

## SPACES OF CONTINUOUS AND MEASURABLE FUNCTIONS INVARIANT UNDER A GROUP ACTION

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**ABSTRACT.** In this paper we characterize spaces of continuous and  $L^p$ -functions on a compact Hausdorff space that are invariant under a transitive and continuous group action. This work generalizes Nagel and Rudin's 1976 results concerning unitarily and Möbius invariant spaces of continuous and measurable functions defined on the unit sphere in  $\mathbb{C}^n$ .

У статті ми характеризуємо простори неперервних і  $L^p$ -функцій на компактї, які є інваріантними відносно неперервної та транзитивної дії групи. Робота узагальнює результати Нагеля і Рудіна 1976 року про інваріантні простори неперервних і вимірних функцій визначений на одиничній сфері в  $\mathbb{C}^n$  відносно дій унітарної групи та групи Мебіуса.

### 1. INTRODUCTION

The idea for this paper came from the realization that much of Nagel and Rudin's work characterizing unitarily invariant spaces of continuous and measurable functions on the unit sphere of  $\mathbb{C}^n$  (originally found in [4] and summarized in [9]) could be generalized to spaces of continuous and measurable functions on a compact Hausdorff space  $X$ , which are invariant under the continuous and transitive action of a compact group  $G$  on  $X$ .

A space of complex functions on  $X$  is  $G$ -invariant if the pre-composition of any function in the set with the action of each element of  $G$  on  $X$  remains in the set. A  $G$ -invariant space is  $G$ -minimal if it contains no proper  $G$ -invariant subspace. Our main result (Theorem 4.1) yields that a collection of closed  $G$ -minimal spaces of continuous functions satisfying certain conditions suffices to characterize all closed  $G$ -invariant spaces of continuous functions on  $X$ : each closed  $G$ -invariant space is the closure of the direct sum of a unique subcollection of the  $G$ -minimal spaces.

A unique regular Borel probability measure  $\mu$  on  $X$  that is  $G$ -invariant in the sense that

$$\int_X f d\mu = \int_X f(\alpha x) d\mu(x),$$

for every continuous function  $f$  on  $X$  and every  $\alpha \in G$  is necessary to define the conditions for the collection of closed  $G$ -minimal spaces. Existence of such a measure is due to André Weil in [10]. Additionally, Theorem 4.1 shows each closed  $G$ -invariant space of measurable functions with respect to  $\mu$  is characterized by the same collection of  $G$ -minimal spaces.

In Section 3, we define the conditions which a collection of  $G$ -minimal spaces must have in order to induce the closed  $G$ -invariant spaces of continuous and measurable functions. In Section 4, we prove our main result, Theorem 4.1. Section 5 is devoted to the proofs of Lemma 4.3 and Lemma 4.4, which are used in establishing Theorem 4.1.

### 2. PRELIMINARIES

Let  $X$  be a compact Hausdorff space and  $C(X)$  the space of continuous complex functions with domain  $X$ . Let  $G$  be a compact group that acts continuously and

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transitively on  $X$ . When we wish to be explicit, the map  $\varphi_\alpha : X \rightarrow X$  shall denote the action of  $\alpha$  on  $X$  for each  $\alpha \in G$ ; otherwise,  $\alpha x$  denotes the action of  $\alpha \in G$  on  $x \in X$ .

Let  $\mu$  denote the unique regular Borel probability measure on  $X$  that is invariant under the action of  $G$ . Specifically,

$$\int_X f d\mu = \int_X f \circ \varphi_\alpha d\mu, \quad (2.1)$$

for all  $f \in C(X)$  and  $\alpha \in G$ . The existence of such a measure is a result of André Weil from [10]. A construction of  $\mu$  can be found in [1] (Theorem 6.2), but existence can be established using the Riesz Representation Theorem (for reference, Theorem 6.19 [7]). Throughout the paper,  $\mu$  shall refer to this measure.

The notation  $L^p(\mu)$  denotes the usual Lebesgue spaces, for  $1 \leq p \leq \infty$ . For  $Y \subset C(X)$ , the uniform closure of  $Y$  is denoted  $\bar{Y}$ , and for  $Y \subset L^p(\mu)$ , the norm-closure of  $Y$  in  $L^p(\mu)$  is denoted  $\bar{Y}^p$ .

The following is an easy consequence of (2.1):

**Remark 2.1.** Let  $1 \leq p < \infty$  and let  $p'$  be its conjugate exponent. Then

$$\int_X (f \circ \varphi_\alpha) \cdot g d\mu = \int_X f \cdot (g \circ \varphi_{\alpha^{-1}}) d\mu,$$

for  $f \in L^p(\mu)$ ,  $g \in L^{p'}(\mu)$ , and  $\alpha \in G$ .

The following definitions are generalizations of definitions found in [9] related to the unitary group. These more specific definitions are given as references.

**Definition 2.2** (12.2.4 [9]). A space of complex functions  $Y$  defined on  $X$  is **invariant under  $G$**  ( **$G$ -invariant**) if  $f \circ \varphi_\alpha \in Y$  for every  $f \in Y$  and every  $\alpha \in G$ .

**Remark 2.3.** Since the action is continuous,  $C(X)$  is  $G$ -invariant. Conversely, if  $C(X)$  is  $G$ -invariant, then each action  $\varphi_\alpha$  must be continuous.

**Remark 2.4.** Explicitly, the invariance property (2.1) means  $\mu(\alpha E) = \mu(E)$  for every Borel set  $E$  and every  $\alpha \in G$ . Consequently, (2.1) holds for every  $L^p$  function, and thus  $L^p(\mu)$  is  $G$ -invariant for all  $1 \leq p \leq \infty$ .

**Definition 2.5** (12.2.4 [9]). If  $Y$  is  $G$ -invariant and  $T$  is a linear transformation on  $Y$ , we say  $T$  **commutes with  $G$**  if

$$T(f \circ \varphi_\alpha) = (Tf) \circ \varphi_\alpha$$

for every  $f \in Y$  and every  $\alpha \in G$ .

**Definition 2.6** (12.2.8 [9]). A space  $Y \subset C(X)$  is  **$G$ -minimal** if it is  $G$ -invariant and contains no nontrivial  $G$ -invariant spaces.

**Example 2.7.** To illustrate these definitions, let  $X = G = T^n$ , the torus in  $\mathbb{C}^n$ , such that the action of  $G$  on  $X$  is given by coordinatewise multiplication. This action is both transitive and continuous.

For each  $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ , we define  $H_k$  to be the space of all complex functions  $f$  on  $T^n$  given by  $f(z) = cz^k$ , where  $c \in \mathbb{C}$  and  $z^k = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}$ ; that is,  $H_k$  is the span of the trigonometric monomial of power  $k$ .

Observe that  $\dim H_k = 1$ , so that each  $H_k$  is closed. Further,  $G$ -invariance of each  $H_k$  is clear, and thus each  $H_k$  is  $G$ -minimal.

Finally, the classical results used in this paper can be found in many texts, with the reference given being one such place.

3.  $G$ -COLLECTIONS

In this section, we introduce the notion of a  $G$ -collection, a collection of closed  $G$ -invariant spaces which characterize all closed  $G$ -invariant subspaces of  $C(X)$  (Definition 3.2). However, we must first give Definition 3.1. The particular case of the unitary group acting on the unit sphere in  $\mathbb{C}^n$  described in [9] inherently satisfies Definition 3.2(\*), so has no need of the following definition, but this is not necessarily true in general for  $G$  acting on  $X$ .

**Definition 3.1.** For each  $x \in X$ , the space  $H(x)$  is the set of all continuous functions that are unchanged by the action of any element of  $G$  which stabilizes  $x$ . That is,

$$H(x) = \{f \in C(X) : f = f \circ \varphi_\alpha, \text{ for all } \alpha \in G \text{ such that } \alpha x = x\}.$$

**Definition 3.2.** Let  $\mathcal{G}$  be a collection of spaces in  $C(X)$  with the following properties:

- (1) Each  $H \in \mathcal{G}$  is a closed  $G$ -minimal space.
- (2) Each pair  $H_1$  and  $H_2$  in  $\mathcal{G}$  is orthogonal (in  $L^2(\mu)$ ): If  $f_1 \in H_1$  and  $f_2 \in H_2$ , then

$$\int_X f_1 \bar{f}_2 d\mu = 0.$$

- (3)  $L^2(\mu)$  is the direct sum of the spaces in  $\mathcal{G}$ .

We say  $\mathcal{G}$  is a  $G$ -collection if it also possesses the following property:

- (\*)  $\dim(H \cap H(x)) = 1$  for each  $x \in X$  and each  $H \in \mathcal{G}$ .

**Remark 3.3.** A collection of spaces in  $C(X)$  lacking at most only property (\*) of Definition 3.2 always exists, as a consequence of the Peter-Weyl theorem from [5].

**Remark 3.4.** Explicitly, Definition 3.2(3) requires each  $f \in L^2(\mu)$  to have a unique expansion  $f = \sum f_i$ , with  $f_i \in H_i$ , that converges unconditionally to  $f$  in the  $L^2$ -norm.

Throughout the remainder of the paper, we assume that a  $G$ -collection  $\mathcal{G}$  exists for  $X$ , indexed by  $I$ . The rest of this section is devoted to establishing results related to  $\mathcal{G}$  and its elements  $H_i$ , beginning with the following theorem, which is a generalization of Theorem 12.2.5 of [9]. Note that we use  $[\cdot, \cdot]$  to denote the inner product on  $L^2(\mu)$ :

$$[f, g] = \int_X f \bar{g} d\mu.$$

**Theorem 3.5.** Suppose  $H$  is a closed  $G$ -invariant subspace of  $C(X)$ , and  $\pi$  is the orthogonal projection of  $L^2(\mu)$  onto  $H$ . Then,  $\pi$  commutes with  $G$ , and to each  $x \in X$  corresponds a unique  $K_x \in H$  such that

$$(\pi f)(x) = [f, K_x] \quad (f \in L^2(\mu)). \quad (3.2)$$

Additionally, the functions  $K_x$  satisfy the following:

- (1)  $K_x(y) = \overline{K_y(x)}$   $(x, y \in X)$ ,
- (2)  $\pi f = \int_X f(x) K_x d\mu(x)$   $(f \in L^2(\mu))$ ,
- (3)  $K_{\varphi_\alpha(x)} = K_x \circ \varphi_{\alpha^{-1}}$   $(\alpha \in G)$ ,
- (4)  $K_x = K_x \circ \varphi_\alpha$ , for all  $\alpha \in G$  such that  $\alpha x = x$ , and
- (5)  $K_x(x) = K_y(y) > 0$   $(x, y \in X)$ .

*Proof.* The projection  $\pi$  commutes with  $G$  due to the  $G$ -invariance of  $H^\perp$ , which follows from Corollary 2.1. The existence and uniqueness of  $K_x$  follows from the fact that  $f \mapsto (\pi f)(x)$  is a bounded linear functional on  $L^2(\mu)$ . Further,  $K_x \in H$  since  $\pi f = 0$  whenever  $f \perp H$ . When  $f \in H$ , we get

$$f(x) = [f, K_x].$$

In particular,  $K_y(x) = [K_y, K_x]$ , which proves (1), and (2) follows naturally. Since  $\pi$  commutes with  $G$ ,

$$[f, K_{\varphi_\alpha(x)}] = (\pi f)(\varphi_\alpha(x)) = \pi(f \circ \varphi_\alpha)(x) = [f \circ \varphi_\alpha, K_x] = [f, K_x \circ \varphi_{\alpha^{-1}}],$$

for every  $f \in L^2(\mu)$  (Corollary 2.1 yields the last equality). This proves (3) and the special case (4). Finally, (3) also yields

$$K_{\varphi_\alpha(x)}(\varphi_\alpha(x)) = (K_x \circ \varphi_{\alpha^{-1}})(\varphi_\alpha(x)) = K_x(x).$$

This and the transitivity of the group action yields (5), with the inequality due to

$$K_x(x) = [K_x, K_x] > 0. \quad \square$$

**Remark 3.6.** Theorem 3.5(4) yields that  $\dim(H(x) \cap H_i) \geq 1$  for each  $x \in X$  and  $i \in I$ , so that Definition 3.2(\*) requires each  $H_i$  to contain a unique (up to a constant multiple) function which satisfies Theorem 3.5(4) for each  $x \in X$ .

**Definition 3.7.** We define  $\pi_i$  to be the projection of  $L^2(\mu)$  onto  $H_i$ . The domain of each  $\pi_i$  is extended to  $L^1(\mu)$  by Theorem 3.5(2).

**Definition 3.8.** If  $\Omega \subset I$ , we define  $E_\Omega$  to be the direct sum of the spaces  $H_i$  for  $i \in \Omega$ .

**Theorem 3.9.** Suppose  $T : H_i \rightarrow H_j$  is linear and commutes with  $G$ . When  $i = j$ ,  $T$  is the identity on  $H_i$  scaled by a constant  $c$ . Otherwise,  $T = 0$ .

*Proof.* For each  $x \in X$ , let  $K_x$  denote the kernel of Theorem 3.5 in  $H_i$  and  $L_x$  the same in  $H_j$ . Then, if  $\alpha \in G$  such that  $\alpha x = x$ , because  $T$  commutes with  $G$ , we get

$$TK_x = T(K_x \circ \varphi_\alpha) = (TK_x) \circ \varphi_\alpha.$$

Thus, by Definition 3.2(\*),  $TK_x = c(x)L_x$  for some constant  $c(x)$ , and hence

$$(TK_x)(x) = c(x)L_x(x).$$

Observe that  $L_x(x)$  is independent of  $x$ . Further, if  $y = \alpha x$ , then

$$(TK_y)(y) = (TK_x \circ \varphi_{\alpha^{-1}})(\alpha x) = (TK_x)(x).$$

Thus,  $c(x) = c$  is the same constant for all  $x \in X$ .

If  $f \in H_i$ , we then get

$$f = \int_X f(x)K_x d\mu(x).$$

Application of  $T$  yields

$$Tf = \int_X f(x)TK_x d\mu(x) = c \int_X f(x)L_x d\mu(x) = c\pi_j f.$$

When  $i = j$ , then  $\pi_j f = f$  for all  $f \in H_i$ . When  $i \neq j$ , then  $\pi_j f = 0$  for all  $f \in H_i$ .  $\square$

#### 4. CHARACTERIZATION OF CLOSED G-INVARIANT SPACES

We now prove our main result, Theorem 4.1. Throughout the section, we let  $\mathcal{X}$  denote any of the spaces  $C(X)$  or  $L^p(\mu)$  for  $1 \leq p < \infty$ .

**Theorem 4.1.** If  $Y$  is a closed  $G$ -invariant subspace of  $\mathcal{X}$ , then  $Y$  is the closure of the direct sum of some subcollection of  $\mathcal{G}$ .

The proof of Theorem 4.1 relies on the particular case when  $\mathcal{X}$  is the space  $L^2(\mu)$  (Theorem 4.2), as well as Lemma 4.3 and Lemma 4.4, which allow the passage from  $L^2(\mu)$  to the other spaces. These lemmas are proved in Section 5.

**Theorem 4.2.** If  $Y$  is a closed  $G$ -invariant subspace of  $L^2(\mu)$ , then  $Y$  is the  $L^2$ -closure of the direct sum of some subcollection of  $\mathcal{G}$ .

*Proof.* Define the set  $\Omega = \{i \in I : \pi_i Y \neq \{0\}\}$  and let  $i \in \Omega$ . Since  $Y$  is  $G$ -invariant and  $\pi_i$  commutes with  $G$ ,  $\pi_i Y$  is a nontrivial  $G$ -invariant space in  $H_i$ . The  $G$ -minimality of  $H_i$  then yields that  $\pi_i Y = H_i$ .

Let  $Y_0$  be the null space of  $\pi_i$  in  $Y$ , with relative orthogonal complement  $Y_1$ . Then  $Y_0$  is  $G$ -invariant, and so is  $Y_1$ . Further,  $\pi_i : Y_1 \rightarrow H_i$  is an isomorphism, whose inverse we denote  $\Lambda$ . If we fix  $j \in I$  such that  $j \neq i$  and define  $T = \pi_j \circ \Lambda$ , then  $T$  maps  $H_i$  into  $H_j$  and commutes with  $G$ . Thus,  $T = 0$ .

We conclude that  $\pi_j Y_1 = \{0\}$  for all  $j \neq i$ , and thus  $Y_1 = H_i$ . Thus,  $H_i \subset Y$  for all  $i \in \Omega$ , and further,  $\overline{E_\Omega} \subset Y$ . Since  $\pi_j Y = \{0\}$  for all  $j \notin \Omega$ , Definition 3.2(3) yields the opposite inclusion.  $\square$

**Lemma 4.3.** *If  $Y$  is a closed  $G$ -invariant space in  $\mathcal{X}$ , then  $Y \cap C(X)$  is dense in  $Y$ .*

**Lemma 4.4.** *If  $Y \subset C(X)$ ,  $Y$  is a  $G$ -invariant space, and some  $g \in C(X)$  is not in the uniform closure of  $Y$ , then  $g$  is not in the  $L^2$ -closure of  $Y$ .*

*Proof of Theorem 4.1.* If  $Y$  is a closed  $G$ -invariant subspace of  $\mathcal{X}$ , define  $\tilde{Y}$  to be the  $L^2$ -closure of  $Y \cap C(X)$ . Lemma 4.4 then yields

$$\tilde{Y} \cap C(X) = Y \cap C(X).$$

We next observe that  $Y \cap C(X)$  is  $L^2$ -dense in  $\tilde{Y}$  and is  $\mathcal{X}$ -dense in  $Y$ , by Lemma 4.3. Each  $\pi_i$  is  $\mathcal{X}$ -continuous as well as  $L^2$ -continuous, so that  $\pi_i Y = \{0\}$  if and only if  $\pi_i \tilde{Y} = \{0\}$ . By Theorem 4.2,  $\tilde{Y}$  is the  $L^2$ -closure of  $E_\Omega$ , where  $\Omega$  is the set of all  $i \in I$  such that  $\pi_i Y \neq \{0\}$ . Another application of Lemma 4.4 yields

$$\tilde{Y} \cap C(X) = \overline{E_\Omega}.$$

The  $\mathcal{X}$ -density of  $Y \cap C(X)$  in  $Y$  then implies  $Y$  is the  $\mathcal{X}$ -closure of  $E_\Omega$ .  $\square$

**Example 4.5.** We now further explore the situation that was set up in Example 2.7. Recall that  $X = G = T^n$ , the torus in  $\mathbb{C}^n$ , such that the action of  $G$  on  $X$  is given by coordinatewise multiplication. This action is both transitive and continuous, and the measure induced by the action is the usual Lebesgue measure  $m$ , normalized so that  $m(T^n) = 1$ .

Further,  $H_k$  is the space of all complex functions  $f$  on  $T^n$  given by  $f(z) = cz^k$ , where  $c \in \mathbb{C}$  and  $z^k = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}$  for  $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ ; that is,  $H_k$  is the span of the trigonometric monomial of power  $k$ .

The collection  $\mathcal{G}$  of spaces  $H_k$  forms a  $G$ -collection: Each  $H_k$  is a closed  $G$ -invariant space of dimension 1, thus is  $G$ -minimal. Further,

$$\int_{T^n} z^k \bar{z}^{k'} dm(z) = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k', \end{cases}$$

so that the spaces  $H_k$  are pairwise orthogonal. Finally,  $L^2(T^n)$  is the direct sum of the spaces  $H_k$  as a consequence of the Stone-Weierstrass theorem (presented in [8] as a special case of Bishop's Theorem, Theorem 5.7). Thus,  $\mathcal{G}$  satisfies the first three properties of Definition 3.2. Lastly, for  $z \in T^n$ , we have  $\dim(H(z) \cap H_k) = 1$  from Theorem 3.5 and the fact that  $\dim H_k = 1$ . Thus,  $\mathcal{G}$  is a  $G$ -collection.

Theorem 4.1 then yields that every closed  $G$ -invariant space of continuous or  $L^p$  functions on  $T^n$  is the closure of the direct sum of some collection of spaces  $H_k$ . Notably, the collection which induces the space of all functions which are restrictions to  $T^n$  of functions holomorphic on the polydisc and continuous on the closed polydisc is the collection of all  $H_k$  such that  $k$  has nonnegative coordinates.

## 5. PROOFS OF LEMMA 4.3 AND LEMMA 4.4

As in Section 4, we let  $\mathcal{X}$  denote any of the spaces  $C(X)$  or  $L^p(\mu)$ , for  $1 \leq p < \infty$ . The proofs of Lemma 4.3 and Lemma 4.4 (given at the end of the section) require Lemma 5.1 and Lemma 5.2, which we now prove.

**Lemma 5.1.** *If  $f \in C(X)$ , then the map  $\alpha \mapsto f \circ \varphi_\alpha$  is a continuous map of  $G$  into  $C(X)$ .*

*Proof.* For  $\alpha \in G$ , we define the map  $\phi : G \rightarrow C(X, X)$  by  $\phi(\alpha) = \varphi_\alpha$ . Then  $\phi$  is continuous when  $C(X, X)$  is given the compact-open topology (Theorem 46.11 of [3]). We note that the continuity of the group action is used here.

We define the map  $T_f : C(X, X) \rightarrow C(X)$  for  $f \in C(X)$  by  $T_f(\varphi) = f \circ \varphi$ , and we endow both spaces with the compact-open topology. Let  $f \circ \varphi \in C(X)$  for  $\varphi \in C(X, X)$  and suppose  $f \circ \varphi \in V$ , where  $V = V(K, U)$  is a subspace element in  $C(X)$ . Explicitly,

$$K \subset (f \circ \varphi)^{-1}(U). \quad \text{That is to say,} \quad K \subset \varphi^{-1}(f^{-1}(U)).$$

Then  $V' = V'(K, f^{-1}(U))$  is a subspace element in  $C(X, X)$  and  $\varphi \in V'$ . Further,  $V' \subset T_f^{-1}(V)$ , so that  $T_f$  is continuous when  $C(X, X)$  and  $C(X)$  are endowed with the respective compact-open topologies. We finally observe that since  $X$  is compact, the norm topology and the compact-open topology on  $C(X)$  coincide.  $\square$

**Lemma 5.2.** *If  $f \in \mathcal{X}$ , then the map  $\alpha \mapsto f \circ \varphi_\alpha$  is a continuous map of  $G$  into  $\mathcal{X}$ .*

*Proof.* For brevity, we let  $\|\cdot\|$  denote the norm of the space  $\mathcal{X}$  and  $\|\cdot\|_\infty$  the uniform norm in  $C(X)$ . If  $\epsilon > 0$ , then  $\|f - g\| < \epsilon/3$  for some  $g \in C(X)$ . There is a neighborhood  $N$  of the identity in  $G$  such that  $\|g - g \circ \varphi_\alpha\|_\infty < \epsilon/3$  for all  $\alpha \in N$  (Lemma 5.1). Since

$$\|f - f \circ \varphi_\alpha\| \leq \|f - g\| + \|g - g \circ \varphi_\alpha\| + \|(g - f) \circ \varphi_\alpha\|,$$

we have  $\|f - f \circ \varphi_\alpha\| < \epsilon$  for all  $\alpha \in N$ .  $\square$

*Proof of Lemma 4.3.* Let  $f \in Y$  and choose  $N$  as in the proof of Lemma 5.2. Let  $\psi : G \rightarrow [0, \infty)$  be continuous, with support in  $N$ , such that  $\int \psi \, dm = 1$ , where  $m$  denotes the Haar measure on  $G$ . Define

$$g(x) = \int_G \psi(\alpha) f(\alpha x) \, dm(\alpha).$$

Since  $\alpha \mapsto \psi(\alpha) f \circ \varphi_\alpha$  is a continuous map into  $Y$ , we have  $g \in Y$ . If  $\beta \in G$  such that  $\beta x_1 = x$ , the invariance of the Haar measure yields

$$g(x) = \int_G \psi(\alpha \beta^{-1}) f(\alpha x_1) \, dm(\alpha).$$

Thus,  $g \in Y \cap C(X)$ .

Finally, the relation

$$f - g = \int_N \psi(\alpha) (f - f \circ \varphi_\alpha) \, dm(\alpha)$$

gives  $\|f - g\| < \epsilon$ , since  $\|f - f \circ \varphi_\alpha\| < \epsilon$  whenever  $\alpha \in N$ .  $\square$

*Proof of Lemma 4.4.* There is a  $\mu' \in M(X)$  such that  $\int f \, d\mu' = 0$  for all  $f \in Y$ , but  $\int g \, d\mu' = 1$ . There is a neighborhood  $N$  of the identity in  $G$  such that  $\operatorname{Re} \int g \circ \varphi_\alpha \, d\mu' > \frac{1}{2}$  for every  $\alpha \in N$ . Associate  $\psi$  to  $N$  as in the proof of Lemma 4.3, and define  $\Lambda \in C(X)^*$  by

$$\Lambda h = \int_X \int_G \psi(\alpha) h(\alpha x) \, dm(\alpha) \, d\mu'(x).$$

By the Schwarz inequality,

$$\left| \int_G \psi(\alpha) h(\alpha x) \, dm(\alpha) \right|^2 \leq \int_G |\psi(\alpha)|^2 \, dm(\alpha) \int_G |h(\alpha x)|^2 \, dm(\alpha) = \|\psi\|_2^2 \int_X |h|^2 \, d\mu,$$

so that

$$|\Lambda h| \leq \|\mu'\| \|\psi\|_2 \|h\|_2.$$

Thus,  $\Lambda$  extends to a bounded linear functional  $\Lambda_1$  on  $L^2(\mu)$ . By interchanging integrals in the definition of  $\Lambda$ , we get  $\Lambda_1 f = 0$  for every  $f \in Y$ , but  $\operatorname{Re} \Lambda_1 g \geq \frac{1}{2}$ . Thus,  $g \notin \bar{Y}^2$ .  $\square$

## 6. FUTURE QUESTIONS

- (1) Does a  $G$ -collection exist for all groups  $G$  acting continuously and transitively on  $X$ ? What about a collection that only lacks  $(*)$ ? What conditions might exist on  $G$  or  $X$  that yield a collection that only lacks  $(*)$ ?
- (2) Under what conditions can the restrictions on  $X$ ,  $G$ , and the action of  $G$  on  $X$  be loosened? Can the compactness of  $X$  and  $G$  be substituted with local compactness? Can the continuity of the action be substituted with separate continuity?
- (3) Suppose  $H$  is a subgroup of  $G$  and  $\mathcal{H}$  is a collection of closed  $H$ -minimal spaces satisfying the same conditions as  $\mathcal{G}$ . What is the relationship between  $\mathcal{H}$  and  $\mathcal{G}$ ? The uniqueness of  $\mu$  shows that the  $H$ -measure is the same as the  $G$ -measure, and further,  $G$ -invariance implies  $H$ -invariance (of a space).

We note that (3) is prompted from the study of  $\mathcal{M}$ -invariant and  $\mathcal{U}$ -invariant spaces of continuous functions on the unit sphere of  $\mathbb{C}^n$  from [4], in which it is shown that there are infinitely many  $\mathcal{U}$ -invariant spaces and only six  $\mathcal{M}$ -invariant spaces. These six  $\mathcal{M}$ -invariant spaces are found by combining the  $\mathcal{U}$ -minimal spaces in a specific way (see Lemma 13.1.2 of [9]), and we are curious if this method can be generalized.

- (4) Can the results of [2] similarly be generalized? That is, can a  $G$ -collection similarly characterize all weak\*-closed  $G$ -invariant subspaces of  $L^\infty(\mu)$ ?
- (5) Under what conditions can a  $G$ -collection characterize the closed  $G$ -invariant algebras of continuous functions? We note that the case for the unitary group acting on the unit sphere of  $\mathbb{C}^n$  is discussed in [6] and is also summarized in [9].

## 7. DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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