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SPECTRA OF ALGEBRAS OF SYMMETRIC ENTIRE FUNCTIONS ON ℓ_p

IRYNA CHERNEGA AND ANDRIY ZAGORODNYUK

Abstract. The paper is devoted to further investigations of algebras of symmetric analytic functions on ℓ_p and their spectra. Using an analog of elementary symmetric polynomials on ℓ_p we propose a description of the spectrum of the algebra of symmetric analytic functions of bounded type on ℓ_p in the form of a multiplicative semigroup of analytic functions on the complex plane. Some applications to the algebra of all symmetric analytic functions on ℓ_p are obtained.

Стаття присвячена подальшим дослідженням алгебр симетричних аналітичних функцій на ℓ_p та їхнього спектру. Використовуючи аналог елементарних симетричних многочленів на ℓ_p, ми пропонуємо опис спектру алгебри симетричних аналітичних функцій обмеженого типу на ℓ_p у вигляді мультиплікативної напівгрупи аналітичних функцій на комплексній площині. Отримано деякі застосування до алгебри всіх симетричних аналітичних функцій на ℓ_p.

1. Introduction

Symmetric analytic functions with respect to various groups or semigroups of operators on Banach spaces were studied by many authors (see e.g. [4, 5, 7, 8, 9, 10, 16] and references therein).

A function $f\colon \ell_p \to \mathbb{C}$ is said to be *symmetric* if it is invariant with respect to permutations of basis vectors. It is well-known that any uniform commutative topological algebra can be represented as a subalgebra of continuous functions on its spectrum (the set of continuous complex homomorphisms). By this reason, spectra of algebras of analytic functions on Banach spaces are typical objects of infinite-dimensional complex analysis. In [25] (see also [23, p. 243]) it was proved that if X is a separable Banach space with the approximation property and h is a bounded complex homomorphism of the algebra $H(X)$ of all analytic functions on X endowed with the compact open topology, then h is a point evaluation functional. That is, there exists a vector $a \in X$ such that $h(f) = f(a)$ for every $f \in H(X)$. Spectra of algebras of analytic functions of bounded type on Banach spaces, in the general case, are much more larger than the set of point evaluation functional (see e.g. [3, 6, 15, 29]).

Algebras of symmetric analytic functions on ℓ_p and their spectra were studied in [1, 11, 12, 13, 20, 27]. In [12] it was observed that the spectrum of the Fréchet algebra $H_{bs}(\ell_1)$ of symmetric analytic functions of bounded type on ℓ_1 can be represented as a multiplicative semigroup of entire functions of exponential type of a complex variable so that the function of the form

$$\prod_{n=1}^{\infty}\Big(1-\frac{t}{a_n}\Big), \quad t\in\mathbb{C}, \quad \sum_{n=1}^{\infty}\frac{1}{|a_n|}<\infty$$

corresponds to the the point evaluation functional $\delta_x(f) = f(x)$, $x \in \ell_1$, $x_n = -1/a_n$ and for every fixed complex λ, the function $e^{\lambda t}$ corresponds to the so-called exceptional

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functionals [11, 12]. Using this approach, in [13] it was proved that any element of the spectrum of $H_{bs}(\ell_1)$ can be represented as a convolution of a point evaluation functional and an exceptional functional. In the paper we consider the case $H_{bs}(\ell_p)$, $1 \le p < \infty$. We propose a representation of the spectrum of $H_{bs}(\ell_p)$ as a multiplicative semigroup of entire functions of a complex variable and find specific representations for point evaluation functionals and exceptional functionals.

In Section 2 we consider basic definitions and preliminary results in the theory of analytic functions on Banach spaces. We refer the reader for details on infinite dimensional complex analysis to [17, 23]. In Section 3 we introduce elementary symmetric polynomials and their generating function on ℓ_p, $p \ge 1$, and examine their properties. In Section 4 we propose a representation of the spectrum of $H_{bs}(\ell_p)$ by the generating functions of the elementary symmetric polynomials. In Section 5, using some special power mappings, we propose some applications to the spectrum of the algebra $H_s(\ell_p)$ of all symmetric analytic functions on ℓ_p.

2. Definitions and preliminary results

Let X be a complex Banach space and $\mathcal{S}$ be a semigroup of operators on X. A function $f\colon X \to \mathbb{C}$ is said to be *symmetric* (with respect to $\mathcal{S}$) if $f(\sigma(x)) = f(x)$ for every $\sigma \in \mathcal{S}$. In the paper we consider the case where $X = \ell_p$ for some $1 \le p < \infty$, $\mathcal{S}$ is the group of all permutations of elements of the standard basis of ℓ_p, and f is in the algebra of analytic functions of bounded type on X. Let us recall that a function f is *analytic* on X if it is continuous and its restriction to any finite dimensional subspace of X is analytic. An analytic function is of *bounded type* if it is bounded on every bounded subset of X. It is well known (see e.g. [2]) that every infinite dimensional Banach space admits an analytic function of unbounded type. Let us denote by $H(X)$ the algebra of all analytic functions on X endowed with the topology of uniform convergence on compact subsets of X and by $H_b(X)$ the Fréchet algebra of analytic functions of bounded type on X endowed with the topology of uniform convergence on bounded subsets of X. Subalgebras of symmetric analytic functions (with respect to permutations of the basis vectors) on ℓ_p are denoted by $H_s(\ell_p)$ and $H_{bs}(\ell_p)$ respectively. Also, we use notation $\mathcal{P}_s(\ell_p)$ for the algebra of symmetric polynomials on ℓ_p. It is known that $H_{bs}(\ell_p)$ is a proper subset in $H_s(\ell_p)$ for every $1 \le p < \infty$ [14, 28].

The spectrum of a given commutative algebra is crucial for understanding its algebraic and topological structures. Let us denote by $M_{bs}(\ell_p)$ the spectrum (the set of continuous complex homomorphisms or characters) of $H_{bs}(\ell_p)$ and by $M_s(\ell_p)$ the set of bounded complex homomorphisms of $H_s(\ell_p)$.

A sequence of (homogeneous) polynomials (P_n) in $\mathcal{P}_s(\ell_p)$ is called a *(homogeneous) algebraic basis* of $\mathcal{P}_s(\ell_p)$ if for every polynomial $Q \in \mathcal{P}_s(\ell_p)$ there exists a unique polynomial q of a finite number m of variables such that

$$Q(x) = q(P_1(x), \ldots, P_m(x)), \quad x \in \ell_p.$$

It is known [18, 24] that the algebra of all symmetric polynomials on ℓ_p, $\mathcal{P}_s(\ell_p)$ admits the power series algebraic basis

$$\Big\{F_n(x) = \sum_{i=1}^{\infty} x_i^n, \quad n = \lceil p \rceil, \lceil p \rceil + 1 \ldots\Big\},$$

where $\lceil p \rceil$ is the minimal integer that is greater or equal than p. The existence of algebraic bases in $\mathcal{P}_s(\ell_p)$ is very important for investigations of the spectrum of $H_{bs}(\ell_p)$ because any character φ in $M_{bs}(\ell_p)$ can be completely defined by its evaluations on elements of an algebraic basis (P_n) of $\mathcal{P}_s(\ell_p)$ (see [11, 13] and [26] for more general case). Thus, using the correspondence

$$\varphi \rightsquigarrow \big(\varphi(P_1), \dots, \varphi(P_n), \dots\big)$$

we can describe $M_{bs}(\ell_p)$ as a subset of complex sequences. We have different descriptions of the spectrum for different algebraic bases, which discover some different properties and structures of it. For example, the representation

$$\varphi \rightsquigarrow \big(\varphi(F_{\lceil p \rceil}), \dots, \varphi(F_n), \dots\big)$$

is additive with respect to a convolution operation "$\star$" on $M_{bs}(\ell_p)$. The convolution can be introduced in the following way (see [12]). For any x and y in ℓ_p we denote

$$x \bullet y = (x_1, y_1, x_2, y_2, \dots, x_n, y_n, \dots).$$

Clearly, $F_k(x \bullet y) = F_k(x) + F_k(y)$, $k \geq \lceil p \rceil$. According to [12], for every $f \in H_{bs}(\ell_p)$ and any fixed $y \in \ell_p$ the function $x \mapsto f(x \bullet y)$ is in $H_{bs}(\ell_p)$. Let $f \mapsto T_y^s(f)$, $T_y^s(f)(x) = f(x \bullet y)$ be the intertwining operator [12]. Then $\theta \star f$ is defined by

$$(\theta \star f)(y) = \theta(T_y^s)(f), \quad \theta \in M_{bs}(\ell_p), \quad f \in H_{bs}(\ell_p),$$

and the convolution of φ, $\theta \in M_{bs}(\ell_p)$ is defined by

$$\varphi \star \theta(f) = \varphi(\theta \star f) = \varphi\big(y \mapsto \theta(T_y^s(f))\big).$$

In particular, we have that $\varphi \star \theta(F_k) = \varphi(F_k) + \theta(F_k)$, $k \geq \lceil p \rceil$.

For every $x \in \ell_p$ there exists a so-called *point evaluation* character $\delta_x \in M_{bs}(\ell_p)$ such that $\delta_x(f) = f(x)$, $f \in H_{bs}(\ell_p)$. Note that $\delta_x = \delta_y$ if and only if $P(x) = P(y)$ for each symmetric polynomial P. If p is integer, then $H_{bs}(\ell_p)$ admits a one-parameter family of so-called exceptional characters ψ_λ, $\lambda \in \mathbb{C}$ such that $\psi_\lambda(F_p) = \lambda$ and $\psi_\lambda(F_k) = 0$ for $k > p$ [11, 12]. Also, we will use notation $\psi_\lambda^{(p)} = \psi_\lambda$ to pay attention that this character acts on $H_{bs}(\ell_p)$. In [13] it was proved that every character in $M_{bs}(\ell_1)$ is of the form $\delta_x \star \psi_\lambda$ for some $x \in \ell_1$ and $\lambda \in \mathbb{C}$. We do not know if it is true for other $p > 1$. Note that a complex homomorphism φ on $\mathcal{P}_s(\ell_p)$ is continuous if and only if the radius function of φ, $R(\varphi)$ is finite [11]. The radius function can be computed as

$$R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{1/n},$$

where φ_n is the restriction of φ to the normed space of n-homogeneous polynomials. In [13] it is proved that $R(\delta_x) = \|x\|$, $x \in \ell_p$ and $R(\psi_\lambda) = |\lambda|$.

3. Elementary symmetric polynomials and their generating function on ℓ_p.

A natural basis which is useful for representation of the spectrum of $H_{bs}(\ell_p)$ in the case $p = 1$ is the basis of *elementary symmetric polynomials*, given by

$$G_n(x) = \sum_{k_1 < \dots < k_n}^{\infty} x_{k_1} \cdots x_{k_n}$$

(see e.g. [1], [12]). The difficulty is that the basis of elementary symmetric polynomials can not be directly extended to the space ℓ_p for $p > 1$ because in this case the right-hand series diverges for every n. However, according to well known Newton's formula, for every $x \in \ell_1$, we can write

$$nG_n(x) = F_1(x)G_{n-1}(x) - F_2(x)G_{n-2}(x) + \dots + (-1)^{n+1}F_n(x), \quad n \in \mathbb{N}. \tag{3.1}$$

Setting in (3.1) $F_k = 0$ for $k < p$ we can define (cf. [19])

$$nG_n^{(p)}(x) = (-1)^{\lceil p \rceil+1}F_{\lceil p \rceil}(x)G_{n-\lceil p \rceil}^{(p)}(x) - F_{\lceil p \rceil+1}(x)G_{n-\lceil p \rceil-1}^{(p)}(x) + \dots + (-1)^{n+1}F_n(x) \tag{3.2}$$

for every $n \geq \lceil p \rceil$.

Proposition 3.1. *The set of polynomials* $\{G_n^{(p)}: n \geq \lceil p \rceil\}$ *is an algebraic basis in* $\mathcal{P}_s(\ell_p)$.

Proof. The proof immediately follows from the fact that polynomials F_n, $n \geq \lceil p \rceil$, form an algebraic basis in $\mathcal{P}_s(\ell_p)$ and from the invertibility of (3.2). $\square$

Let $\mathcal{G}^{(p)}(x)(t)$ be the so-called generating function of the sequence $\big(G_n^{(p)}(x)\big)$,

$$\mathcal{G}^{(p)}(x)(t) = 1 + \sum_{n=\lceil p \rceil}^{\infty} t^n G_n^{(p)}(x), \quad t \in \mathbb{C}.$$

It is well-known (see e.g. [22], p. 3) that if x has only a finite number of nonzero coordinates, then

$$\mathcal{G}(x)(t) = \mathcal{G}^{(1)}(x)(t) = \exp\left(-\sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n}\right).$$

This relation is still true for $x \in \ell_1$ [12]. Thus, for every $x \in \ell_1$,

$$\mathcal{G}^{(p)}(x)(t) = \exp\left(\sum_{k=1}^{\lceil p \rceil - 1} t^k \frac{F_k(-x)}{k}\right) \exp\left(-\sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n}\right). \tag{3.3}$$

On the other hand, we know that

$$\exp\left(-\sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n}\right) = \prod_{n=1}^{\infty}(1 + t x_n).$$

Combining with (3.3), we have

$$\begin{aligned} \mathcal{G}^{(p)}(x)(t) &= \exp\left(\sum_{k=1}^{\lceil p \rceil - 1} t^k \frac{F_k(-x)}{k}\right) \prod_{n=1}^{\infty}(1 + t x_n) \\ &= \prod_{n=1}^{\infty} \exp\Big(-t x_n + \frac{t^2 x_n^2}{2} + \cdots + \frac{t^{\lceil p \rceil}(-x_n)^{\lceil p \rceil}}{\lceil p \rceil}\Big)(1 + t x_n). \end{aligned} \tag{3.4}$$

From the theory of entire function of a single complex variable [21, pp. 29-30] we have that

$$\prod_{n=1}^{\infty} \exp\Big(-t x_n + \frac{t^2 x_n^2}{2} + \cdots + \frac{t^{\lceil p \rceil}(-x_n)^{\lceil p \rceil}}{\lceil p \rceil}\Big)(1 + t x_n)$$

is the canonical Weierstrass product, converging for every $x \in \ell_p$ to an entire function of t of order $\rho = \inf\{q > 0 \colon x \in \ell_q\}$ with zeros $a_n = -\frac{1}{x_n}$, $x_n \neq 0$. Thus, we have proved the following theorem.

Theorem 3.2. *For every fixed* $x \in \ell_p$ *the generating function* (3.4) *is an entire function of* t *of order* $\rho = \inf\{q > 0 \colon x \in \ell_q\}$ *with zeros* $a_n = -\frac{1}{x_n}$ *for* $x_n \neq 0$.

Corollary 3.3. *The function*

$$x \mapsto \mathcal{G}^{(p)}(x)(1) = 1 + \sum_{n=\lceil p \rceil}^{\infty} (G_n^{(p)})(x)$$

is an analytic function on ℓ_p.

Proof. $\mathcal{G}^{(p)}(x)(1)$ is a well-defined convergent series of continuous homogeneous polynomials. So it is analytic. $\square$

Note that $\mathcal{G}^{(1)}(x)(1)$ is of bounded type [12]. But we do not know if it is true for $p > 1$.

4. Representation of the spectrum of $H_{bs}(\ell_p)$

It is easy to check that

$$G_n(x \bullet y) = \sum_{k=0}^{n} G_k(x) G_{n-k}(y), \quad x, y \in \ell_1.$$

In [12] it was shown that

$$\varphi \star \theta(G_n) = \sum_{k=0}^{n} \varphi(G_k)\theta(G_{n-k})$$

for any characters φ, θ in $M_{bs}(\ell_1)$. From this relation it was deduced that

$$\mathcal{G}(\varphi \star \theta) = \mathcal{G}(\varphi)\mathcal{G}(\theta).$$

The following theorem extends this formula for ℓ_p.

Theorem 4.1. *For any characters φ and θ in $M_{bs}(\ell_p)$, $1 \le p < \infty$,*

(i)

$$\varphi \star \theta\big(G_n^{(p)}\big) = \sum_{k=0}^{n} \varphi\big(G_k^{(p)}\big)\theta\big(G_{n-k}^{(p)}\big);$$

(ii)

$$\mathcal{G}^{(p)}(\varphi \star \theta) = \mathcal{G}^{(p)}(\varphi)\mathcal{G}^{(p)}(\theta).$$

Here we suppose that $G_0^{(p)} = 1$ and $G_k^{(p)} = 0$ for $k < p$.

Proof. Let us assume first that $p \in \mathbb{N}$. For a given $x \in \ell_p$ we define the following character $\eta_x^{(p)} \in M_{bs}(\ell_p)$ by

$$\eta_x^{(p)} = \delta_x \star \psi^{(p)}_{-F_p(x)}.$$

Then

$$\eta_x^{(p)}(F_p) = F_p(x) - F_p(x) = 0 \quad \text{and} \quad \eta_x^{(p)}(F_k) = F_k(x), \quad k > p.$$

Note that the radius function

$$R(\eta_x^{(p)})) \le R(\delta_x) + R(\psi^{(p)}_{-F_p(x)}) \le 2\|x\|_p^p.$$

From the definition of $\eta_x^{(p)}$ and formula (3.2) it follows that

$$G_n^{(p+1)}(x) = \eta_x^{(p)}(G_n^{(p)}) = \delta_x \star \psi^{(p)}_{-F_p(x)}(G_n^{(p)}), \quad x \in \ell_p, \quad n \ge p+1, \tag{4.5}$$

and

$$\eta_x^{(p)}(G_p^{(p)}) = \eta_x^{(p)}\Big((-1)^{p+1}\frac{F_p}{p}\Big) = 0.$$

Moreover,

$$\eta^{(p)}_{x\bullet y} = \delta_{x\bullet y} \star \psi^{(p)}_{-F_p(x)-F_p(y)} = \delta_x \star \delta_y \star \psi^{(p)}_{-F_p(x)} \star \psi^{(p)}_{-F_p(y)} = \eta_x^{(p)} \star \eta_y^{(p)},$$

x, $y \in \ell_p$. We claim that for all x, $y \in \ell_p$,

$$G_n^{(p+1)}(x \bullet y) = \sum_{k=0}^{n} \eta_x^{(p)}(G_k^{(p)})\eta_y^{(p)}(G_{n-k}^{(p)}) = \sum_{k=0}^{n} (G_k^{(p+1)}(x))(G_{n-k}^{(p+1)}(y)), \tag{4.6}$$

where $G_0^{(p+1)} = 1$ and $G_1^{(p+1)} = G_2^{(p+1)} = \cdots = G_p^{(p+1)} = 0$. Indeed, for $p = 1$ we have

$$\begin{aligned} G_n^{(2)}(x \bullet y) &= \eta^{(1)}_{x\bullet y}(G_n^{(1)}) = \eta_x^{(1)} \star \eta_y^{(1)}(G_n^{(1)}) \\ &= \sum_{k=0}^{n} \eta_x(G_k^{(1)})\eta_y(G_{n-k}^{(1)}) = \sum_{k=0}^{n} (G_k^{(2)}(x))(G_{n-k}^{(2)}(y)). \end{aligned}$$

Suppose that equation (4.6) holds for every $G_n^{(m+1)}$ such that $m < p$. Then

$$\begin{aligned} G_n^{(p+1)}(x \bullet y) &= \eta^{(p)}_{x\bullet y}(G_n^{(p)}) = \eta^{(p)}_x \star \eta^{(p)}_y(G_n^{(p)}) \\ &= \sum_{k=1}^n \eta^{(p)}_x(G_k^{(p)})\eta^{(p)}_y(G_{n-k}^{(p)}) = \sum_{k=1}^n (G_k^{(p+1)}(x))(G_{n-k}^{(p+1)}(y)). \end{aligned}$$

Thus, (4.6) is proved. Note that both the left and the right parts of equality

$$G_n^{(p+1)}(x \bullet y) = \sum_{k=0}^n (G_k^{(p+1)}(x))(G_{n-k}^{(p+1)}(y))$$

are formally defined if x and y are in ℓ_{p+1}. Since it is true for $x, y \in \ell_p$, it must be true if x and y are in ℓ_{p+1} because ℓ_p is dense in ℓ_{p+1} and the functions $G_k^{(p+1)}$, $k \in \mathbb{N}$, are continuous on ℓ_{p+1}. Also, we know that the equality holds if $p+1 = 1$. Hence, for every integer $1 \le p < \infty$ and $n \ge p$ we have

$$G_n^{(p)}(x \bullet y) = \sum_{k=0}^n \big(G_k^{(p)}(x)\big)\big(G_{n-k}^{(p)}(y)\big). \tag{4.7}$$

If $p > 1$ is noninteger, then $\ell_p \subset \ell_{\lceil p \rceil}$ and so (4.7) is true for $n \ge \lceil p \rceil$.

Let now φ and θ be in $M_{bs}(\ell_p)$. Using the definition of $\varphi \star \theta$ and (4.7) we have

$$\varphi \star \theta\big(G_n^{(p)}\big) = \varphi\big(\theta \star G_n^{(p)}\big) = \varphi\Big(\sum_{k=0}^n G_k^{(p)}\theta\big(G_{n-k}^{(p)}\big)\Big) = \sum_{k=0}^n \varphi\big(G_k^{(p)}\big)\theta\big(G_{n-k}^{(p)}\big) \tag{4.8}$$

for $n \ge \lceil p \rceil$. Thus, item (i) is proved. To prove item (ii) we observe that

$$\sum_{n=0}^\infty t^n \varphi \star \theta\big(G_n^{(p)}\big) = \sum_{n=0}^\infty t^n \sum_{k=0}^n \varphi\big(G_k^{(p)}\big)\theta\big(G_{n-k}^{(p)}\big) = \mathcal{G}^{(p)}(\varphi)\mathcal{G}^{(p)}(\theta). \qquad \square$$

From Theorem 4.1 it follows that $\varphi \mapsto \varphi\big(G_k^{(p)}\big)$ is a representation of $M_{bs}(\ell_p)$ in the form of a multiplicative semigroup of entire functions of a complex variable.

Let us return to the formula (3.1). In the case of integer p, it takes the form

$$\begin{aligned} nG_n^{(p)}(x) &= (-1)^{p+1}F_p(x)G_{n-p}^{(p)}(x) + (-1)^{p+2}F_{p+1}(x)G_{n-p-1}^{(p)}(x) \\ &\quad + \cdots + (-1)^{n-p+1}F_{n-p}(x)G_p^{(p)}(x) + (-1)^{n+1}F_n(x) \end{aligned} \tag{4.9}$$

for $n \ge p$, where $G_0^{(p)} \equiv 1$, $F_0 \equiv 1$ and

$$G_1^{(p)} \equiv G_2^{(p)} \equiv \ldots \equiv G_{p-1}^{(p)} \equiv 0,$$

$$F_1 \equiv F_2 \equiv \ldots \equiv F_{p-1} \equiv 0.$$

In other words, in (4.9) the terms $F_r(x)G_{q-r}^{(p)}(x) = 0$ if $r < p$ or $q - r < p$. Hence, if ξ is a complex homomorphism (not necessarily continuous) on the space of symmetric polynomials $\mathcal{P}_s(\ell_p)$, then

$$\begin{aligned} n\xi(G_n^{(p)}) &= (-1)^{p+1}\xi(F_p)\xi(G_{n-p}^{(p)}) + (-1)^{p+2}\xi(F_{p+1})\xi(G_{n-p-1}^{(p)}) \\ &\quad + \cdots + (-1)^{n-p+1}\xi(F_{n-p})\xi(G_p^{(p)}) + (-1)^{n+1}\xi(F_n). \end{aligned} \tag{4.10}$$

Proposition 4.2. *Let p be a natural number and ξ be a complex homomorphism on $\mathcal{P}_s(\ell_p)$ such that $\xi(F_m) = c \neq 0$ for some $p \le m \le 2p$ and $\xi(F_n) = 0$ for $n \neq m$. Then*

$$\xi(G_{km}^{(p)}) = (-1)^{k(m+1)}\frac{(c/m)^k}{k!}$$

and $\xi(G_n^{(p)}) = 0$ *if* $n \neq km$ *for some* $k \in \mathbb{N}$. *Moreover,*

$$\mathcal{G}^{(p)}(\xi) = \begin{cases} e^{\frac{c}{m}t^m}, & \textit{if } m \textit{ is odd,} \\ 2 - e^{-\frac{c}{m}t^m}, & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. Setting in formula (4.10) $\xi(F_j) = 0$ if $j \neq m$ we can see that

$$\xi(G_{km}^{(p)}) = (-1)^{m+1}\frac{\xi(F_m)\xi(G_{(k-1)m}^{(p)})}{km}$$

and $\xi(G_n^{(p)}) = 0$ if $n \neq km$ for some $k \in \mathbb{N}$. It is easy to check that for $p \leq m \leq 2p$, $G_m^{(p)} = (-1)^{m+1}F_m/m$. Thus, for $k = 1$ we have

$$\xi(G_m^{(p)}) = \frac{(-1)^{m+1}\xi(F_m)}{m} = \frac{(-1)^{m+1}c}{m}.$$

Suppose

$$\xi(G_{(k-1)m}^{(p)}) = \Big((-1)^{m+1}\Big)^{k-1}\frac{(c/m)^{k-1}}{(k-1)!}.$$

Then,

$$\xi(G_{km}^{(p)}) = \frac{\xi(F_m)\Big((-1)^{m+1}\Big)^{k-1}(c/m)^{k-1}}{km(k-1)!} = \Big((-1)^{m+1}\Big)^{k}\frac{(c/m)^{k}}{k!} = (-1)^{k(m+1)}\frac{(c/m)^{k}}{k!}.$$

Therefore,

$$\mathcal{G}^{(p)}(\xi) = 1 + \Big((-1)^{m+1}\Big)^{k}\sum_{k=1}^{\infty}\frac{(c/m)^k t^{km}}{k!} = 1 + (-1)^{m+1}\sum_{k=1}^{\infty}\frac{(-\frac{ct^m}{m})^k}{k!}.$$

Hence,

$$\mathcal{G}^{(p)}(\xi) = \begin{cases} e^{\frac{c}{m}t^m}, & \text{if } m \text{ is odd} \\ 2 - e^{-\frac{c}{m}t^m}, & \text{if } m \text{ is even.} \end{cases}$$

□

Corollary 4.3. *Let* p *be a positive integer. Then*

$$\mathcal{G}^{(p)}\big(\psi_\lambda^{(p)}\big) = \begin{cases} e^{\frac{\lambda}{p}t^p}, & \textit{if } p \textit{ is odd,} \\ 2 - e^{-\frac{\lambda}{p}t^p}, & \textit{if } p \textit{ is even.} \end{cases}$$

and

$$\psi_\lambda^{(p)}\big(G_n^{(p)}\big) = \begin{cases} (-1)^{k(p+1)}\frac{(\lambda/p)^k}{k!}, & \textit{if } n = kp \textit{ for some } k \\ 0, & \textit{otherwise.} \end{cases}$$

Let $x \in \ell_{p-1}$. Then from equation (4.5) and Theorem 4.1 we have

$$G_n^{(p)}(x) = \delta_x \star \psi_{-F_{p-1}(x)}^{(p-1)}\big(G_n^{(p-1)}\big) = \sum_{k=0}^{n} G_k^{(p-1)}(x)\psi_{-F_{p-1}(x)}^{(p-1)}\big(G_{n-k}^{(p-1)}\big).$$

Taking into account Corollary 4.3, we can write

$$\begin{aligned} G_n^{(p)}(x) &= \sum_{j(p-1)\leq n} G_{n-j(p-1)}^{(p-1)}(x)\psi_{-F_{p-1}(x)}^{(p-1)}\big(G_{j(p-1)}^{(p-1)}\big) \\ &= \sum_{j(p-1)\leq n} G_{n-j(p-1)}^{(p-1)}(x)(-1)^{j(p+1)}\frac{(F_p(x))^j}{j!(p-1)^j}. \end{aligned} \tag{4.11}$$

Example 4.4. (c.f. [19]). Let us denote $\mathbf{1} = (1, 0, 0, \dots) \in \ell_p$. From (4.11), for $n \geq p > 1$, we have

$$G_n^{(p)}(\mathbf{1}) = \sum_{j(p-1)\leq n} G^{(p-1)}_{n-j(p-1)}(\mathbf{1})\psi^{(p-1)}_{-1}\big(G^{(p-1)}_{j(p-1)}\big).$$

In particular, for $p = 2$ and $n \geq 2$,

$$G_n^{(2)}(\mathbf{1}) = \sum_{j=0}^{n} G^{(1)}_{n-j}(\mathbf{1})\psi^{(1)}_{-1}\big(G^{(1)}_j\big) = \psi^{(1)}_{-1}\big(G^{(1)}_n\big) + \psi^{(1)}_{-1}\big(G^{(1)}_{n-1}\big) = (-1)^{n+1}\frac{n-1}{n!}.$$

5. Power and root mappings

Let $1 \leq p < \infty$ and $m \in \mathbb{N}$. We define $x^m = (x_1^m, \dots, x_i^m, \dots)$. The mapping $x \mapsto x^m$ is a continuous m-homogeneous polynomial from ℓ_p to ℓ_q for every $q \geq p/m$. Moreover,

$$\|x^m\|_q = \Big(\sum_{i=1}^{\infty} |x_i|^{mq}\Big)^{1/q} \leq \Big(\sum_{i=1}^{\infty} |x_i|^{mp/m}\Big)^{m/p} = \|x\|_p^m.$$

If $q = p/m$, then x^m is surjective and $\|x^m\|_q = \|x\|_p^m$. Clearly, it is not injective if $m > 1$. A right inverse map to x^m (which is not unique) can be defined in the following way:

$$\sqrt[m]{x} = \big(\sqrt[m]{x_1}, \dots, \sqrt[m]{x_i}, \dots\big),$$

where $\sqrt[m]{x_i}$ is the principal value of the mth complex root of x_i. Clearly, $x \mapsto \sqrt[m]{x}$ is an injection from ℓ_q to ℓ_{mq},

$$\big(\sqrt[m]{x}\big)^m = x,$$

and

$$\|\sqrt[m]{x}\|_{mq} = \Big(\sum_{i=1}^{\infty} |x_i|^q\Big)^{1/mq} = \|x\|_q^{1/m}.$$

Let us denote by C^m the composition operator from $H(\ell_q)$ to $H(\ell_{mq})$ defined by

$$C^m \colon f(x) \mapsto f(x^m), \quad f \in H(\ell_q).$$

Also, we denote by C_b^m the restriction of C^m to $H_b(\ell_p)$ by C_s^m the restriction of C^m to $H_s(\ell_p)$, and by C_{bs}^m the restriction of C^m to $H_{bs}(\ell_p)$.

Theorem 5.1.
(i) *The mapping C^m is a continuous homomorphism from $H(\ell_q)$ to $H(\ell_{mq})$.*
(ii) *The mapping C_b^m is a continuous homomorphism from $H_b(\ell_q)$ to $H_b(\ell_{mq})$.*
(iii) *The mapping C_s^m is a continuous homomorphism from $H_s(\ell_q)$ to $H_s(\ell_{mq})$.*
(iv) *The mapping C_{bs}^m is a continuous homomorphism from $H_{bs}(\ell_q)$ to $H_{bs}(\ell_{mq})$.*

Proof. The mapping $x \mapsto x^m$ is a continuous m-homogeneous polynomial from ℓ_{mq} to ℓ_q. So, it is an analytic map of bounded type. Hence, the composition operators C^m and C_b^m are continuous. Moreover, if f is a symmetric function on ℓ_q, then $x \mapsto f(x^m)$ is a symmetric function on ℓ_{mq}. Thus, the operators C_s^m and C_{bs}^m are continuous operators with ranges in $H_s(\ell_{mq})$ and $H_{bs}(\ell_{mq})$, respectively. □

Corollary 5.2. *Let Φ_m be a continuous homomorphism from $H(\ell_{mq})$ to a topological algebra A. Then $C^m \circ \Phi_m$ is a continuous homomorphism from $H(\ell_q)$ to A. Similarly, if Φ_m is a continuous homomorphism from $H_b(\ell_{mq})$ (resp. from $H_s(\ell_{mq})$, from $H_{bs}(\ell_{mq})$) to A, then $C_b^m \circ \Phi_m$ (resp. $C_s^m \circ \Phi_m$, $C_{bs}^m \circ \Phi_m$) is a continuous homomorphism from $H_b(\ell_q)$ (resp. from $H_s(\ell_q)$, from $H_{bs}(\ell_q)$) to A.*

Proposition 5.3. *Let z_n be a sequence of complex numbers such that $z = (z_1, z_2, \dots) \notin \ell_m$ for some $m \in \mathbb{N}$. If $F_k(z)$ is well-defined for every $k \geq m$, then δ_z is a discontinuous complex homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_m)$.*

Proof. Let us suppose that δ_z is continuous. Then it can be extended by continuity to a character on $H_{bs}(\ell_m)$. Thus, by Theorem 5.1, $\delta_{C^m_{bs}(z)}$ is a character $H_{bs}(\ell_1)$. On the other hand, since $z \notin \ell_m$, it follows that $C^m_{bs}(z) \notin \ell_1$ and by [13], $\delta_{C^m_{bs}(z)}$ cannot be a continuous homomorphism. A contradiction. $\square$

Let $\mathcal{P}_0(X)$ be a subalgebra of continuous polynomials $\mathcal{P}(X)$. We denote by $H_{b0}(X)$ (resp. $H_0(X)$) the algebra of analytic functions $f \in H_b(X)$ (resp. $H(X)$) such that all homogeneous polynomials in the Taylor series representation

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

are in $\mathcal{P}_0(X)$. We assume that the space $H_{b0}(X)$ is endowed with the metrizable topology of uniform convergence on bounded sets (induced from $H_b(X)$) and $H_0(X)$ is endowed with the topology of uniform convergence on compact sets (induced from $H(X)$).

Proposition 5.4. *Let φ and ψ be continuous linear functionals on $H_0(X)$. If $\varphi(P) = \psi(P)$ for every $P \in \mathcal{P}_0(X)$, then $\varphi = \psi$.*

Proof. Since the Taylor series of f converges to f in the topology of uniform convergence on compact sets for every $f \in H_0(X)$ and by the continuity of functionals φ and ψ, we have

$$\varphi(f) = \sum_{n=0}^{\infty} \varphi(f_n) = \sum_{n=0}^{\infty} \psi(f_n) = \psi(f)$$

for every $f \in H_0(X)$. $\square$

Corollary 5.5. *The operator of restriction of characters of the algebra $H_0(X)$ to the subalgebra $H_{b0}(X)$ is an injection from the spectrum $M\big(H_0(X)\big)$ of $H_0(X)$ to the spectrum $M\big(H_{b0}(X)\big)$ of $H_{b0}(X)$. In other words, $M\big(H_0(X)\big) \subseteq M\big(H_{b0}(X)\big)$.*

Proof. If the restriction of φ is equal to the restriction of ψ to $H_{b0}(X)$, then $\varphi(P) = \psi(P)$ for every $P \in \mathcal{P}_0(X)$. By Proposition 5.4, $\varphi = \psi$. $\square$

The following theorem gives a complete description of the spectrum of $H_s(\ell_1)$ because by [13], the case $p = 1$ satisfies conditions of this theorem. We do not know if it is true for any other $p > 1$.

Theorem 5.6. *Let p be such that every character in $M_{bs}(\ell_p)$ can be represented as a convolution of a point evaluation functional δ_x, $x \in \ell_p$ and an exceptional functional ψ_λ, $\lambda \in \mathbb{C}$. Then the spectrum $M_s(\ell_p)$ of $H_s(\ell_p)$ consists of point evaluation functionals.*

Proof. It is enough to show that ψ_λ, $\lambda \neq 0$, can not be extended to a continuous homomorphism of $H_s(\ell_p)$ for any positive integer p. In [14] it is proved that $H_s(\ell_1)$ contains analytic functions of unbounded type. This result was generalized for any $H_s(\ell_p)$, $1 \leq p < \infty$ in [28].

For a given $p \in \mathbb{N}$ we denote by $G_n^{\{p\}}$ the np-homogeneous polynomial on ℓ_p defined by

$$G_n^{\{p\}}(x) = G_n(x^p) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1}^p \cdots x_{k_n}^p.$$

From Newton's formula we have

$$\begin{aligned} nG_n^{\{p\}}(x) =& nG_n(x^p) = F_1(x^p)G_{n-1}(x^p) - F_2(x^p)G_{n-2}(x^p) + \cdots + (-1)^{n+1}F_n(x^p) \\ =& F_p(x)G_{n-1}^{\{p\}}(x) - F_{2p}(x)G_{n-2}^{\{p\}}(x) + \cdots + (-1)^{n+1}F_{np}(x). \end{aligned} \tag{5.12}$$

Taking into account $\|G_n\| = 1/n!$ [12] and the following relations

$$\Big(\|x\|_p \le 1\Big) \Leftrightarrow \Big(\sum_{k=1}^{\infty} |x_k^p| \le 1\Big) \Leftrightarrow \Big(\|x^p\|_1 \le 1\Big),$$

we can obtain

$$\|G_n^{\{p\}}\| = \sup_{\|x\|_p \le 1} |G_n(x^p)| = \sup_{\|x^p\|_1 \le 1} |G_n(x^p)| = \sup_{\|x\|_1 \le 1} |G_n(x)| = \frac{1}{n!}.$$

Let us denote by g_r, $r > 0$, the following analytic function on ℓ_p:

$$g_r(x) = \sum_{n=1}^{\infty} \frac{n! G_n^{\{p\}}(x)}{r^{np}}.$$

Then the radius of boundedness of g_r at the origin is equal to

$$\varrho_0(g_r) = \limsup_{n\to\infty} \frac{r}{\|n! G_n^{\{p\}}\|^{1/np}} = r.$$

According to Theorem 2 and Theorem 1 in [28], the function g_r is well-defined on ℓ_p and belongs to $H_s(\ell_p) \setminus H_{bs}(\ell_p)$.

Let $\lambda \ge r$. Then from the equalities $\psi_\lambda(F_p) = \lambda^p$, $\psi_\lambda(F_k) = 0$ for $k > p$, and formula (5.12), we have

$$\psi_\lambda(g_r) = \sum_{n=1}^{\infty} \Big(\frac{\lambda}{r}\Big)^{pn} = \infty.$$

Thus, ψ_λ is not defined on $g_r \in H_s(\ell_p)$ for $\lambda \ge r$. □

6. Conclusions

The paper's main result is the representation of the spectrum of $H_b(\ell_p)$ as a multiplicative semigroup of entire functions on $\mathbb{C}$. However, we have a complete description of the spectrum only for the case $p = 1$. The general case will be the subject of further investigations. For this purpose, it would be useful to have a value of $\|G_n^{(p)}\|$. We know only that $\|G_n^{(1)}\| = 1/n!$. Thus, we have the following question: *What is value of $\|G_n^{(p)}\|$ in ℓ_p for every $p > 1$ and integer $n > p$?*

In addition, it would be interesting to find a structure of an analytic manifold on the spectrum of $H_b(\ell_p)$ such that the Gelfand extension of any function in $H_b(\ell_p)$ is an analytic function on this manifold.

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Iryna Chernega: icherneha@ukr.net
Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, 79060, Lviv, Ukraine

Andriy Zagorodnyuk: andriy.zagorodnyuk@pnu.edu.ua
Vasyl Stefanyk Precarpathian National University, 76018, Ivano-Frankivsk, Ukraine