpha}(A) = \left\{ x \in X / \sup_{\lambda>0} \left\| \lambda^{\alpha} A (\lambda I - A)^{-1} x \right\| < \infty \right\}$$

and $F^{\alpha}(A) = \widetilde{F}^{\alpha}(A)$. This case is well know (see e.g., [8]).

(iii) The Favard class of A with kernel $a(t)$ can alternatively be defined as a subspace of X given by

$$\left\{ x \in X / \limsup_{\lambda\to\infty} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\| < \infty \right\}.$$

As a consequence of $R(t)$ being bounded, the above space coincides with $F^{\alpha}(A)$ in Definition 4.1 and

$$\widetilde{F}^{\alpha}(A) := \left\{ x \in X / \sup_{0<t\leq 1} \frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^{\alpha}} < \infty \right\}.$$

(iv) Let $a = 1 + 1 * l$ with $l \in L^{1}_{loc}(\mathbb{R}^{+})$ and (A,a) being a generator of a bounded k-regularized resolvent family $(R(t))_{t\geq 0}$ with $k(0) \neq 0$ on X. In this case, we have

$$\widetilde{F}^{\alpha}(A) = \left\{ x \in X : \sup_{0<t\leq 1} \frac{\|R(t)x - k(t)x\|}{t^{\alpha}} < \infty \right\}$$

In fact that, by (3.5), we write

$$\begin{aligned}
\widetilde{F}^{\alpha}(A) &= \left\{ x \in X : \sup_{0<t\leq 1} \frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^{\alpha}} < \infty \right\} \\
&= \left\{ x \in X : \sup_{0<t\leq 1} \left(\frac{\|R(t)x - k(t)x\|}{t^{\alpha}} \times \frac{1}{\left| \left(\frac{(k*a)(t)}{t} \right)^{\alpha} \right|} \right) < \infty \right\} \\
&= \left\{ x \in X : \sup_{0<t\leq 1} \frac{\|R(t)x - k(t)x\|}{t^{\alpha}} < \infty \right\}
\end{aligned}$$

(due to $\lim_{t\to 0^{+}} \frac{(k*a)(t)}{t} = constant = k(0)$ with $k(0) \neq 0$).

We prove that $F^{\alpha}(A)$ is stable with respect to $R(t)$ for any scalar kernel a.

Proposition 4.3. *We have $R(t)(F^{\alpha}(A)) \subset F^{\alpha}(A)$, for all $\alpha \in]0,1]$ and $t \geq 0$.*

Proof. For all $x \in D(A)$ and $t \geq 0$, we have by $(R2)$ that

$$AR(t)x = R(t)Ax.$$

Then

$$\frac{1}{\hat{a}(\lambda)} R(t) - AR(t) = \frac{1}{\hat{a}(\lambda)} R(t) - R(t)A,$$

i.e.,

$$\left(\frac{1}{\hat{a}(\lambda)} I - A\right) R(t) = R(t) \left(\frac{1}{\hat{a}(\lambda)} I - A\right).$$

Then it follows from Proposition 2.6 that

$$\hat{a}(\lambda) \neq 0 \,\text{and}\, \tfrac{1}{\hat{a}(\lambda)} \in \rho(A),$$

hence we have

$$\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} R(t) = R(t) \left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1}. \tag{4.6}$$

Now, if $x \in F^{\alpha}(A)$ then

$$\sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\widehat{a}(\lambda)} A \left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x \right\| < \infty.$$

By (4.6) and the boundedness of $R(t)$, we have

$$\begin{aligned}
\sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\widehat{a}(\lambda)} A \left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} R(t)\, x \right\| &= \sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\widehat{a}(\lambda)} AR(t) \left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x \right\| \\
&= \sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\widehat{a}(\lambda)} R(t)\, A \left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x \right\| \\
&\leq \|R(t)\| \sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\widehat{a}(\lambda)} A \left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x \right\| \\
&< +\infty.
\end{aligned}$$

Then $R(t)\, x \in F^{\alpha}(A)$ for all $t \geq 0$, hence we deduce that $R(t)\,(F^{\alpha}(A)) \subset F^{\alpha}(A)$ for all $\alpha \in]0, 1]$ and $t \geq 0$. □

The proof of the following is immediate.

Proposition 4.4. *The Favard classes of order α of A with kernel $a(t)$, $F^{\alpha}(A)$ and $\widetilde{F}^{\alpha}(A)$ are Banach spaces with respect to the norms*

$$\|x\|_{F^{\alpha}(A)} := \|x\| + \sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\|$$

and

$$\|x\|_{\widetilde{F}^{\alpha}(A)} := \|x\| + \sup_{0 < t \leq 1} \frac{\|R(t)\, x - k(t)\, x\|}{|(k * a)(t)|^{\alpha}},$$

respectively.

As for the semigroups and the resolvent families cases (see [8, 3]), we obtain natural inclusions between the Favard classes for different exponents.

Proposition 4.5. *Let (2.1) admit a bounded k-regularized resolvent family $(R(t))_{t \geq 0}$ on X for ω^{+}-exponentially bounded a, $k \in L^1_{loc}(\mathbb{R}^+)$. For all $0 < \beta < \alpha \leq 1$, one has*

(i) $D(A) \subset F^{\alpha}(A) \subset F^{\beta}(A)$.

(ii) $\widetilde{D}(A) \subset \widetilde{F}^{\alpha}(A) \subset \widetilde{F}^{\beta}(A)$.

Proof. (i) Let $x \in F^{\alpha}(A)$, then for all $\lambda > \omega$, one has:

$$\left\| \lambda^{\beta} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\| = \lambda^{\beta-\alpha} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\|$$
$$\leq \frac{1}{\omega^{\alpha-\beta}} \sup_{\lambda > \omega} \left\| \lambda^{\alpha} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\|,$$

which implies that $x \in F^{\beta}(A)$ and from Remark 4.2 (i) we deduce that $D(A) \subset F^{1}(A) \subset F^{\alpha}(A)$.

(ii) Let $x \in \widetilde{F}^{\alpha}(A)$, $0 < t \leq 1$ and $K > 0$ such that $|k(t)| \leq K e^{\omega t}$ for all $0 < t \leq 1$ then

$$\frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^{\beta}} = \frac{1}{|(k*a)(t)|^{\beta-\alpha}} \frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^{\alpha}}$$
$$\leq \left(K e^{\omega} \left(1 * e^{-\omega \cdot} |a(\cdot)| \right)(1) \right)^{\alpha-\beta} \sup_{0<t\leq 1} \frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^{\alpha}}.$$

Hence $x \in \widetilde{F}^{\beta}(A)$ and that $\widetilde{D}(A) \subset \widetilde{F}^{\alpha}(A)$ due to Remark 4.2 (i). □

Note that under Assumption A1 with $p = 1$ and $k(t) = 1$, we have (i) $F^{1}(A) \subset \widetilde{F}^{1}(A)$ (see [11, Assumption 2.3]) where as the inclusion (ii) $\widetilde{F}^{1}(A) \subset F^{1}(A)$ was proved under the a strong assumption in [11, Assumption 3.1].

Note that if the kernel $a \in L^{1}_{loc}(\mathbb{R}^{+})$ is non negative and $k(t) = 1$, then $F^{1}(A) = \widetilde{F}^{1}(A)$ (see [3, Proposition 4.5]).

Now we will prove that $F^{1}(A) = \widetilde{F}^{1}(A)$ holds for all non negative a and k in $L^{1}_{loc}(\mathbb{R}^{+})$ such that k is bounded.

Proposition 4.6. *Let (2.1) admit a bounded k-regularized resolvent family $(R(t))_{t\geq 0}$ on X for ω^{+}-exponentially bounded a, $k \in L^{1}_{loc}(\mathbb{R}^{+})$. We assume that Assumption A1 with $p = 1$ is satisfied and there is a constant N such that $1 \leq k(t) \leq N$ for all $t \geq 0$. Then*

$$F^{1}(A) = \widetilde{F}^{1}(A).$$

Proof. Let Assumption A1 holds with $p = 1$: $\exists \epsilon_{a,k} > 0$ and $t_{a,k} > 0$ such that, for all $0 < t \leq t_{a,k}$,

$$\left| \int_{0}^{t} a(t-s) k(s)\, ds \right| \geq \epsilon_{a,k} \int_{0}^{t} |a(t-s) k(s)|\, ds.$$

Take $x \in F^{1}(A)$, and let $\|R(s)\| \leq M$ for some $M > 0$ and for all $s \in [0, t]$, where $0 < t \leq t_{a,k}$. Since $1 \leq k(t) \leq N$ for all $t \geq 0$, we have for $\lambda > \omega$ that

$$1 \leq \lambda \hat{k}(\lambda) \leq N,$$

which implies that

$$\frac{1}{N} \leq \frac{1}{\lambda \hat{k}(\lambda)} \leq 1.$$

Under Assumption A1 with $p = 1$, we have

$$|(a*k)(t)| = \left| \int_{0}^{t} a(t-s) k(s)\, ds \right| \geq \epsilon_{a,k} \int_{0}^{t} |a(t-s) k(s)|\, ds$$
$$\geq \epsilon_{a,k} \int_{0}^{t} |a(s)|\, ds = \epsilon_{a,k} (1 * |a|)(t).$$

Then

$$\frac{(1*|a|)(t)}{|(k*a)(t)|} \leq \frac{1}{\epsilon_{a,k}}$$

and, by Theorem 2.3 and under the stability of $F^1(A)$ by $R(t)$, we have

$$\begin{aligned}
\frac{\|R(t)x-k(t)x\|}{|(k*a)(t)|} &= \frac{1}{|(k*a)(t)|}\left\|A\int_0^t a(t-s)R(s)xds\right\| \\
&\leq \frac{1}{|(k*a)(t)|}\limsup_{\lambda\mapsto+\infty}\left\|\int_0^t a(t-s)R(s)\frac{1}{\hat{a}(\lambda)}A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}xds\right\| \\
&\leq \frac{1}{|(k*a)(t)|} \\
&\quad\times\limsup_{\lambda\mapsto+\infty}\left\|\int_0^t a(t-s)R(s)\frac{1}{\lambda\hat{k}(\lambda)}\lambda\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}xds\right\| \\
&\leq \frac{M(1*|a|)(t)}{|(k*a)(t)|}\|x\|_{F^1(A)} \\
&\leq \frac{M}{\epsilon_{a,k}}\|x\|_{F^1(A)}.
\end{aligned}$$

Hence, we obtain $x \in \widetilde{F}^1(A)$.

Conversely, let $x \in \widetilde{F}^1(A)$ and set

$$\sup_{0<t\leq 1}\frac{\|R(t)x-k(t)x\|}{|(k*a)(t)|} := J_x < \infty.$$

We write

$$\lambda\cdot\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1} = \lambda AH(\lambda),$$

for all $\lambda > \omega$.

Using the integral representation of the resolvent (see Proposition 2.6) we obtain

$$\begin{aligned}
\lambda AH(\lambda)x &= \frac{\lambda}{\hat{a}(\lambda)}H(\lambda)x - \lambda\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}x \\
&= \frac{\lambda}{\widehat{a}(\lambda)}\left[H(\lambda)x-\hat{k}(\lambda)x\right] \\
&= \frac{\lambda}{\widehat{a}(\lambda)}\left[\hat{R}(\lambda)x-\hat{k}(\lambda)x\right] \\
&= \frac{\lambda}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda s}(R(s)x-k(s)x)ds \\
&= \frac{\lambda}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda s}(k*a)(s)\frac{R(s)x-k(s)x}{(k*a)(s)}ds.
\end{aligned}$$

Since $\left|\lambda\hat{k}(\lambda)\right| \leq N$ for $\lambda > \omega$, $\|R(t)\| \leq M$ for some $M > 0$, for all $t \geq 0$ and under the hypothesis Assumption A1 with $p = 1$ and $1 \leq k(t) \leq N$ for all $t \geq 0$, we have

$$\begin{aligned}
\|\lambda AH(\lambda)x\| \leq & \left|\frac{\lambda}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda s}(k*a)(s)\,ds\right| \sup_{t>0}\frac{\|R(t)x-k(t)x\|}{|(k*a)(t)|} \\
\leq & \left|\frac{\lambda}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda s}(k*a)(s)\,ds\right| (L\|x\| + \sup_{0<t\leq 1}\frac{\|R(t)x-k(t)x\|}{|(k*a)(t)|}) \\
= & \left|\frac{\lambda}{\widehat{a}(\lambda)}\widehat{k*a}(\lambda)\right| (L\|x\| + J_x) \\
= & \left|\frac{\lambda}{\widehat{a}(\lambda)}\widehat{k}(\lambda)\widehat{a}(\lambda)\right| (L\|x\| + J_x) \\
= & \left|\lambda\hat{k}(\lambda)\right| (L\|x\| + J_x) \\
\leq & N(L\|x\| + J_x),
\end{aligned}$$

with $L = \frac{M+N}{\epsilon_{a,k}(1*|a|)(1)}$. This implies that $\sup_{\lambda>\omega}\|\lambda AH(\lambda)x\| < \infty$, which ends the proof. □

Note that in the semigroup case, i.e., for $a(t) = k(t) = 1$, we have the well-known result that $\widetilde{F}^\alpha(A) = F^\alpha(A)$ (see. [8]) and in the resolvent families case, we have $\widetilde{F}^\alpha(A) = F^\alpha(A)$ if the kernel a satisfies Assumption A1 with $p = 1$ and $k(t) = 1$ and if $\lambda\hat{a}(\lambda)$ is bounded for all $\lambda > \omega$ and under another assumption [3, Proposition 4.7].

In what follows, we investigate conditions on the kernels a and k to prove that this is the case for the (A, a) generator of the k-regularized resolvent families.

Note that for all ω^+-exponentially bounded functions a and k, it is clear that $(k*a)^\alpha$ is also ω^+-exponentially bounded (due to $x^\alpha \leq 1 + x$ for $x \geq 0$ and $\alpha \in]0, 1]$).

We will consider the following assumption on a, $k \in L^1_{loc}(\mathbb{R}^+)$ and $0 < \alpha \leq 1$.

Assumption A2: a and k are ω^+-exponentially bounded and there exists $\epsilon_{a,\alpha} > 0$, such that for all $\lambda > \omega$

$$|\hat{a}(\lambda)| \geq \epsilon_{a,\alpha}\lambda^\alpha \int_0^\infty e^{-\lambda t}|(k*a)(t)|^\alpha\,dt.$$

Example 4.7. (i) The famous case $a(t) = k(t) = 1$ satisfies the condition of Assumption A2 for all $\alpha \geq 0$ due to

$$\frac{\lambda^\alpha}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda t}((k*a)(t))^\alpha dt = \Gamma(\alpha+1) \quad \text{for all } \lambda > 0,$$

which corresponds to the semigroup case (here Γ denotes the Gamma function).

(ii) Consider the standard kernel $a(t) = 1$ and $k(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $\beta > 1$. a is nonnegative and, for all $\lambda > 0$,

$$\frac{\lambda^\alpha}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda t}((k*a)(t))^\alpha dt = \frac{\lambda^{\alpha(1-\beta)}\Gamma(\alpha\beta+1)}{(\beta\Gamma(\beta))^\alpha},$$

Thus a and k satisfy Assumption A2 for $\beta > 1$ and $\alpha \in]0, 1]$. This case corresponds to a β-times integrated semigroup. (see [9, 5, 10]).

(iii) Let $a(t) = t^\beta$ and $k(t) = t^\gamma$, $-1 < \beta < 0$, $\gamma > 1$. We have $\widehat{a}(\lambda) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}$ for all $\lambda > 0$ and $(k*a)(t) = \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{\Gamma(\gamma+\beta+1)}t^{\gamma+\beta+1}$ for all $t \geq 0$. Hence

$$\begin{aligned}
\frac{\lambda^\alpha}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda t}((k*a)(t))^\alpha dt = & \lambda^{\alpha+\beta+1}\frac{(\Gamma(\gamma+1)\Gamma(\beta+1))^\alpha}{\Gamma(\beta+1)(\Gamma(\gamma+\beta+1))^\alpha}\int_0^\infty e^{-\lambda t}t^{\alpha(\gamma+\beta+1)}dt \\
= & \lambda^{\beta-\alpha(\gamma+\beta)}\frac{(\Gamma(\gamma+1))^\alpha\Gamma(\alpha\gamma+\alpha\beta+\alpha+1)}{(\Gamma(\beta+1))^{1-\alpha}(\Gamma(\gamma+\beta+1))^\alpha}.
\end{aligned}$$

Then a and k satisfy Assumption A2 for $-1<\beta<0$ and $\gamma>1$.

(iv) Let $a(t)=\mu+\nu t^{\beta}$, $0<\beta<1$, $\mu>0$, $\nu>0$, and $k(t)=1$. Then we have $\widehat{a}(\lambda)=\frac{\mu}{\lambda}+\frac{\nu}{\lambda^{\beta+1}}\Gamma(\beta+1)$ for $\lambda>0$ and $(k*a)(t)=\mu t+\nu\frac{t^{\beta+1}}{\beta+1}$. Further, for $\alpha\in]0,1]$ we have

$$\begin{aligned}\frac{\lambda^{\alpha}}{\widehat{a}(\lambda)}\int_0^{\infty}e^{-\lambda t}((k*a)(t))^{\alpha}dt=&\frac{\lambda^{\alpha}}{\widehat{a}(\lambda)}\int_0^{\infty}e^{-\lambda t}(\mu t+\nu\frac{t^{\beta+1}}{\beta+1})^{\alpha}dt\\ =&\frac{\lambda^{\alpha}}{\widehat{a}(\lambda)}\int_0^{1}e^{-\lambda t}(\mu t+\nu\frac{t^{\beta+1}}{\beta+1})^{\alpha}dt+\frac{\lambda^{\alpha}}{\widehat{a}(\lambda)}\int_1^{\infty}e^{-\lambda t}(\mu t+\nu\frac{t^{\beta+1}}{\beta+1})^{\alpha}dt\\ \le&(\mu+\frac{\nu}{\beta+1})^{\alpha}\frac{\Gamma(\alpha+1)}{\mu}+(\mu+\frac{\nu}{\beta+1})^{\alpha}\frac{\Gamma(\alpha\beta+\alpha+1)}{\mu}\lambda^{-\alpha\beta}.\end{aligned}$$

Thus Assumption A2 is satisfied. Note that, in a particular case where $\beta=1$, $a(t)=\mu+\nu t$, Eq. (2.1) corresponds to a model of a solid of Kelvin-Voigt (see [18]).

(v) Let $a(t)=1$ and $k=1+1*l$ with $l(t)=e^{-t}$. We have $\widehat{a}(\lambda)=\frac{1}{\lambda}$ for all $\lambda>0$ and $(k*a)(t)=2t+e^{-t}-1\le 2t$ for all $t\ge 0$. Hence,

$$\begin{aligned}\frac{\lambda^{\alpha}}{\widehat{a}(\lambda)}\int_0^{\infty}e^{-\lambda t}((k*a)(t))^{\alpha}dt\le&\lambda^{\alpha+1}\int_0^{\infty}e^{-\lambda t}(2t)^{\alpha}dt,\\ =&2^{\alpha}\Gamma(\alpha+1).\end{aligned}$$

Thus a and k satisfy Assumption A2.

(vi) Let $a(t)=1$ and $k=1+1*l$ with $l(t)=-e^{-t}$. We have $\widehat{a}(\lambda)=\frac{1}{\lambda}$ for all $\lambda>0$ and that $(k*a)(t)=1-e^{-t}\le t$ for all $t\ge 0$. Hence

$$\frac{\lambda^{\alpha}}{\widehat{a}(\lambda)}\int_0^{\infty}e^{-\lambda t}((k*a)(t))^{\alpha}dt\le\lambda^{\alpha+1}\int_0^{\infty}e^{-\lambda t}t^{\alpha}dt=\Gamma(\alpha),$$

for all $\lambda>0$. Thus a and k satisfy Assumption A2.

The following result establishes a relation between the spaces $\widetilde{F}^{\alpha}(A)$ and $F^{\alpha}(A)$ and therefore generalizes [8, Proposition 5.12] and [3, Proposition 4.7].

Proposition 4.8. *Let (2.1) admit a bounded k-regularized resolvent family $(R(t))_{t\ge 0}$ on X for ω^{+}-exponentially bounded $a,k\in L^1_{loc}(\mathbb{R}^+)$ and $0<\alpha\le 1$. Assume that $1\le k(t)\le N$ for all $t\ge 0$.*

(i) *If a and k satisfy Assumption A1 and $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ are bounded for $\lambda>\omega$, then $F^{\alpha}(A)\subset\widetilde{F}^{\alpha}(A)$.*

(ii) *If a is nonnegative satisfying Assumption A2, then $\widetilde{F}^{\alpha}(A)\subset F^{\alpha}(A)$.*

Proof. (i) Let $x\in F^{\alpha}(A)$ and $0<t\le 1$. Then $\sup_{\lambda>\omega}\|\lambda^{\alpha}AH(\lambda)x\|=:K_x<\infty$. Using the integral representation of the resolvent (see Proposition 2.6), we obtain

$$\begin{aligned}x=&\frac{1}{\widehat{k}(\lambda)}H(\lambda)x-\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}AH(\lambda)x\quad\text{for }\lambda>\omega,\\ =:&x_{\lambda}-y_{\lambda}.\end{aligned}$$

Since $x_\lambda \in D(A)$ and using $(R2)-(R3)$ we have

$$\begin{aligned}
\|R(t)x_\lambda - k(t)x_\lambda\| &= \left\|\int_0^t a(t-s)R(s)Ax_\lambda ds\right\| \\
&\leq \int_0^t |a(t-s)|\,\|R(s)\|\,\|Ax_\lambda\|\,ds \\
&\leq M\|Ax_\lambda\|\int_0^t |a(s)|\,ds \\
&= M\|\lambda^\alpha AH(\lambda)x\|\left|\frac{1}{\lambda^\alpha \widehat{k}(\lambda)}\right|(1*|a|)(t) \\
&\leq MK_x\left|\frac{1}{\lambda^\alpha \widehat{k}(\lambda)}\right|(1*|a|)(t).
\end{aligned}$$

On the other hand, $(R(t))_{t\geq 0}$ and $k(t)$ are bounded by M and N respectively, and we have

$$\begin{aligned}
\|R(t)y_\lambda - k(t)y_\lambda\| &\leq \|R(t)y_\lambda\| + \|k(t)y_\lambda\| \\
&\leq \|R(t)\|\,\|y_\lambda\| + |k(t)|\,\|y_\lambda\| \\
&\leq (M+N)\|y_\lambda\| \\
&= (M+N)\left\|\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}AH(\lambda)x\right\| \\
&= (M+N)\left|\frac{\widehat{a}(\lambda)}{\lambda^\alpha\widehat{k}(\lambda)}\right|\|\lambda^\alpha AH(\lambda)x\| \\
&\leq (M+N)K_x\left|\frac{\widehat{a}(\lambda)}{\lambda^\alpha\widehat{k}(\lambda)}\right|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{\|R(t)x - k(t)x\|}{|(k*a)(t)|^\alpha} &\leq \frac{MK_x\left|\frac{1}{\lambda^\alpha\widehat{k}(\lambda)}\right|(1*|a|)(t)}{|(k*a)(t)|^\alpha} + \frac{(M+N)K_x\left|\frac{\widehat{a}(\lambda)}{\lambda^\alpha\widehat{k}(\lambda)}\right|}{|(k*a)(t)|^\alpha} \\
&\leq \frac{MK_x}{\epsilon_{a,k}^\alpha}\frac{1}{\lambda^\alpha\left|\widehat{k}(\lambda)\right|}((1*|a|)(t))^{1-\alpha} \\
&\quad + \frac{(M+N)K_x}{\epsilon_{a,k}^\alpha}\left|\frac{\widehat{a}(\lambda)}{\lambda^\alpha\widehat{k}(\lambda)}\right|((1*|a|)(t))^{-\alpha} \\
&\leq \frac{MK_x}{\epsilon_{a,k}^\alpha}\frac{1}{\left|\lambda\widehat{k}(\lambda)\right|}\lambda^{1-\alpha}((1*|a|)(t))^{1-\alpha} \\
&\quad + \frac{(M+N)K_x}{\epsilon_{a,k}^\alpha}\left|\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\right|\lambda^{-\alpha}((1*|a|)(t))^{-\alpha}.
\end{aligned}$$

The third inequality holds, since $1 \leq k(t) \leq N$ for $t \geq 0$ and using Assumption A1 with $p = 1$, $|(k*a)(t)| \geq \epsilon_{a,k}(|k|*|a|)(t)$ and that $\left|\frac{1}{\lambda\widehat{k}(\lambda)}\right| \leq 1$ and $\left|\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\right| \leq K'$ for some $K' > 0$. Substituting $\lambda_t = \frac{N_\omega}{(1*|a|)(t)} > \omega$ for $t \in]0,1]$ ($\lambda_t \to \infty$ as $t \to 0$)

with $N_\omega = 1 + \omega\,(1 * |a|)\,(1)$, we obtain
$$\frac{\|R\,(t)\,x - k\,(t)\,x\|}{|(k*a)\,(t)|^\alpha} \le \frac{MK_xN_\omega^{1-\alpha}}{\epsilon_{a,k}^\alpha} + \frac{(M+N)\,K_xK^{'}N_\omega^{-\alpha}}{\epsilon_{a,k}^\alpha},$$
for all $0 < t \le 1$. Thus $\sup_{0<t\le1}\frac{\|R(t)x-k(t)x\|}{|(k*a)(t)|^\alpha} < \infty$, and hence $x \in \widetilde{F}^\alpha\,(A)$.

(ii) Let $x \in \widetilde{F}^\alpha\,(A)$ be given. Then $\sup_{0<t\le1}\frac{\|R(t)x-k(t)x\|}{|(k*a)(t)|^\alpha} := J_x < \infty$. For $\lambda > \omega$, we write
$$AH\,(\lambda)\,x = \frac{1}{\widehat{a}\,(\lambda)}H\,(\lambda)\,x - \frac{\hat{k}\,(\lambda)}{\hat{a}\,(\lambda)}x.$$
Then
$$\begin{aligned}\lambda^\alpha AH\,(\lambda)\,x =& \frac{\lambda^\alpha}{\widehat{a}\,(\lambda)}H\,(\lambda)\,x - \lambda^\alpha\frac{\hat{k}\,(\lambda)}{\hat{a}\,(\lambda)}x\\ =&\frac{\lambda^\alpha}{\widehat{a}\,(\lambda)}\left[H\,(\lambda)\,x - \hat{k}\,(\lambda)\,x\right]\\ =&\frac{\lambda^\alpha}{\widehat{a}\,(\lambda)}\int_0^\infty e^{-\lambda s}\,(R\,(s)\,x - k\,(s)\,x)\,ds\\ =&\frac{\lambda^\alpha}{\widehat{a}\,(\lambda)}\int_0^\infty e^{-\lambda s}\,((k*a)\,(s))^\alpha\,\frac{R\,(s)\,x - k\,(s)\,x}{((k*a)\,(s))^\alpha}ds.\end{aligned}$$
Using the fact that a and k satisfy Assumption A2, we obtain
$$\|\lambda^\alpha AH\,(\lambda)\,x\| \le \frac{(L_\alpha\,\|x\| + J_x)}{\epsilon_{a,\alpha}} \quad\text{with}\quad L_\alpha = \frac{M+N}{(1*a)^\alpha\,(1)}.$$
Therefore, $\sup_{\lambda>\omega}\|\lambda^\alpha AH\,(\lambda)\,x\| < \infty$ which ends the proof. □

Remark 4.9. Note that conditions Assumption A2 and $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ is bounded for $\lambda > \omega$, are independent.

(i) Let $a\,(t) = t$ and $k\,(t) = 1$. We have $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)} = \frac{\Gamma(2)}{\lambda}$. Then $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ is bounded for all $\lambda > 0$. But
$$\frac{\lambda^\alpha}{\widehat{a}\,(\lambda)}\int_0^\infty e^{-\lambda t}((k*a)\,(t))^\alpha dt = \frac{\lambda^{1-\alpha}}{2^\alpha}\frac{\Gamma\,(2\alpha+1)}{\Gamma\,(2)}$$
Assumption A2 is not satisfying for all $\alpha \in]0,1]$.

(ii) Let $a\,(t) = t^\beta$ and $k\,(t) = t^\gamma,\ -1 < \beta < 0,\ \gamma > 1$. We have $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)} = \lambda^{\gamma-\beta}\frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)}$. Then $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ is not bounded for all $\lambda > 0$, but Assumption A2 is satisfied for all $\alpha \in]0,1]$ (see Example 4.7 (iii)).

Example 4.10. Let $\alpha \in]0,1]$.

(i) Let $a(t) = k\,(t) = 1$. Then $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ is bounded for all $\lambda > 0$ and a satisfies Assumption A1 with $p = 1$. Furthermore a satisfies Assumption A2 (see Example 4.7 (i)) and by virtue of Proposition 4.8 we obtain $F^\alpha\,(A) = \widetilde{F}^\alpha\,(A)$. Hence we recover a result for C_0-semigroups case which corresponds to [8, Proposition 5.12].

(ii) Let $a\,(t) = 1$ and $k = 1 + 1 * l$ with $l\,(t) = e^{-t}$. Then $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)} = \frac{\lambda+1}{\lambda+2}$ which is bounded for all $\lambda > 0$, and a satisfies Assumption A1 with $p = 1$. Furthermore a satisfies Assumption A2 (see Example 4.7 (v)) and by virtue of Proposition 4.8 we obtain $F^\alpha\,(A) = \widetilde{F}^\alpha\,(A)$.

(iii) Let $a\,(t) = t$ and $k\,(t) = 1$. Then $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)} = \frac{\Gamma(2)}{\lambda}$ which is bounded for all $\lambda > 0$ and a satisfies Assumption A1 with $p = 1$ and $k\,(t) = 1$. By virtue of Proposition 4.8 (1) we obtain $F^\alpha\,(A) \subset \widetilde{F}^\alpha\,(A)$.

(iv) Let $a(t) = 1$ and $k = 1 + 1 * l$ with $l \in L^1_{loc}(\mathbb{R}^+)$, ω^+-exponentially bounded. Then $\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ is bounded for all $\lambda > 0$, according to the Riemann-Lebesgue Lemma (see, e.g [1]). If $l(t)$ is negative with $\widehat{l}(0) \geq -1$, then we obtain that $k(t)$ is nonnegative and $0 \leq (k * a)(t) \leq t$. Hence, both the Assumption A1 with $p = 1$ is satisfied, since a and k are nonnegative, and the Assumption A2 is satisfied, since
$$\frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}((k * a)(t))^\alpha dt \leq \Gamma(\alpha + 1)$$
Die to Proposition 4.8, we have $\widetilde{F}^\alpha(A) = F^\alpha(A)$.

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