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SMOOTH BILINEAR FORMS OF $\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ AND $\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})$

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Abstract. We characterize smooth points of unit balls in some spaces of bilinear forms on $\mathbb{R}^2$. We find that for some special cases of hexagonal norms, the set of smooth points of the unit ball of symmetric bilinear forms coincides with the set of those smooth points of the unit ball of bilinear forms that are symmetric.

Надано характеристику гладким точкам одиничних куль в деяких просторах білінійних форм на $\mathbb{R}^2$. Знайдено, що для деяких частинних випадків гексагональних норм множина гладких точок одиничної кулі співпадає з множиною тих гладких точок одиничної кулі білінійних форм, які є симетричними.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E. A point $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\operatorname{sm} B_E$ the set of smooth points of B_E.

A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that $P(x) = L(x, x)$ for every $x \in E$. We denote by $\mathcal{L}(^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x, y)|$. By $\mathcal{L}_s(^2E)$, we donote the closed subspace of $\mathcal{L}(^2E)$ consisting of all continuous symmetric bilinear forms on E. The Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$ is denoted by $\mathcal{P}(^2E)$. For more details on the theory of multilinear mappings and polynomials on a Banach space, we refer to [3].

The main result about smooth points is known as "the Mazur density theorem." Recall that the Mazur density theorem [5, p. 171] says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary.

Choi and Kim [1, 2] initiated and characterized the smooth points of the unit balls of $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_2^2)$. Grecu [4] characterized the smooth 2-homogeneous polynomials on Hilbert spaces. Kim [7] classified the smooth points of the unit ball of $\mathcal{P}(^2d_*(1, w)^2)$, where $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. Kim [6, 8, 9] classified the smooth points of the unit balls of $\mathcal{L}_s(^2\ell^2_\infty), \mathcal{L}_s(^n\ell^m_\infty)$, and $\mathcal{L}_s(^n\ell^m_\infty)$, where $\ell^m_\infty = \mathbb{R}^m$ with the supremum norm. Kim [10] characterized the smooth points of the unit balls of $\mathcal{L}_s(^2\mathcal{L}_s(^2\ell^2_\infty))$. Kim [11] classified the smooth points of the unit ball of $\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm of weight $\frac{1}{2}$.

Let $0 < w_1, w_2 < 1$. We denote by $\mathbb{R}^2_{h(w_1,w_2)}$ the plane with the hexagonal norm

$$\|(x, y)\|_{h(w_1,w_2)} = \max\Big\{|y|, w_1|x| + w_2|y|\Big\}.$$

We denote by $\mathbb{R}^2_{h'(w_1,w_2)}$ the plane with the hexagonal norm

$$\|(x, y)\|_{h'(w_1,w_2)} = \max\Big\{|x|, w_1|x| + w_2|y|\Big\}.$$

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In this paper, we characterize the smooth points of the unit balls of $\mathcal{L}(^2X)$ and $\mathcal{L}_s(^2X)$, where $X = \mathbb{R}^2_{h(w_1,w_2)}$ or $\mathbb{R}^2_{h'(w_1,w_2)}$. Using this, we prove that

$$\mathrm{sm}\, B_{\mathcal{L}_s(^2X)} = \mathrm{sm}\, B_{\mathcal{L}(^2X)} \cap \mathcal{L}_s(^2X).$$

2. Results

Throughout the paper, we let $0 < w_1, w_2 < 1, k_1 = \frac{w_2}{w_1}$ and $k_2 = \frac{1-w_2}{w_1}$. Note that $1 < k_1 + k_2$. Let $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$. Then there are $a, b, c, d \in \mathbb{R}$ such that

$$T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1.$$

For simplicity, we write T by (a, b, c, d). Recently, Kim [12] showed the following Theorems 2.1–2.3.

Theorem 2.1. *Let $0 < w_1, w_2 < 1$ and $T((x_1, y_1),\ (x_2, y_2)) := (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$. Then*

$$\begin{aligned} \|T\| &= \max\Big\{(k_1 + k_2)^2|a|,\ (|a|k_2 + |c|)(k_1 + k_2),\ (|a|k_2 + |d|)(k_1 + k_2), \\ &\qquad |ak_2^2 + b| + |c + d|k_2,\ |ak_2^2 - b| + |c - d|k_2\Big\}. \end{aligned}$$

Note that if $\|T\| = 1$, then $|a| \le w_1^2, |b| \le 1, |c| \le w_1$ and $|d| \le w_1$.

Let $0 < w_1 \le w_2 < 1$. If $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ for some reals a, b, c, d, we denote $T = (a, b, c, d)$.

Note that $\{x_1x_2,\ y_1y_2,\ x_1y_2 + x_2y_1\}$ is a basis for $\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$. If $T = (a, b, c, c) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$, we will write $T = (a, b, c)$.

Theorem 2.2. *Let $0 < w_1, w_2 < 1$ and $f \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Let $\alpha = f(x_1x_2), \beta = f(y_1y_2), u = f(x_1y_2), v = f(x_2y_1)$.*

(a) *If $w_2 < \frac{1}{2}$, then*

$$\begin{aligned} \|f\| &= \max\Big\{|\beta|,\ |w_1^2\alpha \pm w_2^2\beta| + w_1w_2|u \pm v|,\ |w_1^2\alpha \mp w_2(2 - w_2)\beta| + w_1w_2|u \pm v|, \\ &\qquad w_2|\beta| + w_1|u|,\ w_2|\beta| + w_1|v|\Big\}. \end{aligned}$$

(b) *If $\frac{1}{2} \le w_2$, then*

$$\begin{aligned} \|f\| &= \max\Big\{|\beta|,\ |w_1^2\alpha \pm w_2^2\beta| + w_1w_2|u \pm v|,\ w_2|\beta| + w_1|u|,\ w_2|\beta| + w_1|v|, \\ &\qquad (2w_2 - 1)|\beta| + w_1(|u| + |v|),\ |w_1^2\alpha \mp (w_2^2 - w_2 + 1)\beta| + w_1|w_2u \pm (1 - w_2)v|, \\ &\qquad |w_1^2\alpha \mp (w_2^2 - w_2 + 1)\beta| + w_1|(1 - w_2)u \pm w_2v|, \\ &\qquad |w_1^2\alpha \mp w_2(2 - w_2)\beta| + w_1(1 - w_2)|u \pm v|, \\ &\qquad |w_1^2\alpha \mp (3w_2^2 - 4w_2 + 2)\beta| + w_1w_2|u \pm v|\Big\}. \end{aligned}$$

By Theorem 2.1, if $f \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ with $\|f\| = 1$, then

$$|\alpha| \le \frac{1}{w_1^2}, \qquad |\beta| \le 1, \qquad |u| \le \frac{1}{w_1}, \qquad |v| \le \frac{1}{w_1}.$$

Note that $\{x_1x_2,\ y_1y_2,\ x_1y_2 + x_2y_1\}$ is a basis for $\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$. Thus, if $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$, we will write $f = \Big(f(x_1x_2), f(y_1y_2), f(x_1y_2 + x_2y_1)\Big)$.

Theorem 2.3. *Let $0 < w_1, w_2 < 1$ and $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Let $\alpha = f(x_1x_2), \beta = f(y_1y_2), \theta = f(x_1y_2 + x_2y_1)$.*

(a) *If* $w_2 < \frac{1}{2}$, *then*

$$\|f\| = \max\Big\{|\beta|,\ |w_1^2\alpha + w_2^2\beta| + w_1w_2|\theta|,\ |w_1^2\alpha - w_2(2-w_2)\beta| + w_1w_2|\theta|\Big\}.$$

(b) *If* $\frac{1}{2} \leq w_2$, *then*

$$\|f\| = \max\Big\{|\beta|,\ |w_1^2 + \pm w_2^2\beta| + w_1w_2|\theta|,\ (2w_2-1)|\beta| + w_1|\theta|,$$
$$|w_1^2\alpha - w_2(2-w_2)\beta| + w_1(1-w_2)|\theta|,\ |w_1^2\alpha - (3w_2^2 - 4w_2 + 2)\beta| + w_1w_2|\theta|\Big\}.$$

Note that if $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ with $\|f\| = 1$, then

$$|\alpha| \leq \frac{1}{w_1^2}, \qquad |\beta| \leq 1, \qquad |\theta| \leq \frac{1}{w_1}.$$

Theorem 2.4. *Let* $0 < w_1, w_2 < 1$.

(a) *Let* $f = (\alpha, \beta, u, v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$, *and*

$$g_1 = (\alpha, \beta, v, u), \qquad g_2 = (\alpha, -\beta, u, -v), \qquad g_3 = (\alpha, -\beta, -u, v),$$
$$g_4 = (-\alpha, -\beta, u, v), \qquad g_5 = (-\alpha, -\beta, -u, -v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$

Then, $\|f\| = \|g_j\|$ *for* $j = 1, \ldots, 5$.

(b) *Let* $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ *with* $\|T\| = 1, a \geq 0$ *and* $c \geq d \geq 0$. *Then the following are equivalent:*
  1. T *is smooth;*
  2. $T_1 = (a, b, d, c)$ *is smooth;*
  3. $T_2 = (a, -b, c, -d)$ *is smooth;*
  4. $T_3 = (a, -b, -c, d)$ *is smooth;*
  5. $T_4 = (-a, -b, -c, -d)$ *is smooth;*
  6. $T_5 = (a, b, -c, -d)$ *is smooth.*

Proof. (a) It follows that

$$\begin{aligned}
\|g_1\| &= \sup\{|g_1(T)| = |a\alpha + b\beta + dv + cu| : \tilde{T} = (a, b, d, c) \in S_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}\} \\
&= \sup\{|f(T)| = |a\alpha + b\beta + cu + dv| : T = (a, b, c, d) \in S_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}\} \\
&= \|f\|.
\end{aligned}$$

Similarly, $\|f\| = \|g_j\|$ for $j = 2, \ldots, 5$.

(b) We only show that (1)⇔(2) since the proofs of the other cases are similar.

(1)⇒(2): Let $f = (\alpha, \beta, u, v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $f(T_1) = \|f\| = 1$. Let $g_1 = (\alpha, \beta, v, u) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Then $g_1(T) = f(T_1) = 1$. By (a), $\|g_1\| = \|f\| = 1$. Since T is smooth, g_1 is unique. Thus, f is unique. Therefore, T_1 is smooth.

The proof of (2)⇒(1) is similar. □

Theorem 2.5. *Let* $0 < w_1, w_2 < 1$ *and* $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ *with* $\|T\| = 1, a \geq 0$ *and* $c \geq d \geq 0$. *Let*

$$\Omega = \Big\{(k_1+k_2)^2 a,\ (ak_2 + c)(k_1 + k_2),\ |ak_2^2 + b| + (c+d)k_2,\ |ak_2^2 - b| + (c-d)k_2\Big\}.$$

Then, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ *if and only if*

$$\Big((k_1+k_2)^2 a = 1,\ l<1 \text{ for all } l \in \Omega\backslash\{(k_1+k_2)^2 a\}\Big),$$

$$\Big((ak_2+c)(k_1+k_2) = 1,\ a>0,\ c>d,\ l<1 \text{ for all } l\in\Omega\backslash\{(ak_2+c)(k_1+k_2)\}\Big),$$

$$\Big(|ak_2^2+b|+(c+d)k_2=1,\ a>0,\ 0<|ak_2^2+b|<1,\ l<1 \text{ for all } l\in\Omega\backslash\{|ak_2^2+b|+(c+d)k_2\}\Big),$$

$$\Big(|ak_2^2+b|+(c+d)k_2=1,\ a=0,\ w_2\geq\frac{1}{2},\ l<1 \text{ for all } l\in\Omega\backslash\{|ak_2^2+b|+(c+d)k_2\}\Big)$$

$$\text{or}\quad \Big(|ak_2^2-b|+(c-d)k_2=1,\ c>d,\ l<1 \text{ for all } l\in\Omega\backslash\{|ak_2^2-b|+(c-d)k_2\}\Big).$$

Proof. For $X, Y \in \{(k_1+k_2,0),\ (k_2,1),\ (k_2,-1)\}$, we let $\delta_{(X,Y)} \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $\delta_{(X,Y)}(S) = S(X,Y)$ for every $S\in\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$. Note that $\|\delta_{(X,Y)}\| = 1$ and $\delta_{(X_1,Y_1)} \neq \delta_{(X_2,Y_2)}$ if $(X_1,Y_1)\neq(X_2,Y_2)$. Notice that

$$\begin{aligned}
&\delta_{((k_1+k_2,0),(k_1+k_2,0))}(T) = (k_1+k_2)^2 a,\ \delta_{((k_1+k_2,0),(k_2,1))}(T) = (ak_2+c)(k_1+k_2),\\
&\delta_{((k_2,1),(k_1+k_2,0))}(T) = (ak_2+d)(k_1+k_2),\ \delta_{((k_2,1),(k_2,1))}(T) = ak_2^2+b+(c+d)k_2.\\
&\delta_{((k_2,-1),(k_2,1))}(T) = ak_2^2-b+(c-d)k_2,\ \delta_{((k_2,1),(k_2,-1))}(T) = ak_2^2-b+(-c+d)k_2,\\
&\delta_{((k_2,-1),(k_2,-1))}(T) = ak_2^2+b-(c+d)k_2.
\end{aligned}$$

Thus, if there are two $t_1, t_2\in\Omega$ such that $t_1=t_2=1$, then $T\notin \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

By Theorem 2.1,

$$1 \;=\; \|T\| = \max\{l : l\in\Omega\}.$$

Thus, we consider the following four cases:

Case 1. $(k_1+k_2)^2 a = 1, l<1$ for every $l\in\Omega\backslash\{(k_1+k_2)^2 a\}$

Claim. $T\in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha,\beta,u,v)\in\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1=\|f\|=f(T)$.

We will show that $f = \Big(\frac{1}{w_1^2},\ 0,\ 0,\ 0\Big)$. Note that $|b|<1$. Choose $n_0\in\mathbb{N}$ such that

$$\Big|b\pm\frac{1}{n_0}\Big|<1,\quad \Big|ak_2^2+b\pm\frac{1}{n_0}\Big|+(c+d)k_2<1,\quad \Big|ak_2^2-b\pm\frac{1}{n_0}\Big|+(c-d)k_2<1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b\pm\frac{1}{n_0}, c, d\Big)\Big\| = 1.$$

Thus,

$$1 \;\geq\; \Big|f\Big(\Big(a,b\pm\frac{1}{n_0},c,d\Big)\Big)\Big| = \Big|f(T)\pm\Big(\frac{1}{n_0}\Big)\beta\Big| = 1+\Big(\frac{1}{n_0}\Big)|\beta|.$$

Hence, $\beta = 0$.

Note that $c < w_1w_2$ because $1>(ak_2+c)(k_1+k_2)$. Choose $n_0\in\mathbb{N}$ such that

$$\Big|c\pm\frac{1}{n_0}\Big|<w_1w_2,\quad \Big|ak_2+c\pm\frac{1}{n_0}\Big|(k_1+k_2)<1,$$

$$|ak_2^2+b|+\Big|c+d\pm\frac{1}{n_0}\Big|k_2<1,\quad |ak_2^2-b|+\Big|c-d\pm\frac{1}{n_0}\Big|k_2<1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a,b,c\pm\frac{1}{n_0},d\Big)\Big\| = 1.$$

Thus,

$$1 \geq \Big|f\Big(\Big(a, b, c \pm \frac{1}{n_0}, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\frac{1}{n_0}\Big)u\Big| = 1 + \Big(\frac{1}{n_0}\Big)|u|.$$

Hence $u = 0$. Choose $n_0 \in \mathbb{N}$ such that

$$\Big|d \pm \frac{1}{n_0}\Big| < w_1 w_2, \quad \Big|ak_2 + d \pm \frac{1}{n_0}\Big|(k_1 + k_2) < 1,$$

$$|ak_2^2 + b| + \Big|c + d \pm \frac{1}{n_0}\Big|k_2 < 1, \quad |ak_2^2 - b| + \Big|c - d \pm \frac{1}{n_0}\Big|k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b, c, d \pm \frac{1}{n_0}\Big)\Big\| = 1.$$

Thus,

$$1 \geq \Big|f\Big(\Big(a, b, c, d \pm \frac{1}{n_0}\Big)\Big)\Big| = \Big|f(T) \pm \Big(\frac{1}{n_0}\Big)v\Big| = 1 + \Big(\frac{1}{n_0}\Big)|v|.$$

Hence $v = 0$. It follows that

$$1 = a\alpha + b\beta + cu + dv = a\alpha,$$

so, $\alpha = \frac{1}{a} = \frac{1}{w_1^2}$. Thus, $f = \Big(\frac{1}{w_1^2},\ 0,\ 0,\ 0\Big)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Case 2. $(ak_2 + c)(k_1 + k_2) = 1 < l$ for every $l \in \Omega\backslash\{(ak_2 + c)(k_1 + k_2)\}$

Note that $0 \leq a < w_1^2$.

Claim. $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $a > 0$ and $c > d$.

Suppose that $a > 0$ and $c > d$. Then $c < w_1$. We will show that $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha, \beta, u, v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. We will show that $f = \Big(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1},\ 0\Big)$. Note that $|b| < 1$. Indeed, if $|b| = 1$, then $a = c = d = 0$, which is impossible.

Note that $(ak_2 + d)(k_1 + k_2) < l$. Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad d < c - \frac{k_2}{n_0} < c + \frac{k_2}{n_0} < w_1,$$

$$\Big|\Big(a \pm \frac{1}{n_0}\Big)k_2 + d\Big|(k_1 + k_2) < 1, \quad \Big|ak_2^2 + b \pm \frac{k_2^2}{n_0}\Big| + \Big|c + d \pm \frac{k_2}{n_0}\Big|k_2 < 1,$$

$$\Big|ak_2^2 - b \pm \frac{k_2^2}{n_0}\Big| + \Big|c - d \pm \frac{k_2}{n_0}\Big|k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \pm \frac{1}{n_0}, b, c \mp \frac{k_2}{n_0}, d\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a \pm \frac{1}{n_0}, b, c \mp \frac{k_2}{n_0}, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2}{n_0}\Big)u\Big)\Big| \\ &= 1 + \Big|\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2}{n_0}\Big)u\Big|. \end{aligned}$$

Hence, $\alpha = k_2 u$.

Choose $n_0 \in \mathbb{N}$ such that

$$\Big|b \pm \frac{1}{n_0}\Big| < 1, \quad \Big|ak_2^2 + b \pm \frac{1}{n_0}\Big| + (c + d)k_2 < 1, \quad \Big|ak_2^2 - b \pm \frac{1}{n_0}\Big| + (c - d)k_2 < 1.$$

Note that by Theorem 2.1,

$$\left\|\left(a, b \pm \frac{1}{n_0}, c, d\right)\right\| = 1.$$

Thus,

$$1 \geq \left|f\left(\left(a, b \pm \frac{1}{n_0}, c, d\right)\right)\right| = \left|f(T) \pm \left(\frac{1}{n_0}\right)\beta\right| = 1 + \left(\frac{1}{n_0}\right)|\beta|.$$

Hence, $\beta = 0$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < d - \frac{1}{n_0} < d + \frac{1}{n_0} < c, \quad |ak_2^2 + b| + (c + d \pm \frac{1}{n_0})k_2 < 1,$$
$$\left|ak_2 + d \pm \frac{1}{n_0}\right|(k_1 + k_2) < 1, \quad |ak_2^2 - b| + (c - d \mp \frac{1}{n_0})k_2 < 1.$$

Note that by Theorem 2.1,

$$\left\|\left(a, b, c, d \pm \frac{1}{n_0}\right)\right\| = 1.$$

Thus,

$$1 \geq \left|f\left(\left(a, b, c, d \pm \frac{1}{n_0}\right)\right)\right| = \left|f(T) \pm \left(\frac{1}{n_0}\right)v\right| = 1 + \left(\frac{1}{n_0}\right)|v|.$$

Hence, $v = 0$. It follows that

$$1 = a\alpha + b\beta + cu + dv = a\alpha + cu = u(ak_2 + c) = uw_1,$$

so, $u = \frac{1}{w_1}$ and $f = \left(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1},\ 0\right)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $a = 0$.

Let $g_1 = \left(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1},\ 0\right)$, $g_2 = \left(0,\ 0,\ \frac{1}{w_1},\ 0\right) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g_j(T) = 1$ for $j = 1, 2$. By Theorem 2.4, $1 = \|g_j\|$ $j = 1, 2$. Thus, T is not smooth.

Suppose that $c = d$.

We will show that $T \notin \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $g_1 = \left(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1},\ 0\right)$, $g_2 = \left(\frac{1-w_2}{w_1^2},\ 0,\ 0,\ \frac{1}{w_1}\right) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g_j(T) = 1$ for $j = 1, 2$. By Theorem 2.4, $1 = \|g_j\|$ $j = 1, 2$. Thus, T is not smooth.

Case 3. $|ak_2^2 + b| + (c + d)k_2 = 1 > l$ for every $l \in \Omega\backslash\{|ak_2^2 + b| + (c + d)k_2\}$

Note that $0 \leq a < w_1^2$.

Claim. $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $a > 0$ and $0 < |ak_2^2 + b| < 1$.

Let $a > 0$ and $0 < |ak_2^2 + b| < 1$. We will show that $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha, \beta, u, v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. Note that $c+d > 0$.

Suppose that $0 < ak_2^2 + b < 1$.

Note that $ak_2^2 + b + (c + d)k_2 = 1 > l$ for every $l \in A\backslash\{|ak_2^2 + b| + (c + d)k_2\}$. We will show that $f = \left(\frac{(1-w_2)^2}{w_1^2},\ 1,\ \frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\right)$.

Note that $|b| < 1$. Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \left|b \pm \frac{k_2^2}{n_0}\right| < 1, \quad \left|ak_2 + c \pm \frac{k_2}{n_0}\right| < w_1,$$
$$\left|ak_2^2 - b \pm \frac{2k_2^2}{n_0}\right| + (c - d)k_2 < 1.$$

Note that by Theorem 2.1,

$$\left\|\left(a \mp \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c, d\right)\right\| = 1.$$

Thus,

$$\begin{aligned}1 &\geq \Big|f\Big(\Big(a \mp \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(-\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big)\Big| \\ &= 1 + \Big|-\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big|.\end{aligned}$$

Hence, $\alpha = k_2^2\beta$. Choose $n_0 \in \mathbb{N}$ such that

$$\Big|b \pm \frac{k_2}{n_0}\Big| < 1, \quad 0 < c + d - \frac{1}{n_0}, \quad \Big|ak_2 + c \pm \frac{1}{n_0}\Big| < w_1, \quad 0 < ak_2^2 + b \pm \frac{k_2}{n_0} < 1,$$
$$\Big|ak_2^2 - b \pm \frac{k_2}{n_0}\Big| + \Big|c - d \pm \frac{1}{n_0}\Big|k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b \pm \frac{k_2}{n_0}, c \mp \frac{1}{n_0}, d\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned}1 &\geq \Big|f\Big(\Big(a, b \pm \frac{k_2}{n_0}, c \mp \frac{1}{n_0}, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u\Big)\Big| \\ &= 1 + \Big|\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u\Big|.\end{aligned}$$

Hence, $\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u = 0$, so, $u = k_2\beta$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|b \pm \frac{k_2^2}{2n_0}\Big| < 1, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1,$$
$$0 < c + d - \frac{w_1}{2n_0}, \quad 0 < ak_2^2 + b \pm \frac{k_2^2}{2n_0} < 1,$$
$$\Big|ak_2^2 - b \pm \frac{3k_2^2}{2n_0}\Big| + \Big|c - d \pm \frac{k_2}{2n_0}\Big|k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \pm \frac{1}{n_0}, b \mp \frac{k_2^2}{2n_0}, c, d \mp \frac{k_2}{2n_0}\Big)\Big\| = 1.$$

Thus,

$$1 \geq \Big|f\Big(\Big(a \pm \frac{1}{n_0}, b \mp \frac{k_2^2}{2n_0}, c, d \mp \frac{k_2}{2n_0}\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2^2}{2n_0}\Big)\beta - \Big(\frac{k_2}{2n_0}\Big)v\Big)\Big|$$
$$= 1 + \Big|\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2^2}{2n_0}\Big)\beta - \Big(\frac{k_2}{2n_0}\Big)v\Big|.$$

Hence, $\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2^2}{2n_0}\Big)\beta - \Big(\frac{k_2}{2n_0}\Big)v = 0$, so, $v = k_2\beta$. It follows that

$$1 = a\alpha + b\beta + cu + dv = \beta(ak_2^2 + b + (c + d)k_2) = \beta,$$

so, $f = \Big(\frac{(1-w_2)^2}{w_1^2},\ 1,\ \frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\Big)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $-1 < ak_2^2 + b < 0$.

Note that $-ak_2^2 - b + (c + d)k_2 = 1 > l$ for every $l \in A\backslash\{|ak_2^2 + b| + (c + d)k_2\}$. We will show that $f = \Big(-\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\Big)$. Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|b \pm \frac{k_2^2}{n_0}\Big| < 1, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w,$$
$$\Big|ak_2^2 - b \pm \frac{2k_2^2}{n_0}\Big| + (c - d)k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a\mp\frac{1}{n_0}, b\pm\frac{k_2^2}{n_0}, c, d\Big)\Big\|=1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a\mp\frac{1}{n_0}, b\pm\frac{k_2^2}{n_0}, c, d\Big)\Big)\Big| = \Big|f(T)\pm\Big(-\frac{\alpha}{n_0}+\Big(\frac{k_2^2}{n_0}\Big)\beta\Big)\Big| \\ &= 1+\Big|-\frac{\alpha}{n_0}+\Big(\frac{k_2^2}{n_0}\Big)\beta\Big|. \end{aligned}$$

Hence, $\alpha=k_2^2\beta$.

Choose $n_0\in\mathbb{N}$ such that

$$\Big|b\pm\frac{k_2}{n_0}\Big|<1,\quad 0<c+d-\frac{1}{n_0},\quad \Big|ak_2+c\pm\frac{1}{n_0}\Big|<w_1,\quad -1<ak_2^2+b\pm\frac{k_2}{n_0}<0,$$
$$\Big|ak_2^2-b\pm\frac{k_2}{n_0}\Big|+\Big|c-d\pm\frac{1}{n_0}\Big|k_2<1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b\pm\frac{k_2}{n_0}, c\pm\frac{1}{n_0}, d\Big)\Big\|=1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a, b\pm\frac{k_2}{n_0}, c\pm\frac{1}{n_0}, d\Big)\Big)\Big| = \Big|f(T)\pm\Big(\Big(\frac{k_2}{n_0}\Big)\beta+\Big(\frac{1}{n_0}\Big)u\Big)\Big| \\ &= 1+\Big|\Big(\frac{k_2}{n_0}\Big)\beta+\Big(\frac{1}{n_0}\Big)u\Big|. \end{aligned}$$

Hence, $\Big(\frac{k_2}{n_0}\Big)\beta+\Big(\frac{1}{n_0}\Big)u=0$, so, $u=-k_2\beta$.

Choose $n_0\in\mathbb{N}$ such that

$$0<a-\frac{1}{n_0}<a+\frac{1}{n_0}<w_1^2,\quad \Big|b\pm\frac{k_2^2}{2n_0}\Big|<1,\quad \Big|ak_2+c\pm\frac{k_2}{n_0}\Big|<w_1,$$
$$0<c+d-\frac{w_1}{2n_0},\quad -1<ak_2^2+b\pm\frac{k_2^2}{2n_0}<0,$$
$$\Big|ak_2^2-b\pm\frac{3k_2^2}{2n_0}\Big|+\Big|c-d\pm\frac{k_2}{2n_0}\Big|)k_2<1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a\pm\frac{1}{n_0}, b\mp\frac{k_2^2}{2n_0}, c, d\pm\frac{k_2}{2n_0}\Big)\Big\|=1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a\pm\frac{1}{n_0}, b\mp\frac{k_2^2}{2n_0}, c, d\pm\frac{k_2}{2n_0}\Big)\Big)\Big| = \Big|f(T)\pm\Big(\Big(\frac{1}{n_0}\Big)\alpha-\Big(\frac{k_2^2}{2n_0}\Big)\beta+\Big(\frac{k_2}{2n_0}\Big)v\Big)\Big| \\ &= 1+\Big|\Big(\frac{1}{n_0}\Big)\alpha-\Big(\frac{k_2^2}{2n_0}\Big)\beta+\Big(\frac{k_2}{2n_0}\Big)v\Big|. \end{aligned}$$

Hence, $\Big(\frac{1}{n_0}\Big)\alpha-\Big(\frac{k_2^2}{2n_0}\Big)\beta+\Big(\frac{k_2}{2n_0}\Big)v=0$, so, $v=-k_2\beta$. It follows that

$$1=a\alpha+b\beta+cu+dv=\beta(ak_2^2+b-(c+d)k_2)=-\beta,$$

so, $f=\Big(-\frac{(1-w_2)^2}{w_1^2}, -1, \frac{1-w_2}{w_1}, \frac{1-w_2}{w_1}\Big)$. Therefore, $T\in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $|ak_2^2+b|=0$ or 1. We will show that $T\notin \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $|ak_2^2+b|=0$. Let $g^{\pm}=\Big(\pm\frac{(1-w_2)^2}{w_1^2}, \pm 1, \frac{1-w_2}{w_1}, \frac{1-w_2}{w_1}\Big)\in\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g^{\pm}(T)=1$. By Theorem 2.4, $1=\|g^{\pm}\|$. Thus, T is not smooth.

Let $|ak_2^2 + b| = 1$. Then, $c = d = 0$. Let $g^{\pm} = \Big(\pm\frac{(1-w_2)^2}{w_1^2},\ 1,\ \pm\frac{1-w_2}{w_1},\ \pm\frac{1-w_2}{w_1}\Big) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g^{\pm}(T) = 1$. By Theorem 2.4, $1 = \|g^{\pm}\|$. Thus, T is not smooth.

Let $a = 0$.

Claim. $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $w_2 \geq \frac{1}{2}$

Let $w_2 < \frac{1}{2}$ and

$$g_\alpha = \Big(\alpha,\ 1,\ \frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\Big) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)}) \text{ for } \frac{-3w_2^2+4w_2-1}{w_1^2} \leq \alpha \leq \frac{(1-w_2)^2}{w_1^2}.$$

Note that

$$\frac{-3w_2^2+4w_2-1}{w_1^2} < \frac{(1-w_2)^2}{w_1^2}.$$

By Theorem D (a), $\|g_\alpha\| = 1$. Note that $g_\alpha(T) = 1$. Thus, T is not smooth.

Let $w_2 \geq \frac{1}{2}$.

Claim. $b \neq 0$

Suppose not. Then

$$2w_1 \leq c + d = \frac{w_1}{1-w_2} \leq 2c < 2w_1,$$

which is impossible. Thus, the claim holds.

Suppose that $b > 0$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < b \pm \frac{k_2}{n_0} < 1, \quad 0 < c + d - \frac{1}{n_0}, \quad \Big|c \pm \frac{1}{n_0}\Big| < w_1,$$
$$\Big|-b \pm \frac{k_2}{n_0}\Big| + \Big|c - d \pm \frac{1}{n_0}\Big|k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b \pm \frac{k_2}{n_0}, c \mp \frac{1}{n_0}, d\Big)\Big\| = 1 = \Big\|\Big(a, b \pm \frac{k_2}{n_0}, c, d \mp \frac{1}{n_0}\Big)\Big\|.$$

Thus,

$$\begin{aligned}
1 &\geq \Big|f\Big(\Big(a, b \pm \frac{k_2}{n_0}, c \mp \frac{1}{n_0}, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u\Big)\Big| \\
&= 1 + \Big|\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u\Big|, \\
1 &\geq \Big|f\Big(\Big(a, b \pm \frac{k_2}{n_0}, c, d \mp \frac{1}{n_0}\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)v\Big)\Big| \\
&= 1 + \Big|\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)v\Big|,
\end{aligned}$$

Hence, $\Big(\frac{k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u = 0$, so, $u = v = k_2\beta$.

It follows that

$$1 = a\alpha + b\beta + cu + dv = \beta(b + (c+d)k_2) = \beta.$$

Thus, $u = v = \frac{1-w_2}{w_1}$ and

$$f = \Big(\alpha,\ 1,\ \frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\Big).$$

By some calculation, by Theorem 2.4 (b), $\alpha = \frac{(1-w_2)^2}{w_1^2}$. Thus, T is smooth.

Analogous arguments as in the case $b > 0$ lead to

$$f = \Big(-\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\Big).$$

Thus, the claim holds.

Case 4. $|ak_2^2 - b| + (c-d)k_2 = 1 > l$ for every $l \in \Omega \backslash \{|ak_2^2 - b| + (c-d)k_2\}$.

Note that $0 < a < w_1^2$ and $b < 0$.

Claim. $T \in \text{sm}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $c > d$.

Suppose that $c > d$. We will show that $T \in \text{sm}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha, \beta, u, v) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. We will show that $f = \Big(\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{1-w_2}{w_1},\ -\frac{1-w_2}{w_1}\Big)$.

Note that $-1 < b < 0$. Indeed, if $b \geq 0$, then

$$1 > |ak_2^2 + b| + (c+d)k_2 \geq |ak_2^2 - b| + (c-d)k_2 = 1,$$

which is a contradiction. If $b = -1$, then $a = c = 0 = d$, which is a contradiction.

Note that $0 < |ak_2^2 - b| = ak_2^2 - b < 1$, and $ak_2^2 - b + (c-d)k_2 = 1$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad -1 < b - \frac{k_2^2}{n_0} < b + \frac{k_2^2}{n_0} < 0, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1,$$
$$\Big|ak_2^2 + b \pm \frac{2k_2^2}{n_0}\Big| + (c+d)k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \pm \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c, d\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a \pm \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big)\Big| \\ &= 1 + \Big|\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big|. \end{aligned}$$

Hence, $\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta = 0$, so, $\alpha = -k_2^2\beta$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|b \pm \frac{k_2^2}{n_0}\Big| < 1, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1,$$
$$\Big|ak_2^2 + b \pm \frac{k_2^2}{2n_0}\Big| + (c+d)k_2 < 1, \quad 0 < ak_2^2 - b \pm \frac{k_2}{n_0} < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b \pm \frac{k_2}{n_0}, c \pm \frac{1}{n_0}, d\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a, b \pm \frac{k_2}{n_0}, c \pm \frac{1}{n_0}, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{k_2}{n_0}\Big)\beta + \Big(\frac{1}{n_0}\Big)u\Big)\Big| \\ &= 1 + \Big|\Big(\frac{k_2}{n_0}\Big)\beta + \Big(\frac{1}{n_0}\Big)u\Big|. \end{aligned}$$

Hence, $\Big(\frac{k_2}{n_0}\Big)\beta + \Big(\frac{1}{n_0}\Big)u = 0$, so, $u = -k_2\beta$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|b \pm \frac{k_2^2}{2n_0}\Big| < 1, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1,$$
$$\Big|ak_2 + d \pm \frac{3k_2}{2n_0}\Big| < w_1, \quad \Big|ak_2^2 + b \pm \frac{3k_2^2}{2n_0}\Big| + \Big|(c+d)k_2 \pm \frac{k_2^2}{2n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a\pm\frac{1}{n_0}, b\pm\frac{k_2^2}{2n_0}, c, d\pm\frac{k_2}{2n_0}\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a\pm\frac{1}{n_0}, b\pm\frac{k_2^2}{2n_0}, c, d\pm\frac{k_2}{2n_0}\Big)\Big)\Big| = \Big|f(T)\pm\Big(\frac{1}{n_0}\alpha+\frac{k_2^2}{2n_0}\beta+\Big(\frac{k_2}{2n_0}\Big)v\Big)\Big| \\ &= 1+\Big|\frac{1}{n_0}\alpha+\frac{k_2^2}{2n_0}\beta+\Big(\frac{k_2}{2n_0}\Big)v\Big|. \end{aligned}$$

Hence, $\frac{1}{n_0}\alpha+\frac{k_2^2}{2n_0}\beta+\Big(\frac{k_2}{2n_0}\Big)v=0$, so, $v=k_2\beta$. It follows that

$$1=a\alpha+b\beta+cu+dv=\beta(-ak_2^2+b-ck_2+dk_2)=-\beta(ak_2^2-b+ck_2-dk_2)=-\beta,$$

so, $\beta=-1$ and $f=\Big(\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{1-w_2}{w_1},\ -\frac{1-w_2}{w_1}\Big)$. Therefore, $T\in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $c=d$.

We will show that $T\notin \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let

$$\begin{aligned} g_1 &= \Big(\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{1-w_2}{w_1},\ -\frac{1-w_2}{w_1}\Big), \\ g_2 &= \Big(\frac{(1-w_2)^2}{w_1^2},\ -1,\ -\frac{1-w_2}{w_1},\ \frac{1-w_2}{w_1}\Big)\in\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*. \end{aligned}$$

Obviously, $g_j(T)=1$ for $j=1,2$. By Theorem 2.2, $1=\|g_j\|$ for $j=1,2$. Thus, T is not smooth.

This completes the proof. □

We are in a position to characterize the smooth points of the unit ball of $\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$.

Theorem 2.6. *Let* $0<w_1,w_2<1$. *Then*

$$\begin{aligned} \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{ &\pm(a,b,c,d),\ \pm(a,b,d,c),\ \pm(a,-b,-c,d),\ \pm(a,-b,c,-d), \\ &\pm(a,b,-c,-d),\ \pm(a,-b,-d,c),\ \pm(a,-b,d,-c),\ \pm(a,b,-d,-c): \\ &T=(a,b,c,d) \textit{ is smooth in Theorem 2.5 with } a\geq 0, c\geq d\geq 0\Big\}. \end{aligned}$$

Proof. It follows from Theorems 2.4 and 2.5. □

Theorem 2.7. *Let* $0<w_1,w_2<1$ *and* $T=(a,b,c)\in\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$ *with* $\|T\|=1, a\geq 0$ *and* $c\geq 0$. *Let*

$$\Omega' = \Big\{(k_1+k_2)^2a,\ (ak_2+c)(k_1+k_2),\ |ak_2^2+b|+2ck_2,\ |ak_2^2-b|\Big\}.$$

Then, $T\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ *if and only if*

$$\begin{gathered} \Big((k_1+k_2)^2a=1,\ l<1 \textit{ for all } l\in\Omega'\backslash\{(k_1+k_2)^2a\}\Big), \\ \Big((ak_2+c)(k_1+k_2)=1,\ a>0,\ l<1 \textit{ for all } l\in\Omega'\backslash\{(ak_2+c)(k_1+k_2)\}\Big), \\ \Big(|ak_2^2+b|+2ck_2=1,\ a>0,\ 0<|ak_2^2+b|<1,\ l<1 \textit{ for all} \\ l\in\Omega'\backslash\{|ak_2^2+b|+2ck_2\}\Big), \\ \Big(|ak_2^2+b|+2ck_2=1,\ a=0,\ w_2\geq\frac{1}{2},\ l<1 \textit{ for all } l\in\Omega'\backslash\{|ak_2^2+b|+2ck_2\}\Big) \\ \textit{or}\quad \Big(|ak_2^2-b|=1,\ l<1 \textit{ for all } l\in\Omega'\backslash\{|ak_2^2-b|\}\Big). \end{gathered}$$

Proof. For $X, Y \in \{(k_1+k_2, 0),\ (k_2, 1),\ (k_2, -1)\}$, we let $\delta_{(X,Y)} \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $\delta_{(X,Y)}(S) = S(X,Y)$ for every $S \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})$. Note that $\|\delta_{(X,Y)}\| = 1$ and $\delta_{(X_1,Y_1)} \neq \delta_{(X_2,Y_2)}$ if $(X_1,Y_1) \neq (X_2,Y_2)$ and $(X_1,Y_1) \neq (Y_2,X_2)$.

Thus, if there are two $t_1, t_2 \in \Omega'$ such that $t_1 = t_2 = 1$, then $T \notin \operatorname{sm} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

By Theorem 2.1,

$$1 \;=\; \|T\| = \max\{l : l \in \Omega'\}.$$

Thus, we consider the following four cases:

Case 1. $(k_1+k_2)^2 a = 1, l < 1$ for every $l \in \Omega' \setminus \{(k_1+k_2)^2 a\}$

Claim. $T \in \operatorname{sm} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha, \beta, \theta) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. We will show that $f = \left(\frac{1}{w_1^2},\ 0,\ 0\right)$. Note that $|b| < 1$. Choose $n_0 \in \mathbb{N}$ such that

$$\left|b \pm \frac{1}{n_0}\right| < 1, \quad \left|ak_2^2 + b \pm \frac{1}{n_0}\right| + 2ck_2 < 1, \quad \left|ak_2^2 - b \pm \frac{1}{n_0}\right| < 1.$$

Note that by Theorem 2.1,

$$\left\|\left(a, b \pm \frac{1}{n_0}, c\right)\right\| = 1.$$

Thus,

$$1 \;\geq\; \left|f\left(\left(a, b \pm \frac{1}{n_0}, c\right)\right)\right| = \left|f(T) \pm \left(\frac{1}{n_0}\right)\beta\right| = 1 + \left(\frac{1}{n_0}\right)|\beta|.$$

Hence, $\beta = 0$.

Note that $c < w_1 w_2$ because $1 > (ak_2 + c)(k_1 + k_2)$. Choose $n_0 \in \mathbb{N}$ such that

$$\left|c \pm \frac{1}{n_0}\right| < w_1 w_2, \quad \left|ak_2 + c \pm \frac{1}{n_0}\right|(k_1 + k_2) < 1,$$
$$|ak_2^2 + b| + 2\left|c \pm \frac{1}{n_0}\right|k_2 < 1, \quad |ak_2^2 - b| < 1.$$

Note that by Theorem 2.1,

$$\left\|\left(a, b, c \pm \frac{1}{n_0}\right)\right\| = 1.$$

Thus,

$$1 \;\geq\; \left|f\left(\left(a, b, c \pm \frac{1}{n_0}\right)\right)\right| = \left|f(T) \pm \left(\frac{1}{n_0}\right)\theta\right| = 1 + \left(\frac{1}{n_0}\right)|\theta|.$$

Hence $\theta = 0$.

It follows that

$$1 = a\alpha + b\beta + c\theta = a\alpha,$$

so, $\alpha = \frac{1}{a} = \frac{1}{w_1^2}$. Thus, $f = \left(\frac{1}{w_1^2},\ 0,\ 0\right)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Case 2. $(ak_2 + c)(k_1 + k_2) = 1 < l$ for every $l \in \Omega' \setminus \{(ak_2 + c)(k_1 + k_2)\}$.

Note that $0 \leq a < w_1^2$.

Claim. $T \in \operatorname{sm} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $a > 0$.

Suppose that $a > 0$.

Then $c < w_1$. We will show that $T \in \operatorname{sm} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha, \beta, \theta) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. We will show that $f = \left(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1}\right)$. Note that $|b| < 1$. Indeed, if $|b| = 1$, then $a = c = 0$, which is impossible.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|ak_2^2 + b \pm \frac{2k_2^2}{n_0}\Big| + 2\Big|c \mp \frac{k_2}{n_0}\Big|k_2 < 1, \quad \Big|ak_2^2 - b \pm \frac{k_2^2}{n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \pm \frac{1}{n_0}, b, c \mp \frac{k_2}{n_0}\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a \pm \frac{1}{n_0}, b, c \mp \frac{k_2}{n_0}\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2}{n_0}\Big)\theta\Big)\Big| \\ &= 1 + \Big|\Big(\frac{1}{n_0}\Big)\alpha - \Big(\frac{k_2}{n_0}\Big)\theta\Big|. \end{aligned}$$

Hence, $\alpha = k_2\theta$.

Choose $n_0 \in \mathbb{N}$ such that

$$\Big|b \pm \frac{1}{n_0}\Big| < 1, \quad \Big|ak_2^2 + b \pm \frac{1}{n_0}\Big| + 2ck_2 < 1, \quad \Big|ak_2^2 - b \pm \frac{1}{n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b \pm \frac{1}{n_0}, c\Big)\Big\| = 1.$$

Thus,

$$1 \geq \Big|f\Big(\Big(a, b \pm \frac{1}{n_0}, c\Big)\Big)\Big| = \Big|f(T) \pm \Big(\frac{1}{n_0}\Big)\beta\Big| = 1 + \Big(\frac{1}{n_0}\Big)|\beta|.$$

Hence, $\beta = 0$.

It follows that

$$1 = a\alpha + b\beta + c\theta = a\alpha + c\theta = \theta(ak_2 + c) = \theta w_1,$$

so, $\theta = \frac{1}{w_1}$ and $f = \Big(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1}\Big)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $a = 0$.

Let $g_1 = \Big(\frac{1-w_2}{w_1^2},\ 0,\ \frac{1}{w_1}\Big),\ g_2 = \Big(0,\ 0,\ \frac{1}{w_1}\Big) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g_j(T) = 1$ for $j = 1, 2$. By Theorem 2.3, $1 = \|g_j\|\ j = 1, 2$. Thus, T is not smooth.

Case 3. $|ak_2^2 + b| + 2ck_2 = 1 > l$ for every $l \in \Omega^{'}\backslash\{|ak_2^2 + b| + 2ck_2\}$

Note that $0 \leq a < w_1^2$.

Claim. $T \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $a > 0$ and $0 < |ak_2^2 + b| < 1$.

Let $a > 0$ and $0 < |ak_2^2 + b| < 1$. We will show that $T \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $f = (\alpha, \beta, \theta) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. Note that $c > 0$.

Suppose that $0 < ak_2^2 + b < 1$. Note that $ak_2^2 + b + 2ck_2 = 1 > l$ for every $l \in A\backslash\{|ak_2^2 + b| + 2ck_2\}$. We will show that $f = \Big(\frac{(1-w_2)^2}{w_1^2},\ 1,\ \frac{1-w_2}{w_1}\Big)$.

Note that $|b| < 1$. Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|b \pm \frac{k_2^2}{n_0}\Big| < 1, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1,$$
$$\Big|ak_2^2 - b \pm \frac{2k_2^2}{n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \mp \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a \mp \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c\Big)\Big)\Big| = \Big|f(T) \pm \Big(-\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big)\Big| \\ &= 1 + \Big|-\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big|. \end{aligned}$$

Hence, $\alpha = k_2^2\beta$.

Choose $n_0 \in \mathbb{N}$ such that

$$\Big|b \pm \frac{2k_2}{n_0}\Big| < 1, \quad 0 < 2c - \frac{2}{n_0}, \quad \Big|ak_2 + c \pm \frac{1}{n_0}\Big| < w_1, \quad 0 < ak_2^2 + b \pm \frac{2k_2}{n_0} < 1,$$
$$\Big|ak_2^2 - b \pm \frac{2k_2}{n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b \pm \frac{2k_2}{n_0}, c \mp \frac{1}{n_0}\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a, b \pm \frac{2k_2}{n_0}, c \mp \frac{1}{n_0}, d\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{2k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u\Big)\Big| \\ &= 1 + \Big|\Big(\frac{2k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)u\Big|. \end{aligned}$$

Hence, $\Big(\frac{2k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)\theta = 0$, so, $\theta = 2k_2\beta$.

It follows that

$$1 = a\alpha + b\beta + c\theta = \beta(ak_2^2 + b + 2ck_2) = \beta,$$

so, $f = \Big(\frac{(1-w_2)^2}{w_1^2},\ 1,\ \frac{2(1-w_2)}{w_1}\Big)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $-1 < ak_2^2 + b < 0$. Note that $-ak_2^2 - b + 2ck_2 = 1 > l$ for every $l \in A\backslash\{|ak_2^2 + b| + 2ck_2\}$. We will show that $f = \Big(-\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{2(1-w_2)}{w_1}\Big)$.

Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad \Big|b \pm \frac{k_2^2}{n_0}\Big| < 1, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w,$$
$$\Big|ak_2^2 - b \pm \frac{2k_2^2}{n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \mp \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a \mp \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c\Big)\Big)\Big| = \Big|f(T) \pm \Big(-\frac{\alpha}{n_0} + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big)\Big| \\ &= 1 + \Big|-\frac{\alpha}{n_0} + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big|. \end{aligned}$$

Hence, $\alpha = k_2^2\beta$.

Choose $n_0 \in \mathbb{N}$ such that

$$\Big|b \pm \frac{2k_2}{n_0}\Big| < 1, \quad 0 < 2c - \frac{2}{n_0}, \quad \Big|ak_2 + c \pm \frac{1}{n_0}\Big| < w_1, \quad -1 < ak_2^2 + b \pm \frac{2k_2}{n_0} < 0,$$
$$\Big|ak_2^2 - b \pm \frac{2k_2}{n_0}\Big| < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b\pm\frac{2k_2}{n_0}, c\pm\frac{1}{n_0}\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a, b\pm\frac{2k_2}{n_0}, c\pm\frac{1}{n_0}\Big)\Big)\Big| = \Big|f(T)\pm\Big(\Big(\frac{2k_2}{n_0}\Big)\beta+\Big(\frac{1}{n_0}\Big)\theta\Big)\Big| \\ &= 1+\Big|\Big(\frac{2k_2}{n_0}\Big)\beta+\Big(\frac{1}{n_0}\Big)\theta\Big|. \end{aligned}$$

Hence, $\Big(\frac{2k_2}{n_0}\Big)\beta+\Big(\frac{1}{n_0}\Big)\theta = 0$, so, $\theta=-2k_2\beta$. It follows that

$$1 = a\alpha+b\beta+c\theta = \beta(ak_2^2+b-2ck_2) = -\beta,$$

so, $f=\Big(-\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{2(1-w_2)}{w_1}\Big)$. Therefore, $T\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Suppose that $|ak_2^2+b|=0$ or 1. We will show that $T\notin \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $|ak_2^2+b|=0$. Let $g^{\pm}=\Big(\pm\frac{(1-w_2)^2}{w_1^2},\ \pm 1,\ \frac{2(1-w_2)}{w_1}\Big)\in\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g^{\pm}(T)=1$. By Theorem 2.3, $1=\|g^{\pm}\|$. Thus, T is not smooth.

Let $|ak_2^2+b|=1$. Then, $c=0$. Let $g^{\pm}=\Big(\pm\frac{(1-w_2)^2}{w_1^2},\ 1,\ \pm\frac{2(1-w_2)}{w_1}\Big)\in\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Obviously, $g^{\pm}(T)=1$. By Theorem 2.3, $1=\|g^{\pm}\|$. Thus, T is not smooth.

Let $a=0$.

Claim. $T\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $w_2\geq\frac{1}{2}$

Let $w_2<\frac{1}{2}$ and

$$g_\alpha=\Big(\alpha,\ 1,\ \frac{2(1-w_2)}{w_1}\Big)\in\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)}) \text{ for } \frac{-3w_2^2+4w_2-1}{w_1^2}\leq\alpha\leq\frac{(1-w_2)^2}{w_1^2}.$$

Note that

$$\frac{-3w_2^2+4w_2-1}{w_1^2}<\frac{(1-w_2)^2}{w_1^2}.$$

By Theorem 2.3, $\|g_\alpha\|=1$. Note that $g_\alpha(T)=1$. Thus, T is not smooth.

Let $w_2\geq\frac{1}{2}$.

Claim. $b\neq 0$

Suppose not. Then

$$2w_1\leq\frac{w_1}{1-w_2}=2c<2w_1,$$

which is impossible. Thus, the claim holds.

Suppose that $b>0$. Choose $n_0\in\mathbb{N}$ such that

$$0<b\pm\frac{2k_2}{n_0}<1,\quad 0<2c-\frac{2}{n_0},\quad \Big|c\pm\frac{1}{n_0}\Big|<w_1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b\pm\frac{2k_2}{n_0}, c\mp\frac{1}{n_0}\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a, b\pm\frac{2k_2}{n_0}, c\mp\frac{1}{n_0}\Big)\Big)\Big| = \Big|f(T)\pm\Big(\Big(\frac{2k_2}{n_0}\Big)\beta-\Big(\frac{1}{n_0}\Big)\theta\Big)\Big| \\ &= 1+\Big|\Big(\frac{2k_2}{n_0}\Big)\beta-\Big(\frac{1}{n_0}\Big)\theta\Big|, \end{aligned}$$

Hence, $\Big(\frac{2k_2}{n_0}\Big)\beta - \Big(\frac{1}{n_0}\Big)\theta = 0$, so, $\theta = 2k_2\beta$.

It follows that

$$1 = a\alpha + b\beta + c\theta = \beta(b + 2ck_2) = \beta.$$

Thus, $\theta = \frac{2(1-w_2)}{w_1}$ and

$$f = \Big(\alpha,\ 1,\ \frac{2(1-w_2)}{w_1}\Big).$$

By the some calculation, by Theorem 2.3(b), $\alpha = \frac{(1-w_2)^2}{w_1^2}$. Thus, T is smooth.

By analogous arguments as in the case $b > 0$, we have

$$f = \Big(-\frac{(1-w_2)^2}{w_1^2},\ -1,\ \frac{2(1-w_2)}{w_1}\Big).$$

Thus, the claim holds.

Case 4. $|ak_2^2 - b| = 1 > l$ for every $l \in \Omega^{'}\backslash\{|ak_2^2 - b|\}$

Note that $0 < a < w_1^2$.

Claim. $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let $f = (\alpha, \beta, \theta) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $1 = \|f\| = f(T)$. We will show that $f = \Big(\frac{(1-w_2)^2}{w_1^2},\ -1,\ 0\Big)$.

Note that $0 \le c < w_1$. Indeed, if $c = w_1$, then

$$1 > (ak_2 + c)(k_1 + k_2) \ge c(k_1 + k_2) = w_1(k_1 + k_2) = 1,$$

which is a contradiction. Note that $-1 < b < 0$. Indeed, if $b \ge 0$, then

$$1 > |ak_2^2 + b| + (c + d)k_2 \ge |ak_2^2 - b| = 1,$$

which is a contradiction. If $b = -1$, then $a = c = 0$ and $1 > |ak_2^2 + b| + 2ck_2 = 1$, which is a contradiction.

Note that $0 < |ak_2^2 - b| = ak_2^2 - b < 1$. Choose $n_0 \in \mathbb{N}$ such that

$$0 < a - \frac{1}{n_0} < a + \frac{1}{n_0} < w_1^2, \quad -1 < b - \frac{k_2^2}{n_0} < b + \frac{k_2^2}{n_0} < 0, \quad \Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1,$$

$$\Big|ak_2^2 + b \pm \frac{2k_2^2}{n_0}\Big| + 2ck_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a \pm \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\ge \Big|f\Big(\Big(a \pm \frac{1}{n_0}, b \pm \frac{k_2^2}{n_0}, c\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big)\Big| \\ &= 1 + \Big|\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta\Big|. \end{aligned}$$

Hence, $\Big(\frac{1}{n_0}\Big)\alpha + \Big(\frac{k_2^2}{n_0}\Big)\beta = 0$, so, $\alpha = -k_2^2\beta$.

Choose $n_0 \in \mathbb{N}$ such that

$$\Big|ak_2 + c \pm \frac{k_2}{n_0}\Big| < w_1, \quad |ak_2^2 + b| + 2\Big|c \pm \frac{1}{n_0}\Big|k_2 < 1.$$

Note that by Theorem 2.1,

$$\Big\|\Big(a, b, c \pm \frac{1}{n_0}\Big)\Big\| = 1.$$

Thus,

$$\begin{aligned} 1 &\geq \Big|f\Big(\Big(a, b \pm \frac{k_2}{n_0}, c \pm \frac{1}{n_0}\Big)\Big)\Big| = \Big|f(T) \pm \Big(\Big(\frac{k_2}{n_0}\Big)\beta + \Big(\frac{1}{n_0}\Big)\theta\Big)\Big| \\ &= 1 + \Big|\Big(\frac{k_2}{n_0}\Big)\beta + \Big(\frac{1}{n_0}\Big)\theta\Big|. \end{aligned}$$

It follows that

$$1 = a\alpha + b\beta + c\theta = \beta(-ak_2^2 + b) = -\beta(ak_2^2 - b) = -\beta,$$

so, $\beta = -1$ and $f = \Big(\frac{(1-w_2)^2}{w_1^2},\ -1,\ 0\Big)$. Therefore, $T \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

This completes the proof. □

We are in a position to characterize the smooth points of the unit ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$.

Note that $T = (a, b, c) \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $-(a, b, c), \pm(a, b, -c) \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Theorem 2.8. *Let $0 < w_1, w_2 < 1$. Then,*

$$\operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{ \pm (a, b, c),\ \pm(a, b, -c) : T = (a, b, c) \text{ is smooth in Theorem 2.7 with } a \geq 0, c \geq 0\Big\}.$$

Proof. It follows from Theorem 2.7. □

Theorem 2.9. *Let $0 < w_1, w_2 < 1$ and $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})$. Let $\tilde{T} = (b, a, d, c) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})$. Then,*

(*a*) $\|T\|_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} = \|\tilde{T}\|_{\mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})}$;

(*b*) $\operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{(a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)}) : (b, a, d, c) \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})}\Big\}$.

Proof. (a). Note that for $(x, y) \in \mathbb{R}^2$,

$$\|(x, y)\|_{h'(w_1,w_2)} = \|(y, x)\|_{h(w_2,w_1)}.$$

It follows that

$$\begin{aligned} \|T\|_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} &= \sup_{\|(x_j,y_j)\|_{h'(w_1,w_2)}=1, j=1,2} |T((x_1, y_1),\ (x_2, y_2))| \\ &= \sup_{\|(x_j,y_j)\|_{h'(w_1,w_2)}=1, j=1,2} |\tilde{T}((y_1, x_1),\ (y_2, x_2))| \\ &= \sup_{\|(y_j,x_j)\|_{h(w_2,w_1)}=1, j=1,2} |\tilde{T}((y_1, x_1),\ (y_2, x_2))| \\ &= \|\tilde{T}\|_{\mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})}. \end{aligned}$$

(b) follows from (a). □

Theorem 2.10. (*a*) $\operatorname{sm} B_{\mathcal{L}(^2E)} \cap \mathcal{L}_s(^2E) \subseteq \operatorname{sm} B_{\mathcal{L}_s(^2E)}$ *for a real Banach space E.*

(*b*) $\operatorname{sm} B_{\mathcal{L}_s(^2X)} = \operatorname{sm} B_{\mathcal{L}(^2X)} \cap \mathcal{L}_s(^2X)$, *where $X = \mathbb{R}^2_{h(w_1,w_2)}$ or $\mathbb{R}^2_{h'(w_1,w_2)}$.*

Proof. (a). Let $T \in \operatorname{sm} B_{\mathcal{L}(^2E)} \cap \mathcal{L}_s(^2E)$.

Claim. $T \in \operatorname{sm} B_{\mathcal{L}_s(^2E)}$

Suppose not. There are $f_1 \neq f_2 \in \mathcal{L}_s(^2E)^*$ such that $f_j(T) = \|f_j\| = 1$ for $j = 1, 2$. Note that $\mathcal{L}_s(^2E)$ is a closed subspace of $\mathcal{L}(^2E)$. By the Hahn-Banach theorem, there are extensions $\tilde{f}_j \in \mathcal{L}(^2E)^*$ for $j = 1, 2$ such that $\tilde{f}_j|_{\mathcal{L}_s(^2E)} = f_j$ and $\|\tilde{f}_j\| = \|f_j\| = 1$ for $j = 1, 2$. Since $\tilde{f}_j(T) = f_j(T) = 1$ for $j = 1, 2$ and $T \in \operatorname{sm} B_{\mathcal{L}(^2E)}$, we have $\tilde{f}_1 = \tilde{f}_2$. Thus, $f_1 = \tilde{f}_1|_{\mathcal{L}_s(^2E)} = \tilde{f}_2|_{\mathcal{L}_s(^2E)} = f_2$, which is a contradiction.
Thus, the claim holds.

(b) follows from Theorems 2.5–2.9. □

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