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THE q-ANALOG OF THE RODRIGUES FORMULA FOR SYMMETRIC q-DUNKL-CLASSICAL ORTHOGONAL q-POLYNOMIALS

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Abstract. The purpose of this paper is to establish a Rodrigues type formula for q-Dunkl-classical symmetric orthogonal q-polynomials.

Нашою метою є встановити формулу типу Родрігеса для q-класичних симетричних ортогональних q-поліномів Данкла.

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients. Assume $\mathcal{O}$ is a lowering operator on $\mathcal{P}$ satisfying:

$$\mathcal{O}(\mathcal{P}) = \mathcal{P}, \quad \mathcal{O}(1) = 0, \quad \text{and} \quad \deg\{\mathcal{O}\left(x^n\right)\} = n - 1 \quad (n \in \mathbb{N}),$$

where $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the theory of orthogonal polynomials, certain lowering operators are used to classify orthogonal polynomials. Specifically, we can define a monic orthogonal polynomial sequence (MOPS) $\{P_n\}_{n\geq 0}$ as an $\mathcal{O}$-classical polynomial sequence if the sequence $\{\frac{\mathcal{O}Pn+1}{\omega_{n+1}}\}_{n\geq 0}$ is a MOPS, where ω_n is a constant factor such that $\{\frac{\mathcal{O}Pn+1}{\omega_{n+1}}\}_{n\geq 0}$ is monic. We can enumerate some of the lowering operators such as: the derivative operator denoted by D, the difference operator denoted by Δ, where $\Delta p(x)$ equals $p(x+1)$ minus $p(x)$, the Hahn operator denoted by H_q, where $H_q(x)$ equals $\frac{f(qx)-f(x)}{(q-1)x}$, and the Dunkl operator denoted by T_μ, where $T_\mu p(x)$ equals $p'(x)$ plus $2\mu H_{-1}(x)$.

The $\mathcal{O}$-classical polynomial sequences encompass the most celebrated orthogonal polynomial sequences. For instance, when $\mathcal{O} = D$, we obtain the continuous orthogonal polynomial sequences such as Hermite, Laguerre, Bessel, and Jacobi [2, 18]. On the other hand, when $\mathcal{O} = \Delta$, we get the classical discrete orthogonal polynomial sequences like Charlier, Meixner, Krawtchouk, and Hahn (see [12]).

Let us consider $D_w p(x)$ as a natural extension of the fundamental difference operator, where $D_w p(x) = \frac{p(x+w)-p(x)}{w}$ for $w \neq 0$. The classical orthogonal polynomials belonging to the D_w class are discussed in [1] along with their essential properties. According to [4], the generalized Hermite and generalized Gegenbauer polynomial sequences are the only symmetric T_μ-classical polynomial sequences for the Dunkl operator. In the domain of interest, there have been some noteworthy contributions by various authors, including [3, 5, 6, 13, 25, 26].

Previously, a new lowering operator has been employed to address similar problems, as detailed in references [1, 17]. This has led to the introduction of a concept called $T_{\theta,q}$-classical orthogonal polynomials (also known as q-Dunkl-classical orthogonal polynomials), where $T_{\theta,q}$ represents the q-Dunkl operator, which can be defined as follows:

$$\left(T_{\theta,q} f\right)(x) = \left(H_q f\right)(x) + \theta\left(H_{-1} f\right)(x), \quad f \in \mathcal{P},\ \theta \in \mathbb{C}.$$

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The classification of the $T_{\theta,q}$-classical symmetric orthogonal polynomials is available in [3, 7, 9]. In the symmetric case, the $T_{\theta,q}$-classical is defined as the regular form of u that satisfies the Pearson differential equation:

$$T_{\theta,q}(\Phi u)+\Psi u=0,$$

where Φ even and monic and Ψ odd are fixed polynomials of degree at most 2 and 1, respectively.

In a recent publication ([3, 9]), it was demonstrated that, with the exception of a dilatation factor, the only symmetric orthogonal q-polynomials q-Dunkl-classical are the q^2-analogue of generalized Hermite and q^2-analogue of generalized Gegenbauer (as defined in [15]). Therefore, it is natural to ask for a Rodrigues-type formula for these q-Dunkl-classical symmetric orthogonal q-polynomials.

This paper is organized as follows. Section 2 provides an introduction to some initial findings and notations that will be used in the subsequent sections. In Section 3, we present a fresh characterization of q-Dunkl-classical symmetric orthogonal q-polynomials.

2. Preliminaries

Let $\mathcal{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathcal{P}'$ its dual space, whose elements are forms. We denote by $\langle u,p\rangle$ the action of $u\in\mathcal{P}'$ on $p\in\mathcal{P}$. In particular, we denote by $(u)_n:=\langle u,x^n\rangle,\ n\geq 0$, the moments of u. Moreover, a form (linear functional) u is called *symmetric* if $(u)_{2n+1}=0$, $n\geq 0$.

Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any polynomial g and any $a\in\mathbb{C}\backslash\{0\}$ and any $q\neq 1$, we let $Du:=u'$, gu, $h_a u$ and $H_q u$, be the forms defined by duality [17, 20]

$$\langle u',f\rangle:=-\langle u,f'\rangle,\qquad \langle gu,f\rangle:=\langle u,gf\rangle,\ f,\ g\in\mathcal{P},$$
$$\langle h_a u,f\rangle:=\langle u,h_a f\rangle=\langle u,f(ax)\rangle,\qquad \langle H_q u,f\rangle:=-\langle u,H_q f\rangle,\ f\in\mathcal{P},$$

where $(H_q f)(x)=\frac{f(qx)-f(x)}{(q-1)x}$, $q\in\widetilde{\mathbb{C}}:=\mathbb{C}\backslash\bigcup\limits_{n\geq 0}U_n$ with

$$U_n=\begin{cases}\{0\}\ ,\ n=0\\ \{z\in\mathbb{C}\ ,\ z^n=1\}\ ,\ n\geq 1.\end{cases}$$

The following formulas hold [17]

$$H_q(fu)=(h_{q^{-1}}f)H_q u+q^{-1}(H_{q^{-1}}f)u,\ f\in\mathcal{P},\ u\in\mathcal{P}',\tag{2.1}$$
$$(H_{q^{-1}}\circ h_q)(f)=qH_q(f),\ f\in\ \mathcal{P},\tag{2.2}$$
$$h_a(gu)=(h_{a^{-1}}g)\circ(h_a u),\quad g\in\mathcal{P},\ u\in\mathcal{P}'.\tag{2.3}$$

A form u is called *normalized*, if it satisfies $(u)_0=1$. We assume that the forms used in this paper are normalized.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n=n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n\in\mathcal{P}'$, defined by $\langle u_n,P_m\rangle=\delta_{n,m}$, $n,\ m\geq 0$, where $\delta_{n,m}$ is the Kronecker's symbol. Notice that u_0 is said to be the canonical functional associated with the MPS $\{P_n\}_{n\geq 0}$. The sequence $\{P_n\}_{n\geq 0}$ is called *symmetric* when $P_n(-x)=(-1)^nP_n(x),\ n\geq 0$.

Let us recall the following result.

Lemma 2.1. [20, 19]. *For any $u\in\mathcal{P}'$ and any integer $m\geq 1$, the following statements are equivalent:*

(i) $\langle u,P_{m-1}\rangle\neq 0,\ \langle u,P_n\rangle=0,\ n\geq m;$

(ii) $\exists\lambda_\nu\in\mathbb{C},\ 0\leq\nu\leq m-1,\ \lambda_{m-1}\neq 0$ *such that* $u=\sum\limits_{\nu=0}^{m-1}\lambda_\nu u_\nu.$

The form u is called *regular* if we can associate with it a MPS $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_n P_m\rangle = r_n\delta_{n,m}, \quad r_n \neq 0, \quad n,\ m \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called a monic *orthogonal* polynomial sequence (MOPS) with respect to u. Note that $u = (u)_0 u_0$, with $(u)_0 \neq 0$. When u is regular, let F be a polynomial such that $Fu = 0$, then $F = 0$, [22].

Proposition 2.2. [20, 19]. *Let $\{P_n\}_{n\geq 0}$ be a MPS with $\deg P_n = n$, $n \geq 0$, and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent.*

(i) *$\{P_n\}_{n\geq 0}$ is orthogonal with respect to u_0.*

(ii) *For all $n \geq 0$*

$$u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0. \tag{2.4}$$

(iii) *$\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation*

$$(\text{TTRR})\ : \begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{cases} \tag{2.5}$$

where $\beta_n = \langle u_0, xP_n^2\rangle\langle u_0, P_n^2\rangle^{-1}$, $n \geq 0$ and $\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle\langle u_0, P_n^2\rangle^{-1} \neq 0$, $n \geq 0$.

If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to the regular form u_0, then $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) = a^{-n}P_n(ax)$, $n \geq 0$, $a \neq 0$, is a MOPS with respect to the regular form $\tilde{u}_0 = h_{a^{-1}}u_0$, and satisfies [19]

$$\begin{cases} \tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x),\ n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}\beta_n$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

The following lemma is necessary.

Lemma 2.3. [13]. *A MOPS satisfying (2.5) is symmetric, if and only if, $\beta_n = 0$, $n \geq 0$.*

Next, we recall the concept of H_q-*semiclassical* form that we will need in the sequel. A form u is called H_q-*semiclassical* if it is regular, and there exist two polynomials Φ and Ψ, Φ monic, $\deg\Phi = t \geq 0$, $\deg\Psi = p \geq 1$ such that u fulfills the q-analogue of the distributional equation of Pearson type

$$(\text{PE}):\ \ H_q(\Phi u) + \Psi u = 0, \tag{2.6}$$

where the pair (Φ, Ψ) is admissible, i.e., when $p = t - 1$, writing $\Psi(x) = a_p x^p + \dots$, then $a_p \neq n + 1$, $\quad n \in \mathbb{N}$. The corresponding orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called H_q-*semiclassical* [16]. Moreover, if u is semiclassical satisfying (2.6), the class of u, denoted s is defined by

$$s = \min\big(\max(\deg\Phi - 2, \deg\Psi - 1)\big) \geq 0,$$

where the minimum is taken over all pairs (Φ, Ψ) satisfying (2.6). In particular, if $s = 0$ the form u is usually called H_q-classical [17].

The H_q-semiclassical character is kept by a dilatation [16]. In fact, when u satisfies (2.6), then $h_{a^{-1}}u$ fulfills the following PE

$$H_q\big(a^{-t}\Phi(ax)h_{a^{-1}}u\big) + a^{1-t}\Psi(ax)h_{a^{-1}}u = 0,$$

with the recurrence coefficients, $\tilde{\beta}_n$ and $\tilde{\gamma}_{n+1}$ are given above.

Let us introduce the q-Dunkl operator

$$T_{\theta,q}(f)(x) = (H_q f)(x) + \theta(H_{-1}f)(x), \quad f \in \mathcal{P}, \theta \in \mathbb{C}, \tag{2.7}$$

where

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

Note that, $T_{0,q}$ is reduced to the q-derivative operator (for more details, see [17]). We, also, have

$$\lim_{q\longrightarrow 1} T_{\theta,q}f(x) = f'(x) + \theta\frac{f(x)-f(-x)}{2x} = T_\theta f(x),$$

where T_θ is called Dunkl operator, introduced by Dunkl [11] (see also [4, 8, 25]).

The transposed ${}^tT_{\theta,q}$ of $T_{\theta,q}$ is ${}^tT_{\theta,q} = -H_q - \theta H_{-1} = -T_{\theta,q}$, leaving out a slight abuse of notation without consequence. Thus, we have

$$\langle T_{\theta,q}u, f\rangle = -\langle u, T_{\theta,q}f\rangle, \quad u\in\mathcal{P}', \quad f\in\mathcal{P}, \quad \theta\in\mathbb{C}.$$

In particular, this yields

$$(T_{\theta,q}u)_n = -\theta_{n,q}(u)_{n-1}, \quad n\geq 0,$$

with the convention $(u)_{-1}=0$, where

$$\theta_{n,q} = [n]_q + \theta\frac{1-(-1)^n}{2}, \quad n\geq 0.$$

Here, $[n]_q,\ n\geq 0$, denotes the basic q-number defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1},\ n>0,\ [0]_q=0.$$

It is easy to see that

$$h_a\circ T_{\theta,q} = aT_{\theta,q}\circ h_a \quad \text{in } \mathcal{P}', \quad a\in\mathbb{C}\backslash\{0\}. \tag{2.8}$$

$$h_a(fu) = (h_{a^{-1}}f)(h_au), \quad f\in\mathcal{P},\ u\in\mathcal{P}',\ a\in\mathbb{C}\backslash\{0\}. \tag{2.9}$$

Remark 2.4. [9] When u is a symmetric form, we obtain

$$\begin{aligned}T_{\theta,q}(fu) =&(h_{q^{-1}}f)(T_{\theta,q}u) + \big(T_{\theta,q}(h_{q^{-1}}f)\big)u\\ &+\theta\frac{q+1}{2}\Big((H_{-q}(h_{q^{-1}}f)) + (H_{-q}(h_{-q^{-1}}f))\Big)u,\ f\in\mathcal{P},\ u\in\mathcal{P}'.\end{aligned} \tag{2.10}$$

Now, consider a MPS $\{P_n\}_{n\geq 0}$ and let

$$P_n^{[1]}(x,\theta,q) = \frac{1}{\theta_{n+1,q}}\left(T_{\theta,q}P_{n+1}\right)(x), \quad \theta\neq -[2n+1]_q, \quad n\geq 0.$$

Definition 2.5. [3, 7, 9] A MOPS $\{P_n\}_{n\geq 0}$ is called q-Dunkl-classical or $T_{\theta,q}$-classical if $\left\{P_n^{[1]}(.,\theta,q)\right\}_{n\geq 0}$ is also a MOPS. In this case, the form u_0 is called q-Dunkl-classical or $T_{\theta,q}$-classical form.

3. Rodrigues type formula

The following was proved in [9].

Theorem 3.1. *For any symmetric MOPS $\{P_n\}_{n\geq 0}$, the following statements are equivalent:*

(a) *The sequence $\{P_n\}_{n\geq 0}$ is q-Dunkl-classical.*
(b) *There exist two polynomials Φ (monic and even) and Ψ with $\deg\Phi\leq 2$ and $\deg\Psi = 1$ such that the associated regular form u_0 satisfies*

$$T_{\theta,q}(\Phi u_0) + \Psi u_0 = 0, \tag{3.11}$$

$$q^{-n}\Psi'(0) - \frac{1}{2}\big(\theta_{n,q} + q^{-1}[n]_{q^{-1}} - \theta + \theta q^{-n} - [n]_q\big)\Phi''(0)\neq 0, \quad n\geq 0. \tag{3.12}$$

Proposition 3.2. *If $\{P_n\}_{n\geq 0}$ is q-Dunkl-classical symmetric MOPS, then*

$$\left\{P_n^{[m]}(.,\theta,q)=\frac{T^m_{\theta,q}P_{n+m}}{\prod_{k=1}^m\theta_{n+k,q}}\right\}_{n\geq 0},\ \ m\geq 1,$$

is also a q-Dunkl-classical symmetric MOPS and we have

$$T_{\theta,q}\left(\Phi_m u_0^{[m]}(\theta,q)\right)+\Psi_m u_0^{[m]}(\theta,q)=0, \tag{3.13}$$

$$u_0^{[m]}(\theta,q)=q^{\frac{-m(m-1)}{2}\deg\Phi}\xi_m\Big(\prod_{i=0}^{m-1}h_{q^i}\Phi\Big)u_0,\quad m\geq 1, \tag{3.14}$$

$$q^{m\deg\Phi}\Phi_m(x)=(h_{q^m}\Phi)(x), \tag{3.15}$$

$$q^{m\deg\Phi}\Psi_m(x)=\Psi(x)-\sum_{i=0}^{m-1}\big(T_{\theta,q}\circ h_{q^i}\Phi-\theta(q+1)H_{-q}\circ h_{q^i}\Phi\big)(x). \tag{3.16}$$

where Φ and Ψ are the same polynomials as in (3.11), $\left\{u_n^{[m]}(\theta,q)\right\}_{n\geq 0}$ is the dual sequence of $\left\{P_n^{[m]}(.,\theta,q)\right\}_{n\geq 0}$ and ξ_m is defined by the condition $\left(u_0^{[m]}(\theta,q)\right)_0=1$.

For the proof, the following lemma is needed.

Lemma 3.3. [3, 9] *If $\{P_n\}_{n\geq 0}$ is q-Dunkl-classical symmetric MOPS, then*

$$u_0^{[1]}(\theta,q)=k\Phi u_0 \tag{3.17}$$

where k is a normalization factor and Φ is the same polynomials as in (3.11).

Proof of Proposition 3.2. Suppose $m=1$. The form u_0 satisfies (3.11). Multiplying both sides by Φ and on account of (2.10) and (3.17), we get

$$T_{\theta,q}\left(\Phi_1 u_0^{[1]}(\mu)\right)+\Psi_1 u_0^{[1]}(\theta,q)=0.$$

Therefore, (3.13)-(3.16) are valid for $m=1$. By induction, we easily obtain the general case. □

The main result of this paper is the following.

Theorem 3.4. *The symmetric MOPS $\{P_n\}_{n\geq 0}$ is q-Dunkl-classical if and only if there exist a monic polynomial Φ, $\deg\Phi\leq 2$ and a sequence $\{\Lambda_n\}_{n\geq 0},\Lambda_n\neq 0, n\geq 0$, such that*

$$P_n u_0=\Lambda_n T^n_{\theta,q}\left(\Big(\prod_{i=0}^{m-1}h_{q^i}\Phi\Big)u_0\right),\quad n\geq 0. \tag{3.18}$$

We may call (3.18) a (functional) Rodrigues type formula for the q-Dunkl-classical symmetric orthogonal polynomials.

Proof. Necessity. Consider $\left\langle T^n_{\theta,q}u_0^{[n]},P_m\right\rangle=(-1)^n\left\langle u_0^{[n]},T^n_{\theta,q}P_m\right\rangle$, $n,m\geq 0$. For $0\leq m\leq n-1, n\geq 1$, we have $T^n_{\theta,q}P_m=0$. For $m\geq n$, put $m=n+k, k\geq 0$. Then

$$\left\langle u_0^{[n]},T^n_{\theta,q}P_{n+k}\right\rangle=\left(\prod_{v=1}^n\theta_{k+v,q}\right)\left\langle u_0^{[n]},P_k^{[n]}\right\rangle=\left(\prod_{v=1}^n\theta_{v,q}\right)\delta_{0,k}$$

following the definitions. Consequently

$$T^n_{\theta,q}u_0^{[n]}=(-1)^n\left(\prod_{v=1}^n\theta_{v,q}\right)u_n,\quad n\geq 0.$$

But from (2.4) so that, in accordance with (3.14), we obtain (3.18) where

$$\Lambda_n = (-1)^n q^{\frac{-n(n-1)}{2}\deg \Phi} \xi_n \frac{\langle u_0, P_n^2 \rangle}{\prod_{v=1}^{n} \theta_{v,q}}, n \geq 0. \tag{3.19}$$

Sufficiency. Making $n = 1$ in (3.18), we have $P_1 u_0 = \Lambda_1 T_{\theta,q}(\Phi u_0)$ and (3.12) is satisfied since u_0 is regular. Therefore, the sequence $\{P_n\}_{n\geq 0}$ is q-Dunkl-classical according to Theorem 3.1. □

Next, we recall some properties of: *q^2-analogue of the symmetrical generalized Hermite form* $\mathcal{H}(\mu, q^2)$, and *q^2-analogue of the symmetrical generalized Gegenbauer form* $\mathcal{G}(\alpha, \beta, q^2)$, (see [15]).

Proposition 3.5. *The q^2-analogue of the symmetrical generalized Hermite form $\mathcal{H}(\mu, q^2)$ is regular if and only if $\mu \neq -[n]_{q^2} - \frac{1}{2}$, $n \geq 0$. It is a H_q-semiclassical form of class one for $\mu \neq \frac{1}{q(q+1)} - \frac{1}{2}$, $\mu \neq -[n]_{q^2} - \frac{1}{2}$, $n \geq 0$, satisfying the H_q-Pearson equation*

$$H_q(x\mathcal{H}(\mu, q^2)) + (q+1)\left(x^2 - \mu - \frac{1}{2}\right)\mathcal{H}(\mu, q^2) = 0. \tag{3.20}$$

The recurrence coefficients of the MOPS $\{H_n^{\mu,q^2}\}_{n\geq 0}$ are given by

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = q^{2n}\left([n]_{q^2} + \mu + \frac{1}{2}\right), \\ \gamma_{2n+2} = q^{2n}[n+1]_{q^2}, \quad n \geq 0. \end{cases} \tag{3.21}$$

One can see that for $\mu = 0$, these polynomials are reduced to q-Hermite polynomials (see [17]).

The set $\left\{\mathcal{H}_n^{\mu,q^2}(x)\right\}_{n\geq 0}$ is an MOPS with respect to the regular form $\mathcal{H}(\mu, q^2)$.

This last form is $T_{\theta,q}$-classical and satisfies

$$T_{\theta,q}(\mathcal{H}(\mu, q^2)) = -q(q+1)x\mathcal{H}(\mu, q^2).$$

Proposition 3.6. *The q^2-analogue of the symmetrical generalized Gegenbauer form $\mathcal{G}(\alpha, \beta, q^2)$ is regular if and only if $\alpha + \beta \neq \frac{3-2q^2}{q^2-1}$, $\alpha + \beta \neq -[n]_{q^2} - 2$, $\beta \neq -[n]_{q^2} - 1$, $\alpha + \beta + 2 - (\beta+1)q^{2n} + [n]_{q^2} \neq 0$, $n \geq 0$. It is H_q-semiclassical of class one for $\alpha+\beta \neq \frac{3-2q^2}{q^2-1}$, $\alpha+\beta \neq -[n]_{q^2}-2$, $\beta \neq -[n]_{q^2}-1$, $\alpha+\beta+2-(\beta+1)q^{2n}+[n]_{q^2} \neq 0$, $n \geq 0$, $\beta \neq \frac{1}{q(q+1)} - 1$ satisfying H_q-Pearson equation*

$$H_q\big(x(x^2-1)\mathcal{G}(\alpha, \beta, q^2)\big) - (q+1)\big((\alpha+\beta+2)x^2 - (\beta+1)\big)\mathcal{G}(\alpha, \beta, q^2) = 0. \tag{3.22}$$

The recurrence coefficients of the MOPS $\{S_n^{(\alpha,\beta,q^2)}\}_{n\geq 0}$ are given by

$$\begin{cases} \beta_n &= 0, \quad n \geq 0, \\ \gamma_{2n+1} &= q^{2n}\frac{(\alpha+\beta+2+[n-1]_{q^2})(\beta+1+[n]_{q^2})}{(\alpha+\beta+2+[2n-1]_{q^2})(\alpha+\beta+2+[2n]_{q^2})}, \quad n \geq 0, \\ \gamma_{2n+2} &= q^n[n+1]_{q^2}\frac{\alpha+\beta+2-(\beta+1)q^{2n}+[n]_{q^2}}{(\alpha+\beta+2+[2n]_{q^2})(\alpha+\beta+2+[2n+1]_{q^2})}, \quad n \geq 0. \end{cases} \tag{3.23}$$

The set $\left\{\mathrm{S}_n^{(\alpha,\mu-\frac{1}{2},q^2)}\right\}_{n\geq 0}$ is an MOPS with respect to the regular form $\mathcal{G}(\alpha, \mu - \frac{1}{2}, q^2)$. This form is $T_{\theta,q}$-classical and satisfies

$$T_{\theta,q}\left(\left(x^2-1\right)\mathcal{G}^{(\alpha,\mu-\frac{1}{2},q^2)}\right) = q(q+1)(\alpha+1)x\mathcal{G}^{(\alpha,\mu-\frac{1}{2},q^2)}.$$

Lemma 3.7. [9] *If u_0 is a symmetric q-Dunkl-classical form, then $\tilde{u}_0 = h_{a^{-1}}u_0$ is also for every $a \neq 0$.*

Theorem 3.8. [3, 9] *Up to a dilatation, the only q-Dunkl-classical symmetric MOPS are:*

(a) *The generalized q^2-Hermite polynomials* $\left\{H_n^{\mu,q^2}(x)\right\}_{n\geq 0}$ *for* $\mu=\frac{\theta+1}{q(q+1)}-\frac{1}{2}$ *and* $\mu\neq -[n]_{q^2}-\frac{1}{2},\ n\geq 0.$
Moreover,

$$T_{\theta,q}(\mathcal{H}(\mu))+2x\mathcal{H}(\mu)=0.$$

(b) *The q^2-analogue of the generalized Gegenbauer polynomials* $\left\{S_n^{(\alpha,\beta,q^2)}(x)\right\}_{n\geq 0}$ *for*

$$\beta=\mu-\frac{1}{2}=\frac{\theta+1}{q(q+1)}-1;\quad \alpha+\beta\neq\frac{3-2q^2}{q^2-1};\quad \alpha+\beta\neq -[n]_{q^2}-2;\quad \beta\neq -[n]_{q^2}-1,$$

$$\alpha+\beta+2-(\beta+1)q^{2n}+[n]_{q^2}\neq 0,\quad n\geq 0;\quad \beta\neq\frac{1}{q(q+1)}-1.$$

Moreover,

$$T_{\theta,q}\left(\left(x^2-1\right)\mathcal{G}\left(\alpha,\mu-\frac{1}{2},q^2\right)\right)-2(\alpha+1)x\mathcal{G}\left(\alpha,\mu-\frac{1}{2},q^2\right)=0.$$

Finally, we characterize the q^2-analogue of generalized Hermite polynomials and the q^2-analogue of generalized Gegenbauer ones in terms of the Rodrigues type formula as follows.

Theorem 3.9. *We may write*

(1) $H_n^{\mu,q^2}(x)\mathcal{H}(\mu,q^2)=(-1)^n\prod_{v=1}^{n}\frac{\gamma_v^{\mathcal{H}}}{\theta_{v,q}}T_{\theta,q}^{n}(\mathcal{H}(\mu,q^2)),\quad n\geq 0.$ *with*

$$\begin{aligned}\gamma_{2n+1}^{\mathcal{H}} &= q^{2n}([n]_{q^2}+\mu+\frac{1}{2}),\\ \gamma_{2n+2}^{\mathcal{H}} &= q^{2n}[n+1]_{q^2},\ n\geq 0.\end{aligned}$$

(2) $S_n^{(\alpha,\mu-\frac{1}{2},q^2)}(x)\mathcal{G}\left(\alpha,\mu-\frac{1}{2},q^2\right)=\Lambda_n T_{\theta,q}^{n}\left(\left(\prod_{i=0}^{n-1}h_{q^i}\left(x^2-1\right)\right)\mathcal{G}\left(\alpha,\mu-\frac{1}{2},q^2\right)\right),\quad n\geq 0$

with

$$\begin{aligned}\Lambda_n &= (-1)^n q^{-n(n-1)}\xi_n\prod_{v=1}^{n}\frac{\gamma_v^{\mathcal{G}}}{\theta_{v,q}},\quad n\geq 0,\\ \gamma_{2n+1}^{\mathcal{G}} &= q^{2n}\frac{(\alpha+\mu+\frac{3}{2}+[n-1]_{q^2})(\mu+\frac{1}{2}+[n]_{q^2})}{(\alpha+\mu+\frac{3}{2}+[2n-1]_{q^2})(\alpha+\mu+\frac{3}{2}+[2n]_{q^2})},\\ \gamma_{2n+2}^{\mathcal{G}} &= q^{2n}[n+1]_{q^2}\frac{\alpha+\mu+\frac{3}{2}-(\mu+\frac{1}{2})q^{2n}+[n]_{q^2}}{(\alpha+\mu+\frac{3}{2}+[2n]_{q^2})(\alpha+\mu+\frac{3}{2}+[2n+1]_{q^2})},\ n\geq 0.\end{aligned}$$

Proof. Use Theorems 3.4 and 3.8, Propositions 3.5 and 3.6 and equation (3.19). ☐

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