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WEAVING OPERATOR FRAMES FOR $B(\mathcal{H})$

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Abstract. This paper aims to study the concept of weaving operator frames within Hilbert spaces $\mathcal{H}$. Properties of weaving operator frames are explored. An investigation into the dual aspect of weaving operator frames within $B(\mathcal{H})$ spaces is presented. The behavior and characteristics of weaving operator responses within the context of Hilbert spaces are discuted. Finally, perturbation results concerning weaving operator frames are obtained.

В статті вивчається концепція фреймів сплітаючих операторів в гільбертових просторах $\mathcal{H}$. Досліджуються властивості фреймів сплітаючих операторів. Вивчено подвійний аспект фреймів сплітаючих операторів в просторах $B(\mathcal{H})$. Обговорено поведінку та характеристики реакцій сплітаючого оператора в контексті гільбертових просторів. Отримано результати збурення фреймів сплітаючих операторів.

1. Introduction

Frames in Hilbert spaces has been introduced by Duffin and Schaeffer [7] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [5] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frame, and Gabor frame [8]. Frames have been used in signal processing, image processing, data compression, and sampling theory. However, Vashisht et al. [16] introduced weaving frames and many authors [1, 3, 4, 6, 15] have studied their properties in light of recent technological advancements, wireless communications, and weaving frames. For more about frames, see [9, 11, 12, 13, 14] and the references therein. Recently, Bemrose et al. [1] introduced a new concept of *weaving frames* in separable Hilbert spaces. This notion has potential applications in distributed signal processing and wireless sensor networks, see for example [2, 4].

The paper is organized as follows. We continue this section by giving the definitions and some basic results about frames in a Hilbert space. In Section 2 the concept of operator frames in Hilbert spaces is introduced, and we give some of their properties. Subsequently, we introduce the concept of a dual weaving operator frame in Section 3. In Section 4, we introduce the notion of weaving operator responses of elements of Hilbert spaces to show that the concept of operator frames is a generalization of the usual frames for Hilbert spaces. Sufficient conditions for perturbations of weaving operator frames are given in Section 5.

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces and $\{\mathcal{H}_i\}_{i\in I}$ be a sequence of closed subspaces of $\mathcal{K}$, where I is a subset of $\mathbb{N}$. Let $B(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{K}$. We write $B(\mathcal{H}) = B(\mathcal{H},\mathcal{H})$ in the case where $\mathcal{K} = \mathcal{H}$. We denote by $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we denote $T^\dagger$ for pseudo-inverse of T. Let

$$[m] = \{1, 2, \cdots, m\} \text{ and } [m]^c = \mathbb{N} \setminus [m] = \{m+1, m+2, \dots\}$$

for a given positive integer m.

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Definition 1.1 ([7]). A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is said to be a frame if there are constants $0 < A \leq B < \infty$ such that, for every $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \tag{1.1}$$

where A and B are lower frame bound and upper frame bound, respectively.

If $A = B$, the frame is termed a tight frame. If $A = B = 1$, it is referred to as a normalized tight or Parseval frame. Additionally, if a sequence $\{f_i\}_{i\in I}$ fulfills only the upper bound condition in (1.1), it is also denoted as a Bessel sequence.

Definition 1.2 ([10]). A set of bounded linear operators $\{T_i\}_{i\in I}$ defined on a Hilbert space $\mathcal{H}$ is termed an operator frame for $B(\mathcal{H})$ if there exist positive constants A and B such that for all $f \in \mathcal{H}$, the inequality

$$A\|f\|^2 \leq \sum_{i\in I} \|T_i f\|^2 \leq B\|f\|^2 \tag{1.2}$$

holds, where A and B represent the lower and upper bounds for the operator frame, respectively.

An operator frame $\{T_i\}_{i\in I}$ is called tight if the constants A and B can be chosen to be equal. It is called a Parseval operator frame when $A = B = 1$. Moreover, if every operator T_i is self-adjoint, i.e., $T_i = T_i^*$, it is called a self-adjoint operator frame. For each sequence $\{\mathcal{H}_i\}_{i\in I}$, we define the space $\oplus_{i\in I}\mathcal{H}_i$ by

$$\oplus_{i\in I}\mathcal{H}_i = \{\{f_i\}_{i\in I} | f_i \in \mathcal{H}_i, \|\{f_i\}_{i\in I}\|_2^2 = \sum_{i\in I} \|f_i\|^2 < \infty\},$$

with the inner product defined by

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle .$$

The synthesis operator of $\{\Phi_i\}_{i\in I}$ is given by

$$T_\Phi : \oplus\mathcal{H}_i \longrightarrow \mathcal{H}; \quad T_\Phi\{g_i\}_{i\in I} = \sum_{i\in I} \Phi_i^* g_i, \quad \text{for all } g_i \in \mathcal{H}_i.$$

We call the adjoint of T_Φ the analysis operator which is given by $T_\Phi^* f = \{\Phi_i f\}_{i\in I}$.

By composing T_Φ and T_Φ^*, we obtain the frame operator

$$S_\Phi f = T_\Phi T_\Phi^* f = \sum_{i\in I} \Phi_i^* \Phi_i f$$

which is bounded, positive and invertible. Then, the following reconstruction formula takes place for all $f \in \mathcal{H}$

$$f = S_\Phi^{-1} S_\Phi f = S_\Phi S_\Phi^{-1} f = \sum_{i\in I} \Phi_i^* \Phi_i S_\Phi^{-1} f = \sum_{i\in I} S_\Phi^{-1} \Phi_i^* \Phi_i f.$$

We call $\{\Phi_i S_\Phi^{-1}\}_{i\in I}$ the canonical dual operator frame of $\{\Phi_i\}_{i\in I}$.

2. Weaving operator frames for $B(\mathcal{H})$

This section introduces the concept of woven operator frames in the context of a Hilbert space $\mathcal{H}$. Woven operator frames extend the notion of classical operator frames by giving two families of operator frames. We will formally define woven operator frames and explore some of their properties.

Definition 2.1. Two operator frames $\{\Phi_i \in B(\mathcal{H})\}_{i\in I}$ and $\{\Psi_i \in B(\mathcal{H})\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ are said to be woven operator frames if there exist universal constants $0 < A \leq B < \infty$ such that for each partition σ of I, the family $\{\Phi_i\}_{i\in\sigma} \cup \{\Psi_i\}_{i\in\sigma^c}$ is an operator frame in $\mathcal{H}$ with bounds A and B, respectively, that is,

$$A\|f\|^2 \leq \sum_{i\in\sigma} \|\Phi_i f\|^2 + \sum_{i\in\sigma^c} \|\Psi_i f\|^2 \leq B\|f\|^2. \tag{2.3}$$

This definition introduces the concept of woven operator frames in the context of a Hilbert space.

Definition 2.2. A family of operator frames $\{\Phi_{ij}\}_{j\in\sigma_i}$ for $i \in I$ in a Hilbert space $\mathcal{H}$ is said to be woven operator frames if there exist universal positive constants A and B such that for any partition $(\sigma_i)_{i\in I}$ of $\mathbb{N}$, the family $\{\cup_{i\in I}\Phi_{ij}\}_{j\in\sigma_i}$ is an operator frame in $\mathcal{H}$ with bounds A and B, respectively, that is,

$$A\|f\|^2 \leq \sum_{i\in I} \sum_{j\in\sigma_i} \|\Phi_{ij} f\|^2 \leq B\|f\|^2. \tag{2.4}$$

Note that if I is a countably infinite set, the family $\{\Phi_{ij}\}_{j=1}^{\infty}$ for $i \in I$ is referred to as infinitely woven.

In the context of discrete Hilbert frames, the concept of *infinitely woven frames* was extensively studied by Deepshikha and L. K. Vashisht in their paper [6].

Proposition 2.3. *Let $\{\Phi_{ij}\}_{j\in\sigma_i}$ for $i \in I$ be a family of woven operator frames in a Hilbert space $\mathcal{H}$. Then the frame operator S is self-adjoint, positive, bounded and invertible on $\mathcal{H}$.*

Proof. Since $S_\Phi^* = (T_\Phi^* T_\Phi)^* = T_\Phi^* T_\Phi = S_\Phi$, the frame operator S_Φ is self adjoint.

Let $\{\Phi_{ij}\}_{j=1}^{\infty}$ be woven operator frames in $\mathcal{H}$ with universal lower and upper frame bounds A and B, respectively. Then

$$\begin{aligned}\langle S_\Phi f, f\rangle = \langle \sum_{i\in I} \sum_{j\in\sigma_i} \Phi_{ij}^* \Phi_{ij} f, f\rangle &= \sum_{i\in I} \sum_{j\in\sigma_i} \langle \Phi_{ij}^* \Phi_{ij} f, f\rangle \\ &= \sum_{i\in I} \sum_{j\in\sigma_i} \langle \Phi_{ij} f, \Phi_{ij} f\rangle = \sum_{i\in I} \sum_{j\in\sigma_i} \|\Phi_{ij} f\|^2\end{aligned}$$

and, hence,

$$AI \leq S_\Phi \leq BI.$$

Therefore, the frame operator S is positive, bounded and invertible. □

Proposition 2.4. *If each $\Phi_j = \{\Phi_{ij}\}_{i\in I}$ is a g-Bessel sequence for a Hilbert space $\mathcal{H}$ with bounds B_j for all $j \in [m]$, then every weaving is a g-Bessel sequence with $\sum_{j=1}^{m} B_j$ as a Bessel bound.*

Proof. Let $\{\sigma_j\}_{j\in[m]}$ be any partition of I. Then, for every $f \in \mathcal{H}$, we have

$$\sum_{j=1}^{M} \sum_{i\in\sigma_j} \|\Phi_{ij} f\|^2 \leq \sum_{j=1}^{M} \sum_{i\in I} \|\Phi_{ij} f\|^2 \leq \sum_{j=1}^{M} B_j \|f\|^2.$$

This completes the proof. □

Remark 2.5. Proposition 2.4 also holds for infinitely woven frames, given that the sequence $\{B_j\}_{j=1}^{\infty}$ belongs to $\ell^1(I)$.

Proposition 2.6. *Let $\{\Phi_i\}_{i\in\mathbb{N}}$ and $\{\Psi_i\}_{i\in\mathbb{N}}$ be g-Bessel sequences in a Hilbert space $\mathcal{H}$ with g-Bessel bounds A_1 and A_2, respectively. If $J \subset \mathbb{N}$ and $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are woven operator frames, then Φ and Ψ are woven operator frames for $\mathcal{H}$.*

Proof. We have

$$A\|f\|^2\leq \sum_{j\in\sigma\cap J}\|\Phi_j f\|^2+\sum_{j\in\sigma^c\cap J}\|\Psi_j f\|^2\leq \sum_{j\in\sigma}\|\Phi_j f\|^2+\sum_{j\in\sigma^c}\|\Psi_j f\|^2\leq (A_1+A_2)\|f\|^2.$$

Hence Φ and Ψ are woven operator frames for $\mathcal{H}$. □

Proposition 2.7. *Let $J\subset I$. If a family of operator frames $\{\Phi_{ij}\}_{i\in J,j\in[m]}$ is woven, then $\{\Phi_{ij}\}_{i\in I,j\in[m]}$ is also woven.*

Proof. For any $\sigma_j\subset I$, $\sigma_j\cap J\subset J$. Let A be a lower bound of $\{\Phi_{ij}\}_{i\in\sigma_j\cap J,j\in[m]}$. Then for any $f\in\mathcal{H}$ we have

$$A\|f\|^2\leq \sum_{j=1}^{m}\sum_{i\in\sigma_j\cap J}\|\Phi_{ij}f\|^2\leq \sum_{j=1}^{m}\sum_{i\in\sigma_j}\|\Phi_{ij}f\|^2.$$

Since $\{\Phi_{ij}\}_{i\in I}$ is a g-Bessel sequence for all $j\in[m]$ for $\mathcal{H}$, the upper bound of $\{\Phi_{ij}\}_{i\in I,j\in[m]}$ is always given. This implies that $\{\Phi_{ij}\}_{i\in I,j\in[m]}$ is woven for $\mathcal{H}$. □

Proposition 2.8. *Let $\{\Phi_{ij}\}_{i\in I,j\in[m]}$ be a woven family of g-frames for a Hilbert space $\mathcal{H}$ with common frame bounds A and B. Let $S_{\Phi}^{(j)}$ be the frame operator of $\{\Phi_{ij}\}_{i\in I}$ for each $j\in[m]$. For any partition σ_j of I, if S_Ψ represents the frame operator of $\Psi=\{\Phi_{ij}\}_{i\in\sigma_j,j\in[m]}$, then for any $f\in\mathcal{H}$,*

$$\sum_{j\in[m]}\|(S_{\Phi}^{(j)})_{\sigma_j}f\|^2\leq B\|S_\Psi\|\|f\|^2,$$

where $(S_{\Psi}^{(j)})_{\sigma_j}$ denotes the frame operator $S_{\Psi}^{(j)}$ with sum restricted to σ_j.

Proof. Let $(T_{\Phi}^{(j)})_{\sigma_j}$ be the synthesis operator of $\{\Phi_{ij}\}_{i\in I}$ restricted to the sum over σ_j. Since $S_{\Phi}^{(j)}\geq AI_{\mathcal{H}}$, for any $f\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in[m]}\|\langle S_{\Phi,\sigma_j}^{(j)}f,S_{\Phi,\sigma_j}^{(j)}f\rangle_{\mathcal{A}}\| &= \sum_{j\in[m]}\|S_{\Phi,\sigma_j}^{(j)}f\|^2\\ &=\sum_{j\in[m]}(\sup_{\|g\|=1}\|\langle S_{\Phi,\sigma_j}^{(j)}f,g\rangle_{\mathcal{A}}\|)^2\\ &=\sum_{j\in[m]}(\sup_{\|g\|=1}\|\langle T_{\Phi,\sigma_j}^{(j)}(T_{\Phi,\sigma_j}^{(j)})^*f,g\rangle_{\mathcal{A}}\|)^2\\ &\leq\sum_{j\in[m]}B\|\langle (T_{\Phi,\sigma_j}^{(j)})^*f,(T_{\Phi,\sigma_j}^{(j)})^*f\rangle_{\mathcal{A}}\|\\ &=B\sum_{j\in[m]}\|\sum_{\sigma_j}\langle\Phi_{ij}f,\Phi_{ij}f\rangle_{\mathcal{A}}\|\\ &\leq B\|\langle S_{\Phi,\sigma_j}f,f\rangle_{\mathcal{A}}\|\\ &\leq B\|S_{\Phi,\sigma_j}\|\|\langle f,f\rangle_{\mathcal{A}}\|.\end{aligned}$$

This completes the proof. □

3. Dual of weaving operator frames for $B(\mathcal{H})$

This section explores the dual aspects of weaving operator frames in $B(\mathcal{H})$,

Definition 3.1. Let $T=\{T_i\}_{i\in I}$ be an weaving operator frame for $B(\mathcal{H})$. A family of weaving operators $\widetilde{T}=\left\{\widetilde{T}_i\right\}_{i\in I}$ on $\mathcal{H}$ is called a dual of weaving operator frame $T=\{T_i\}_{i\in I}$ if it satisfies

$$x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\widetilde{T}_i x,\quad \forall x\in\mathcal{H}. \tag{3.5}$$

Furthermore, we call $\left\{\widetilde{T}_i\right\}_{i\in I}$ a dual of weaving operator frame $T=\{T_i\}_{i\in I}$ if $\left\{\widetilde{T}_i\right\}_{i\in I}$ is also a weaving operator frame for $B(\mathcal{H})$ and satisfies the condition (3.5).

Definition 3.2. A weaving operator sequence $T=\{T_i\}_{i\in I}$ on $\mathcal{H}$ is a weaving operator Riesz basis for $B(\mathcal{H})$ if it satisfies

(i) $\overline{span}\,\{T_i^*\}_{i\in I}=\mathcal{H}$;
(ii) there exist constants $C,D>0$ such that

$$C\left\|\{x_i\}_{i\in I}\right\|^2\le\left\|\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*x_i\right\|^2\le D\left\|\{x_i\}_{i\in I}\right\|^2,\quad \forall\{x_i\}_{i\in I}\in\oplus_{i\in I}\mathcal{H}_i. \tag{3.6}$$

Theorem 3.3. *Every weaving frame for $B(\mathcal{H})$ has a weaving dual frame.*

Proof. If $T=\{T_i\}_{i\in I}$ is a weaving operator frame for $B(\mathcal{H})$ with bounds A,B, then the operator sequence $\widetilde{T}=\left\{T_iS_T^{-1}\right\}_{i\in I}$ is a weaving dual frame of $T=\{T_i\}_{i\in I}$. So we have

$$x=S_TS_T^{-1}x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*T_iS_T^{-1}x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\tilde{T}_ix,\quad \forall x\in\mathcal{H}$$

and $\widetilde{T}=\left\{T_iS_T^{-1}\right\}_{i\in I}$ satisfies

$$A\left\|S_T\right\|^{-2}\cdot\|x\|^2\le\sum_{i\in I}\sum_{i\in\sigma_i}\left\|\widetilde{T}_ix\right\|^2=\sum_{i\in I}\sum_{i\in\sigma_i}\left\|T_iS_T^{-1}x\right\|^2\le B\left\|S_T^{-1}\right\|^2\cdot\|x\|^2,\quad \forall x\in\mathcal{H}.$$

Hence $\left\{T_iS_T^{-1}\right\}_{i\in I}$ is a canonical weaving dual frame of $\{T_i\}_{i\in I}$. □

Assume that $T=\left\{T_i\right\}_{i\in I}$ is a weaving operator frame for $B(\mathcal{H})$ with analytic operator R_T and $\widetilde{T}=\left\{\widetilde{T}_i\right\}_{i\in I}$ is a weaving dual frame of T with analytic operator $R_{\widetilde{T}}$. Then for any x in $\mathcal{H}$, we have

$$x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\widetilde{T}_ix=R_T^*R_{\widetilde{T}}x. \tag{3.7}$$

This shows that every element of $\mathcal{H}$ can be reconstructed with a weaving operator frame for $B(\mathcal{H})$ and its weaving dual frame. Moreover, we also have another fact that for any operator A on $\mathcal{H}$, we get

$$Ax=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\widetilde{T}_iAx,\quad \forall x\in\mathcal{H}. \tag{3.8}$$

That is, an association of the weaving operator frame and its dual frame can reconstruct pointwisely every weaving operator on $\mathcal{H}$ and so we can write

$$A\doteq\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\tilde{T}_iA,$$

where $\sum_{i\in I}T_i^*\widetilde{T}_iA$ converges strongly to A.

Definition 3.4. A family of weaving operators $\{T_i\}_{i\in I}$ on $\mathcal{H}$ is called a weaving operator Bessel sequence in $B(\mathcal{H})$ if

$$\overline{span}\,\{T_i^*\}_{i\in I}=\mathcal{H},$$

where

$$\overline{span}\{T_i^*\}_{i\in I}=\text{ the cloure of }\Big\{\sum_{i\in L}T_i^*x_i:\{x_i\}_{i\in I}\in S(\mathcal{H}),\forall L\in\mathcal{F}(\Phi)\Big\}.$$

Theorem 3.5. *Let $T=\{T_i\}_{i\in I}$ be a weaving operator Bessel sequence in $B(\mathcal{H})$, then*

(1) *$T=\{T_i\}_{i\in I}$ is a weaving operator frame for $B(\mathcal{H})$ if and only if R_T is bounded below;*
(2) *$T=\{T_i\}_{i\in I}$ is an independent weaving operator frame for $B(\mathcal{H})$ if and only if R_T is invertible*

Proof. The proof of (1) is easy and so we omit it. Assume that $T=\{T_i\}_{i\in I}$ is an independent weaving operator frame. Then we now prove that R_T is invertible. From the condition and the definition of independent weaving operator frame, we know that R_T^* is injective, and so $R\left(R_T\right)=\operatorname{Ker}\left(R_T^*\right)=\{0\}$.

This shows that the range of R_T is dense in $\mathcal{H}$. On the other hand, from (1), we know that R_T is bounded below, and so $R\left(R_T\right)$ is closed. Hence R_T is invertible. Conversely, if R_T is invertible, then R_T is bounded below. Thus $T=\{T_i\}_{i\in I}$ is a weaving operator frame.

Now, suppose that $T=\{T_i\}_{i\in I}$ is not an independent weaving operator frame. Then there exist a non-zero sequence $\{x_i\}_{i\in I}\subset\mathcal{H}$ and some $i_0\in I$ such that $x_{i_0}\neq 0$. Thus

$$T_{i_0}^*x_{i_0}=\sum_{i\neq i_0}T_i^*x_i. \tag{3.9}$$

Since R_T is also surjective, there exists $x\in\mathcal{H}$ such that $R_Tx=\{T_ix\}_{i\in I}=\eta_{i_0}\in\oplus_{i\in I}\mathcal{H}_i$, where $\eta_{i_0}=\{y_i\}_{i\in I},y_{i_0}=x_{i_0}$ and $y_i=0,i\neq i_0$. Hence $T_{i_0}x=x_{i_0}$ and so $\left\langle x,T_{i_0}^*x_{i_0}\right\rangle=\langle T_{i_0}x,x_{i_0}\rangle=\|x_{i_0}\|^2\neq 0$. But (3.9) implies that

$$\left\langle x,T_{i_0}^*x_{i_0}\right\rangle=\Big\langle x,\sum_{i\neq i_0}T_i^*x_i\Big\rangle=\sum_{i\neq i_0}\langle x,T_i^*x_i\rangle=\sum_{i\neq i_0}\langle T_ix,x_i\rangle=0.$$

This is a contradiction. So $T=\{T_i\}_{\forall i\in I}$ is independent. □

Theorem 3.6. *Let $T=\{T_i\}_{i\in I}$ be a sequence of weaving operators on $\mathcal{H}$. Then the following statements are equivalent.*

(1) *$T=\{T_i\}_{i\in I}$ is a weaving operator Riesz basis.*
(2) *$T=\{T_i\}_{i\in I}$ is an independent weaving operator frame.*

Proof. If $T=\{T_i\}_{i\in I}$ is a weaving operator Riesz basis, then R_T^* is bounded below by Theorem 3.5 and so the range, denoted by $R\left(R_T^*\right)$, of R_T^* is closed. In addition, $\overline{span}\,\{T_i^*\}_{i\in}=R\left(R_T^*\right)=\mathcal{H}$. Thus R_T^* is bijective. By the Banach Inverse Theorem, R_T^* is invertible, and so R_T is also invertible. Hence (2) in Theorem 3.5 implies that $T=\{T_i\}_{i\in I}$ is an independent weaving operator frame.

Conversely, assume that $T=\{T_i\}_{i\in I}$ is an independent weaving operator frame. Then (2) in Theorem 3.5 shows that R_T is invertible. Thus R_T^* is invertible. For any $\{x_i\}_{i\in i}\in\oplus_{i\in I}\mathcal{H}_i$, we have

$$\left\|\{x_i\}_{i\in I}\right\|^2=\left\|R_T^{*-1}R_T^*\left(\{x_i\}_{i\in I}\right)\right\|^2\leq\left\|R_T^{*-1}\right\|^2\left\|R_T^*\left(\{x_i\}_{i\in I}\right)\right\|^2.$$

Put $C = \left\|R_T^{*-1}\right\|^{-2}$ and $D = \|R_T^*\|^2$. Then

$$C\left\|\{x_i\}_{i\in I}\right\|^2 \leq \left\|R_T^*\left(\{x_i\}_{i\in I}\right)\right\|^2 = \Big\|\sum_{i\in I}\sum_{i\in\sigma_i} T_i^* x_i\Big\|^2$$
$$\leq D\left\|\{x_i\}_{i\in I}\right\|^2, \quad \forall \{x_i\}_{i\in I} \in \oplus_{i\in I}\mathcal{H}_i.$$

Thus, the condition (3.6) holds. Since R_T^* is invertible, the condition (i) of Definition 3.2 holds. Hence $T = \{T_i\}_{i\in I}$ is an operator Riesz basis. □

Lemma 3.7. *Let $T = \{T_i\}_{i\in I}$ be a weaving operator frame for $B(\mathcal{H})$ with bounds A, B. If $Q = \{Q_i\}_{i\in I}$ is a weaving operator Bessel sequence in $B(\mathcal{H})$ with a bound $M < \frac{A^2}{4B}$, then $T \pm Q := \{T_i \pm Q_i\}_{i\in I}$ is a weaving operator frame for $B(\mathcal{H})$.*

Proof. We only prove the case where $T + Q = \{T_i + Q_i\}_{i\in I}$. The other case is similar.

For any $x \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in I}\sum_{i\in\sigma_i}\|(T_i+Q_i)x\|^2 &\leq \sum_{i\in I}\sum_{i\in\sigma_i}(\|T_ix\| + \|Q_ix\|)^2 \\
&= \sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|^2 + \sum_{i\in I}\sum_{i\in\sigma_i}\|Q_ix\|^2 + 2\sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|\,\|Q_ix\| \\
&\leq B\|x\|^2 + M\|x\|^2 + 2\Big(\sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i\in I}\sum_{i\in\sigma_i}\|Q_ix\|^2\Big)^{\frac{1}{2}} \\
&\leq (B+M)\|x\|^2 + 2\sqrt{B}\sqrt{M}\|x\|^2 \\
&\leq (B + M + 2\sqrt{B}\sqrt{M})\|x\|^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i\in I}\sum_{i\in\sigma_i}\left\|(T_i+Q_i)x\right\|^2 &\geq \sum_{i\in I}\sum_{i\in\sigma_i}(\|T_ix\| - \|Q_ix\|)^2 \\
&= \sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|^2 + \sum_{i\in I}\sum_{i\in\sigma_i}\|Q_ix\|^2 - 2\sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|\,\|Q_ix\| \\
&\geq A\|x\|^2 + \sum_{i\in I}\sum_{i\in\sigma_i}\|Q_ix\|^2 - 2\Big(\sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i\in I}\sum_{i\in\sigma_i}\|Q_ix\|^2\Big)^{\frac{1}{2}} \\
&\geq A\|x\|^2 - 2\Big(\sum_{i\in I}\sum_{i\in\sigma_i}\|T_ix\|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i\in I}\sum_{i\in\sigma_i}\|Q_ix\|^2\Big)^{\frac{1}{2}} \\
&\geq (A - 2\sqrt{B}\sqrt{M})\|x\|^2.
\end{aligned}$$

Hence $T + Q = \{T_i + Q_i\}_{i\in I}$ is a weaving operator frame for $B(\mathcal{H})$. □

Theorem 3.8. *Let $T = \{T_i\}_{i\in I}$ be a weaving operator frame for $B(\mathcal{H})$ with bounds A, B. Then the following statements are equivalent.*

1. *$T = \{T_i\}_{i\in I}$ is independent.*
2. *$T = \{T_i\}_{i\in I}$ is a weaving operator Riesz basis.*
3. *$R(R_T) = \oplus_{i\in I}\mathcal{H}_i$.*
4. *$T = \{T_i\}_{i\in I}$ has a weaving unique dual frame.*

Proof. Theorems 3.8 and 3.6 yield that (1) ⇔ (2) ⇔ (3). We only need to prove (1) ⇔ (4).

(1) ⇒ (4) Suppose that an independent operator frame $\{T_i\}_{i\in I}$ has two weaving dual frames $\widetilde{T} = \left\{\widetilde{T}_i\right\}_{i\in I}$ and $\widetilde{Q} = \left\{\widetilde{Q}_i\right\}_{i\in I}$. Then $R_{\widetilde{T}}$ and $R_{\widetilde{Q}}$ are left inverses of R_T^* from (3.7). Thus $\widetilde{T} = \widetilde{Q}$.

(4) $\Rightarrow$ (1) Assume that $T=\{T_i\}_{i\in I}$ has a unique dual frame $\tilde{T}=\{\widetilde{T}_i\}_{i\in I}$. Suppose that $T=\{T_i\}_{i\in I}$ is not independent. Then $R(R_T)\neq\oplus_{i\in I}\mathcal{H}_i$, i.e., $R(R_T)^{\perp}\neq\{0\}$.

Thus there exists a nonzero element $\{x_i\}_{i\in I}\in R(R_T)^{\perp}$ such that $\left\|\{x_i\}_{i\in I}\right\|<\dfrac{A}{2\sqrt{B}}$.

Take a unit vector $e\in\mathcal{H}$, define a sequence of bounded linear operators $\tilde{U}=\{\widetilde{U}_i\}_{i\in I}$ in such a way that $\widetilde{U}_i x=\langle x,e\rangle x_i, \forall x\in\mathcal{H}$. Put $\widetilde{Q}=\widetilde{U}+\widetilde{T}$. Then for all x in $\mathcal{H}$,

$$\sum_{i\in \mathrm{E}}\sum_{i\in\sigma_i}\left\|\tilde{U}_i x\right\|^2=\sum_{i\in I}\sum_{i\in\sigma_i}\|\langle x,e\rangle x_i\|^2\leq\sum_{i\in I}\sum_{i\in\sigma_i}\|\langle x,e\rangle\|^2\|x_i\|^2\leq\|x\|^2\sum_{i\in I}\sum_{i\in\sigma_i}\|x_i\|^2.$$

Thus the sequence $\{\widetilde{U}_i\}_{i\in I}$ is a weaving operator Bessel sequence with a Bessel bound less than $\dfrac{A^2}{4B}$. By Lemma 3.7, we know that $\widetilde{Q}$ is a weaving operator frame for $B(\mathcal{H})$. For any $x\in\mathcal{H}$, since $\{\widetilde{U}_i x\}_{i\in I}=\{\langle x,e\rangle x_i\}_{i\in I}\in R(R_T)^{\perp}=\operatorname{Ker}(R_T^*)$, we see that $R_T^*(\{\widetilde{U}_i x\}_{i\in I})=\sum_{i\in I}T_i^*\widetilde{U}_i x=0$, and so

$$x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\widetilde{T}_i x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\left(\widetilde{T}_i+\widetilde{U}_i\right)x=\sum_{i\in I}\sum_{i\in\sigma_i}T_i^*\widetilde{Q}_i x.$$

Thus $\widetilde{Q}$ is also a weaving dual frame of $T=\{T_i\}_{i\in I}$. Clearly, $\widetilde{Q}\neq\tilde{T}$. This contradicts the uniqueness of the weaving dual frame of T. Hence $T=\{T_i\}_{i\in I}$ is independent. □

For a frame of subspaces $\{W_i\}_{i\in I}$ with respect to the family of weights $\{v_i\}_{i\in I}$ for $\mathcal{H}$ with synthesis operator $T_{W,v}$, the sequence $\{u_i\}_{i\in I}=\{S_{W,v}^{-1}W_i\}_{i\in I}$ is called a weaving dual frame of $\{W_i\}_{i\in I}$, where the operator $S_{W,v}=T_{W,v}T_{W,v}^*$.

Theorem 3.9. *For a frame of subspaces $\{W_i\}_{i\in I}$ with respect to the family of weights $\{v_i\}_{i\in I}$ for $\mathcal{H}$, define $T_i=v_iS_{W,v}\pi_{W_i}S_{W,v}^{-1}$ and $Q_i=v_i\pi_{u_i}S_{W,v}^{-1}$. Then $Q=\{Q_i\}_{i\in I}$ and $T=\{T_i\}_{i\in I}$ are all weaving operator frames for $B(\mathcal{H})$, and Q is a weaving dual frame of T.*

Proof. Assume that $\{W_i\}_{i\in I}$ has frame bounds A,B.

Claim 1. $T=\{T_i\}_{i\in I}$ is a weaving operator frame for $B(\mathcal{H})$. For any $x\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I}\sum_{i\in\sigma_i}\|T_i x\|^2&=\sum_{i\in I}\sum_{i\in\sigma_i}\left\|v_iS_{W,v}\pi_{W_i}S_{W,v}^{-1}x\right\|^2\\&\leq\|S_{W,v}\|^2B\left\|S_{W,v}^{-1}x\right\|^2\\&\leq B\|S_{W,v}\|^2\left\|S_{W,v}^{-1}\right\|^2\|x\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i\in I}\sum_{i\in\sigma_i}\|T_i x\|^2&=\sum_{i\in I}\sum_{i\in\sigma_i}\left\|v_iS_{W,v}\pi_{W_i}S_{W,v}^{-1}x\right\|^2\\&\geq\sum_{i\in I}\sum_{i\in\sigma_i}\left\|S_{W,v}^{-1}\right\|^{-2}\left\|v_i\pi_{W_i}S_{W,v}^{-1}x\right\|^2\\&\geq\left\|S_{W,v}^{-1}\right\|^{-2}A\left\|S_{W,v}^{-1}x\right\|^2\\&\geq A\left\|S_{W,v}^{-1}\right\|^{-2}\|S_{W,v}\|^{-2}\|x\|^2.\end{aligned}$$

Thus $T=\{T_i\}_{i\in I}$ is a weaving operator frame.

Claim 2. $Q=\{Q_i\}_{i\in I}$ is also a weaving operator frame for $B(\mathcal{H})$. The proof is similar to Claim 1.

Claim 3. $T_{U,v} = S_{W,v}^{-1} T_{W,v} S_{W,v}, T_{U,v}^* = S_{W,v}^{-1} T_{W,v}^* S_{W,v}, S_{U,v} = S_{W,v}$. It is easy to check that $\pi_{u_i} = S_{W,v}^{-1} \pi_{W_i} S_{W,v}$. Thus $T_{U,v} = S_{W,v}^{-1} T_{W,v} S_{W,v}$, $T_{U,v}^* = S_{W,v}^{-1} T_{W,v}^* S_{W,v}$ and so

$$\begin{aligned} S_{U,v} &= T_{U,v} T_{U,v}^* \\ &= S_{W,v}^{-1} T_{W,v} S_{W,v} S_{W,v}^{-1} T_{W,v}^* S_{W,v} \\ &= S_{W,v}^{-1} T_{W,v} T_{W,v}^* S_{W,v} \\ &= S_{W,v}^{-1} S_{W,v} S_{W,v} \\ &= S_{W,v}. \end{aligned}$$

Hence, for any $x \in \mathcal{H}$, we compute

$$\begin{aligned} \sum_{i \in \mathrm{E}} \sum_{i \in \sigma_i} T_i^* Q_i x &= \sum_{i \in I} v_i S_{W,v}^{-1} \pi_{W_i} S_{W,v} \cdot v_i S_{W,v}^{-1} \pi_{W_i} S_{W,v} S_{W,v}^{-1} x \\ &= S_{W,v}^{-1} \Big(\sum_{i \in I} \sum_{i \in \sigma_i} v_i^2 \pi_{W_i} x \Big) \\ &= S_{W,v}^{-1} \left(S_{W,v} x \right) \\ &= x. \end{aligned}$$

This shows that $Q = \{Q_i\}_{i \in I}$ is a weaving dual operator frame of the weaving operator frame $T = \{T_i\}_{i \in I}$. □

Remark. In (1.1), if $A = B$, we call $\{W_i\}_{i \in I}$ a weaving Parseval frame of subspaces for $\mathcal{H}$.

Theorem 3.10. *Assume that $\{W_i\}_{i \in I}$ is a weaving Parseval frame of subspaces for a Hilbert space $\mathcal{H}$. Then $\{v_i \pi_{W_i}\}_{i \in I}$ is a weaving operator frame for $B(\mathcal{H})$ and a weaving dual frame of itself.*

Proof. If $\{W_i\}_{i \in I}$ is a weaving Parseval frame of subspaces, then $S_{W,v} = I$. So the theorem is a consequence of Theorem 3.9. □

4. Weaving operator responses

The following terminology is given by Li and Cao [10]. Let e be a unit vector in $\mathcal{H}$. For every $f \in \mathcal{H}$, define $T_f^e x = \langle x, f \rangle e$, for all $x \in \mathcal{H}$. Then T_x^e is a bounded linear operator on $\mathcal{H}$ and T_x^e is called *operator response* of f with respect to e. The set $\mathcal{R}_e^{\mathcal{H}} = \left\{ T_f^e : f \in \mathcal{H} \right\}$ is called an *operator response space* of $\mathcal{H}$ with respect to e.

Theorem 4.1. *Assume that $\{f_i\}_{i \in I}$ is a sequence in a Hilbert space $\mathcal{H}$ and $\{e_i\}_{i \in I}$ is a sequence of unit vectors in $\mathcal{H}$. Then the following statements are valid.*

1. *$\{f_i\}_{i \in I}$ is complete, i.e., $\overline{\operatorname{span}}\{f_i : i \in I\} = \mathcal{H}$ if and only if $\left\{T_{f_i}^e\right\}_{i \in I}$ is complete.*
2. *$\{f_i\}_{i \in I}$ is a frame for $\mathcal{H}$ if and only if $\left\{T_{f_i}^{e_i}\right\}_{i \in I}$ is a weaving operator frame for $B(\mathcal{H})$.*
3. *$\{f_i\}_{i \in I}$ is a tight frame for $\mathcal{H}$ if and only if $\left\{T_{f_i}^{e_i}\right\}_{i \in I}$ is a tight weaving operator frame for $B(\mathcal{H})$.*
4. *$\{f_i\}_{i \in I}$ is a normalized tight frame for $\mathcal{H}$ if and only if $\left\{T_{f_i}^{e_i}\right\}_{i \in I}$ is a weaving Parseval operator frame for $B(\mathcal{H})$.*
5. *If $\{e_i\}$ is either not complete, or orthogonal, then $\left\{T_{f_i}^{e_i}\right\}_{i \in I}$ is not independent.*

Proof. If $\{f_i\}_{i \in I}$ is complete, then for all $x \in \mathcal{H}$ and all $\varepsilon > 0$, there exist a sequence $\{c_i\}_{i \in I} \in \mathbb{C}$ and a finite set $L \in \mathcal{F}(\Phi)$ such that

$$\Big\| \sum_{i \in L} \sum_{i \in \sigma_i} c_i f_i - x \Big\| < \varepsilon.$$

Take $x_i = c_i e_i$, forall i in I, then $\langle x_i, e_i\rangle = c_i$, forall i in I and so

$$\Big\|\sum_{i\in L}\sum_{i\in\sigma_i} T_{f_i}^{e_i*}x_i - x\Big\| = \Big\|\sum_{i\in L}\sum_{i\in\sigma_i}\langle x_i, e_i\rangle f_i - x\Big\| = \Big\|\sum_{i\in L}\sum_{i\in\sigma_i} c_i f_i - x\Big\| < \varepsilon.$$

Thus $\overline{span}\{T_{f_i}^{e_i}\} = \mathcal{H}$, that is, $\{T_{f_i}^{e_i}\}_{i\in I}$ is complete.

On the other hand, if $\{T_{f_i}^{e_i}\}_{i\in I}$ is complete, then for all $x \in \mathcal{H}$ and all $\varepsilon > 0$, there exist a sequence $\{x_i\}_{i\in I} \in S(\mathcal{H})$ and a finite set $L \in \mathcal{F}(\Phi)$ such that

$$\Big\|\sum_{i\in L}\sum_{i\in\sigma_i} T_{f_i}^{e_i*}x_i - x\Big\| < \varepsilon.$$

That is,

$$\Big\|\sum_{i\in L}\sum_{i\in\sigma_i}\langle x_i, e_i\rangle f_i - x\Big\| < \varepsilon.$$

Thus $\{f_i\}_{i\in I}$ is complete. Moreover, for every $x \in \mathcal{H}$, we have

$$\sum_{i\in I}\sum_{i\in\sigma_i}\|T_{f_i}^{e_i}x\|^2 = \sum_{i\in I}\sum_{i\in\sigma_i}\|\langle x, f_i\rangle e_i\|^2 = \sum_{i\in I}\sum_{i\in\sigma_i}|\langle x, f_i\rangle|^2.$$

Thus (2) through (4) are valid.

Assume that $\{e_i\}$ is not complete. Then $\{e_i :$ forall i in $I\}^\perp \neq \{0\}$. Take a nonzero sequence $\{x_i\}_{i\in I} \subset \{e_i : i \in I\}^\perp\backslash\{0\}$. Then

$$\sum_{i\in I}\sum_{i\in\sigma_i} T_{f_i}^{e_i*}x_i = \sum_{i\in I}\sum_{i\in\sigma_i}\langle x_i, e_i\rangle f_i = 0. \tag{4.10}$$

This shows that the sequence $\{T_{f_i}^{e_i}\}_{i\in I}$ is not independent.

Next, we suppose that $\{e_i\}$ is orthogonal. Take a mapping $\phi : I \leftarrow I$ such that $\phi(i) \neq i$ for all $i \in I$ and define $x_i = e_{\phi(i)}$. Then

$$\sum_{i\in I} T_{f_i}^{e_i*}x_i = \sum_{i\in i}\sum_{i\in\sigma_i}\langle e_{\phi(i)}, e_i\rangle f_i = 0. \tag{4.11}$$

Hence the sequence $\{T_{f_i}^{e_i}\}_{i\in I}$ is not independent. □

Theorem 4.2. *Let $\{f_i\}_{i\in I} \subset \mathcal{H}$, $\{\widetilde{f}_i\}_{i\in I} \subset \mathcal{H}$ and $\{e_i\}_{i\in I}$ be a sequence of unit vectors in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *$\{f_i\}_{i\in I}$ and $\{\widetilde{f}_i\}_{i\in I}$ are a pair of weaving dual frames for $\mathcal{H}$.*
(2) *$\{T_{\widetilde{f}_i}^{e_i}\}_{i\in I}$ and $\{T_{f_i}^{e_i}\}_{i\in I}$ are weaving dual frames of each other.*

Proof. (1) ⇒ (2) Let (1) hold. Then Theorem 4.1 implies that $\{T_{f_i}^{e_i}\}_{i\in I}$ and $\{T_{\widetilde{f}_i}^{e_i}\}_{i\in I}$ are weaving operator frames for $B(\mathcal{H})$. For any $x \in \mathcal{H}$, we may compute

$$\begin{aligned}\sum_{i\in I}\sum_{i\in\sigma_i} T_{f_i}^{e_i*}T_{\widetilde{f}_i}^{e_i}x &= \sum_{i\in I}\sum_{i\in\sigma_i} T_{f_i}^{e_i*}\langle x, \widetilde{f}_i\rangle e_i\\ &= \sum_{i\in I}\sum_{i\in\sigma_i}\langle\langle x, \widetilde{f}_i\rangle e_i, e_i\rangle f_i\\ &= \sum_{i\in I}\sum_{i\in\sigma_i}\langle x, \widetilde{f}_i\rangle\langle e_i, e_i\rangle f_i\\ &= \sum_{i\in I}\sum_{i\in\sigma_i}\langle x, \widetilde{f}_i\rangle f_i = x.\end{aligned}$$

Hence $\{T_{\widetilde{f}_i}^{e_i}\}_{i\in I}$ is a weaving dual frame of the operator frame $\{T_{f_i}^{e_i}\}_{i\in I}$.

Similarly, we can prove that $\{T^{e_i}_{f_i}\}_{i\in I}$ is a weaving dual frame of the operator frame $\{T^{e_i}_{\tilde{f}_i}\}_{i\in I}$.

(2) $\Rightarrow$ (1) Suppose that $\{T^{e_i}_{\tilde{f}_i}\}_{i\in I}$ and $\{T^{e_i}_{f_i}\}_{i\in I}$ are weaving dual frames of each other. Then we know from Theorem 4.1 that $\{f_i\}_{i\in I}$ and $\{\tilde{f}_i\}_{i\in I}$ are frames for $\mathcal{H}$ and

$$\sum_{i\in I}\sum_{i\in\sigma_i} T^{e_i*}_{f_i}T^{e_i}_{\tilde{f}_i}x = x, \sum_{i\in I}\sum_{i\in\sigma_i} T^{e_i*}_{\tilde{f}_i}T^{e_i}_{f_i}x = x, \quad \forall x\in\mathcal{H}.$$

Furthermore, for any $x\in\mathcal{H}$, we get

$$\begin{aligned}
x &= \sum_{i\in I}\sum_{i\in\sigma_i} T^{e_i*}_{f_i}T^{e_i}_{\tilde{f}_i}x = \sum_{i\in I}\sum_{i\in\sigma_i} T^{e_i*}_{f_i}\langle x,\tilde{f}_i\rangle e_i\\
&= \sum_{i\in I}\sum_{i\in\sigma_i}\langle\langle x,\tilde{f}_i\rangle e_i, e_i\rangle f_i = \sum_{i\in I}\sum_{i\in\sigma_i}\langle x,\tilde{f}_i\rangle\langle e_i,e_i\rangle f_i\\
&= \sum_{i\in I}\sum_{i\in\sigma_i}\langle x,\tilde{f}_i\rangle f_i
\end{aligned}$$

and

$$\begin{aligned}
x &= \sum_{i\in I}\sum_{i\in\sigma_i} T^{e_i*}_{\tilde{f}_i}T^{e_i}_{f_i}x = \sum_{i\in I}\sum_{i\in\sigma_i} T^{e_i*}_{\tilde{f}_i}\langle x,f_i\rangle e\\
&= \sum_{i\in I}\sum_{i\in\sigma_i}\langle\langle x,f_i\rangle e_i, e_i\rangle \tilde{f}_i = \sum_{i\in I}\sum_{i\in\sigma_i}\langle x,f_i\rangle\langle e_i,e_i\rangle \tilde{f}_i\\
&= \sum_{i\in I}\sum_{i\in\sigma_i}\langle x,f_i\rangle \tilde{f}_i.
\end{aligned}$$

Thus $\{f_i\}_{i\in I}$ and $\{\tilde{f}_i\}_{i\in I}$ are a pair of weaving dual frames for $\mathcal{H}$. □

5. Perturbation theorem for weaving operator frames

In this section, we explore the perturbation theorem for weaving operator frames.

The following theorem is an adaptation of Theorem 4.1 in the continuous weaving frames setting, providing a variant of Paley-Wiener-type perturbation, as presented by L.K. Vashisht and Deepshikha in [15].

Theorem 5.1. *For each $j\in[m]$, let $\Phi_j = \{\Phi_{ij}\}_{i\in I}$ be a weaving operator frame for $\mathcal{H}$ with frame bounds A_j and B_j. Assume that there exist nonnegative scalars c_j, η_j, μ_j, $(j\in[m])$ such that for some fixed $n\in[m]$,*

$$A = A_n - \sum_{j\in[m]\setminus\{n\}}(c_j+\eta_j\sqrt{B_n}+\mu_j\sqrt{B_j})(\sqrt{B_n}+\sqrt{B_j}) > 0$$

and

$$\Big\|\sum_{i\in J}(\Phi^*_{in}-\Phi^*_{ij})g_i\Big\| \le \eta_j\Big\|\sum_{i\in J}\Phi^*_{in}g_i\Big\| + \mu_j\Big\|\sum_{i\in J}\Phi^*_{ij}g_i\Big\| + \Phi_j\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2}$$

for any finite subset $J\subset I$, $g_i\in\mathcal{H}_i$ and $j\in[m]\setminus\{n\}$. Then for any partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\Phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a weaving operator frame for $\mathcal{H}$ with the universal frame bounds A and $\sum_{j\in[m]}B_j$. Furthermore, the family of weaving operator frames $\{\Phi_j\}_{j\in[m]}$ for $\mathcal{H}$ is woven.

Proof. By Proposition 2.4, for any partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\Phi_{ij}\}_{i\in\sigma_j,j\in[m]}$ is a g-Bessel sequence with Bessel bound $\sum_{j\in[m]}B_j$.

For the lower frame inequality, let $T_{\Phi}^{(i)}$ be a synthesis operator associated with the operator frame $\{\Phi_{ij}\}_{i\in I}$ for $j \in [m]$. Since

$$\begin{aligned}
\|T_{\Phi}^{(j)} g_i\| &= \Big\|\sum_{i\in J} \Phi_{ij}^* g_i\Big\| = \sup_{\|g\|=1}\Big|\big\langle g, \sum_{i\in J}\Phi_{ij}^* g_i\big\rangle\Big| \\
&\le \sup_{\|g\|=1}\Big(\sum_{i\in J}\|\Phi_{ij} g\|^2\Big)^{1/2}\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2} \\
&= \|T_{\Phi}^{(j)}\|\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2} \\
&\le \sqrt{B_j}\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2}
\end{aligned}$$

for any finite subset $J \subset I$, $g_i \in \mathcal{H}_i$, for $j \in [m]\setminus\{n\}$, we have

$$\begin{aligned}
\|(T_{\Phi}^{(n)} - T_{\Phi}^{(j)})g_i\| &= \sup_{\|g\|=1}\big|\big\langle g, (T_{\Phi}^{(n)} - T_{\Phi}^{(j)})g_i\big\rangle\big| \\
&= \sup_{\|g\|=1}\Big|\big\langle g, \sum_{i\in J}(\Phi_{in}^* - \Phi_{ij}^*)g_i\big\rangle\Big| \\
&= \Big\|\sum_{i\in J}(\Phi_{in}^* - \Phi_{ij}^*)g_i\Big\| \\
&\le \eta_j\Big\|\sum_{i\in J}\Phi_{in}^* g_i\Big\| + \mu_j\Big\|\sum_{i\in J}\Phi_{ij}^* g_i\Big\| + c_j\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2} \\
&\le \eta_j\|T_{\Phi}^{(n)}\|\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2} + \mu_j\|T_{\Phi}^{(j)}\|\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2} + c_j\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2} \\
&\le (c_j + \eta_j\sqrt{B_n} + \mu_j\sqrt{B_j})\Big(\sum_{i\in J}\|g_i\|^2\Big)^{1/2}.
\end{aligned}$$

This gives

$$\|T_{\Phi}^{(n)} - T_{\Phi}^{(j)}\| \le c_j + \eta_j\sqrt{B_n} + \mu_j\sqrt{B_j}. \tag{5.12}$$

For $j \in [m]$ and $\sigma \subset I$, we define

$$T_{\Phi}^{(j\sigma)} : \oplus_{i\in\sigma}\mathcal{H}_i \longrightarrow \mathcal{H}, \quad T_{\Phi}^{(j\sigma)}\{g_i\} = \sum_{i\in\sigma}\Phi_{ij}^* g_i, \quad g_i \in \mathcal{H}_i.$$

It is easy to show that

$$\|T_{\Phi}^{(j\sigma)} g_i\| \le \|T_{\Phi}^{(j)} g_i\| \le \sqrt{B_j}(\sum_{i\in J}\|g_i\|^2)^{1/2}.$$

Thus $\|T_{\Phi}^{(j\sigma)}\| \le \sqrt{B_j}$ for all $j \in [m]$.

Similarly, by using (5.12) one can show that for any $j \in [m]\setminus\{n\}$,

$$\|T_{\Phi}^{(n\sigma)} - T_{\Phi}^{(j\sigma)}\| \le c_j + \eta_j\sqrt{B_n} + \mu_j\sqrt{B_j}.$$

For any $f \in \mathcal{H}$ and $j \in [m]\setminus\{n\}$, we have

$$\begin{aligned}
&\|(T_{\Phi}^{(n\sigma)}(T_{\Phi}^{(n\sigma)})^* - T_{\Phi}^{(j\sigma)}(T_{\Phi}^{(j\sigma)})^*)f\| \\
&\le \|(T_{\Phi}^{(n\sigma)}(T_{\Phi}^{(n\sigma)})^* - T_{\Phi}^{(n\sigma)}(T_{\Phi}^{(j\sigma)})^*)f\| + \|(T_{\Phi}^{(n\sigma)}(T_{\Phi}^{(j\sigma)})^* - T_{\Phi}^{(j\sigma)}(T_{\Phi}^{(j\sigma)})^*)f\| \\
&\le \|T_{\Phi}^{(n\sigma)}\|\|((T_{\Phi}^{(n\sigma)})^* - (T_{\Phi}^{(j\sigma)})^*)f\| + \|(T_{\Phi}^{(j\sigma)})^*\|\|(T_{\Phi}^{(n\sigma)} - T_{\Phi}^{(j\sigma)})f\| \\
&\le (c_j + \eta_j\sqrt{B_n} + \mu_j\sqrt{B_j})(\sqrt{B_n} + \sqrt{B_j})\|f\|.
\end{aligned} \tag{5.2}$$

Let $\{\sigma_j\}_{j\in[m]}$ be any partition of I and T_Φ be the synthesis operator associated with the Bessel operator sequence $\{\Phi_{ij}\}_{i\in\sigma_j, j\in[m]}$. By (5.2), we have

$$\begin{aligned}
\|T_\Phi^* f\|^2 &= |\langle f, T_\Phi T_\Phi^* f\rangle| \\
&= \Big|\Big\langle f, \sum_{i\in I} \Phi_{ij}^*\Phi_{ij} f\Big\rangle\Big| \\
&= \Big|\Big\langle f, \sum_{i\in I} \Phi_{in}^*\Phi_{in} f - \sum_{j\in[m]\setminus\{n\}} \sum_{i\in\sigma_j} (\Phi_{in}^*\Phi_{in} - \Phi_{ij}^*\Phi_{ij}) f\Big\rangle\Big| \\
&\geq \Big|\Big\langle f, \sum_{i\in I} \Phi_{in}^*\Phi_{in} f\Big\rangle\Big| - \sum_{j\in[m]\setminus\{n\}} \Big|\Big\langle f, \sum_{i\in\sigma_j} (\Phi_{in}^*\Phi_{in} - \Phi_{ij}^*\Phi_{ij}) f\Big\rangle\Big| \\
&\geq |\langle f, T_\Phi^{(n)} (T_\Phi^{(n)})^* f\rangle| - \sum_{j\in[m]\setminus\{n\}} \|f\| \sup_{\|f_0\|=1} \Big|\Big\langle f_0, \sum_{i\in\sigma_j} (\Phi_{in}^*\Phi_{in} - \Phi_{ij}^*\Phi_{ij}) f\Big\rangle\Big| \\
&\geq A_n\|f\|^2 - \sum_{j\in[m]\setminus\{n\}} \|f\| \|(T_\Phi^{(n\sigma_j)} (T_\Phi^{(n\sigma_j)})^* - T_\Phi^{(j\sigma_j)} (T_\Phi^{(j\sigma_j)})^*) f\| \\
&= (A_n - \sum_{j\in[m]\setminus\{n\}} (c_j + \eta_j\sqrt{B_n} + \mu_j\sqrt{B_j})(\sqrt{B_n} + \sqrt{B_j}))\|f\|^2 > 0.
\end{aligned}$$

Hence the $\{\Phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is an operator frame for $\mathcal{H}$ with required universal frame bounds. □

Proposition 5.2. *Let $\{\Phi_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds A and B and T_i be a bounded, invertible operator for all $i \in I$. If $\|I_\mathcal{H} - T_i\|^2 < \frac{A}{B}$, then $\{\Phi_i\}_{i\in I}$ and $\{\Phi_i T_i\}_{i\in I}$ are woven.*

Proof. Note that T_j is invertible. Thus $\{\Phi_i T_i\}_{i\in I}$ is automatically a g-frame. It is easy to compute that $(1 + \|T_i\|^2)B$ is an upper frame bound of $\{\Phi_i\}_{i\in\sigma} \cup \{\Phi_i T_i\}_{i\in\sigma^c}$. For every $\sigma \in I$ and for every $\in \mathcal{H}$, we have, by Minkowski's inequality and subadditivity of the square root function,

$$\begin{aligned}
&\Big(\sum_{i\in\sigma} \|\Phi_i f\|^2 + \sum_{i\in\sigma^c} \|\Phi_i T_i f\|^2\Big)^{1/2} \\
&\quad= \Big(\sum_{i\in\sigma} \|\Phi_i f\|^2 + \sum_{i\in\sigma^c} \|\Phi_i (f - (f - T_i f))\|^2\Big)^{1/2} \\
&\quad= \Big(\sum_{i\in\sigma} \|\Phi_i f\|^2 + \sum_{i\in\sigma^c} \|\Phi_i f - \Phi_i (I_\mathcal{H} - T_i) f\|^2\Big)^{1/2} \\
&\quad\geq \Big(\sum_{i\in\sigma} \|\Phi_i f\|^2 + \sum_{i\in\sigma^c} \|\Phi_i f\|^2 - \sum_{i\in\sigma^c} \|\Phi_i (I_\mathcal{H} - T_i) f\|^2\Big)^{1/2} \\
&\quad\geq \Big(\sum_{i\in I} \|\Phi_i f\|^2\Big)^{1/2} - \Big(\sum_{i\in\sigma^c} \|\Phi_i (I_\mathcal{H} - T_i) f\|^2\Big)^{1/2} \\
&\quad\geq \sqrt{A}\|f\| - \sqrt{B}\|(I_\mathcal{H} - T_i) f\| \\
&\quad\geq (\sqrt{A} - \sqrt{B}\|I_\mathcal{H} - T_i\|)\|f\|.
\end{aligned}$$

Thus $\{\Phi_i\}_{i\in\sigma} \cup \{\Phi_i T_i\}_{i\in\sigma^c}$ forms an operator frame having

$$A - B\|I_\mathcal{H} - T_i\|^2 > 0$$

as its lower frame bound. □

Corollary 5.3. *Let $\{\Phi_i\}_{i\in I}$ be an operator frame for $\mathcal{H}$ with frame bounds A and B and frame operator S_Φ. If $B/A < 2$, then Φ and the scaled canonical dual operator frame $\widetilde{\Phi} = \{\frac{2AB}{A+B}\Phi_i S_\Phi^{-1}\}_{i\in I}$ are woven.*

Proof. We apply Proposition 5.2 to the operator $T = T_i = T_j = \frac{2AB}{A+B} S_\Phi^{-1}$ for all $i, j \in I$. Since the spectrum of S_Φ is contained in the interval $[A, B]$, the spectrum of $I_\mathcal{H} - T$ is contained in the interval $[\frac{A-B}{A+B}, \frac{B-A}{A+B}]$ and thus

$$\|I_\mathcal{H} - T\| \leq \frac{B-A}{B+A}.$$

This norm is majorized by $\sqrt{(A/B)}$, whenever $B/A \leq 2$. □

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