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ON THE CODISK-CYCLIC LINEAR RELATIONS

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Abstract. In this paper we extend and study the notions of codisk-cyclicity and codisk transitivity, studied in [5, 16, 17, 21, 22] for linear operators, to linear relations (multivalued linear operators) on a complex Hilbert space H. Among other things, we show that if a closed and bounded linear relation T is codisk-cyclic then its range is dense in H and T^p is also codisk-cyclic for every $p \in \mathbb{N}$. We also show that the codisk-cyclicity is equivalent to codisk-transtivity. A codisk-cyclicity criterion is given. Some examples that illustrate our results are presented.

У цій статті ми розширюємо та вивчаємо поняття кодиск-циклічності та кодиск-транзитивності, що досліджувались в [5, 16, 17, 21, 22] для лінійних операторів, до лінійних відношень (багатозначних лінійних операторів) на комплексному гільбертовому просторі H. Серед іншого, ми показуємо, що якщо замкнене та обмежене лінійне відношення T є кодиск-циклічним, то його область значень щільна в H, а T^p також є кодиск-циклічним для кожного $p \in \mathbb{N}$. Ми також показуємо, що кодиск-циклічність еквівалентна кодиск-транзитивності. Наведено критерій кодиск-циклічності. Надано деякі приклади, що ілюструють наші результати.

1. Introduction

For two separable infinite dimensional Hilbert spaces H and K over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote the set of all bounded linear operators from H to K by $\mathcal{B}(H, K)$. If $K = H$, we use the shorthand notation $\mathcal{B}(H) := \mathcal{B}(H, H)$. For a subset Λ of H, we use $int(\Lambda)$ and $\overline{\Lambda}$ to represent the interior and the closure of Λ, respectively. We recall some important concepts in the study of linear dynamical properties, with a specific focus on the notions of hypercyclicity and codisk-cyclicity. We say that $T \in \mathcal{B}(H)$ is *hypercyclic* if there exists a non-zero vector x in H such that the set $Orb(T, x) := \{x, Tx, T^2x, ...\}$ is norm dense in H. In this case, x is called a *hypercyclic vector for* T. In addition, we say that T satisfies the *hypercyclicity criterion* if there exist two subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ dense in H, an increasing sequence of integers $\{n_k\}$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \longrightarrow H$ such that:

i) $T^{n_k}x \longrightarrow 0$, for every $x \in \mathcal{D}_1$;
ii) $S_{n_k}y \longrightarrow 0$, for every $y \in \mathcal{D}_2$;
iii) $T^{n_k}S_{n_k}y \longrightarrow y$, for every $y \in \mathcal{D}_2$.

If T satisfies the hypercyclicity criterion, then T is hypercyclic. Also, if A is the unilateral backward shift on ℓ_2, then λA is hypercyclic if and only if $|\lambda| > 1$, (see [14]). This motivates the following notion introduced in [22] and studied by [5, 16, 17, 21, 22]. A linear operator $T \in \mathcal{B}(H)$ is said to be *codisk-cyclic* if there exists a non-zero vector x in H such that

$$\overline{\mathbb{U}Orb(T, x)} := \overline{\{\alpha T^n x : \alpha \in \mathbb{U}, n \geqslant 0\}}^{\|\cdot\|} = H,$$

where $\mathbb{U} := \{\alpha \in \mathbb{C} : |\alpha| \geqslant 1\}$. In this case, the vector x is said to be a *codisk-cyclic vector* for T. We say that T is *codisk transitive* if for any pair (U, V) of non-empty open

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subsets of H, there exist $\alpha \in \mathbb{U}$ and $n \geqslant 0$ such that

$$\alpha T^n(U) \cap V \neq \emptyset.$$

The codisk-cyclicity criterion stands as a fundamental outcome of linear dynamics [5, 22]. A bounded linear operator T on H is said to satisfy the *codisk-cyclicity criterion* if there exist an increasing sequence of integers $\{n_k\}$, a sequence $\{\alpha_{n_k}\}$ in $\mathbb{U}$, two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset H$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \longrightarrow H$ such that:

i) $\alpha_{n_k} T^{n_k} x \longrightarrow 0$, for every $x \in \mathcal{D}_1$;

ii) $\alpha_{n_k}^{-1} S_{n_k} y \longrightarrow 0$, for every $y \in \mathcal{D}_2$;

iii) $T^{n_k} S_{n_k} y \longrightarrow y$, for every $y \in \mathcal{D}_2$.

If T satisfies the codisk-cyclicity criterion, then T is codisk-cyclic. For a more comprehensive understanding of hypercyclicity, codisk-cyclicity and their interconnected properties in the context of linear dynamics, we refer the reader to the references [5, 13, 16, 17, 22, 21].

Recently, a study of linear dynamics of linear relations appeared in [11]. Moreover, Abakumov et al. 2018 have also studied hypercyclicity of linear relations on an infinite separable Hilbert space, see [1]. Additionally, in [6], we study the notion of disk-cyclicitiy of linear relations. This paper is a continuation of the study of dynamics of linear relations. We present and study the concept of codisk-cyclicity in the context of linear relations.

This paper is structured as follows. In Section 2, we recall the fundamental definitions and the symbols used for linear relations. In Section 3, we introduce and study the notion of codisk-cyclicity for a linear relation, which extends the concept of codisk-cyclicity for a bounded linear operator. We also show that the same properties known for a codisk-cyclic linear operator hold true for codisk-cyclicity of linear relations. Section 4 is devoted to present the concept of a codisk transitive linear relation and provide several characterizations for it. Among these characterizations, we prove that a linear relation is codisk-cyclic if and only if it is codisk transitive if and only if the set of codisk-cyclic vectors is a dense G_δ-set. In the last section, we conclude by presenting two criteria for determining the codisk-cyclicity of linear relations and giving a relationship between them.

2. Preliminaries of Linear relations

A *linear relation* or *multivalued linear operator* T on H is a mapping from a subspace $\mathcal{D}(T) := \{x \in H : Tx \text{ is a non-empty subset of } H\}$ called the domain of T into $2^H \setminus \emptyset$ the set of all non-empty subsets of H, such that

$$T(x + \lambda y) = T(x) + \lambda T(y),$$

for all $x, y \in \mathcal{D}(T)$ and all non-zero scalar λ [12]. We denoted by $\mathcal{LR}(H)$ the set of all linear relations on H. If $T \in \mathcal{LR}(X)$ then it is uniquely determined by its graph $G(T)$ which is defined by

$$G(T) := \{(x, y) \in H \times H : x \in \mathcal{D}(T) \text{ and } y \in T(x)\}.$$

The inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) := \{(y, x) \in H \times H : (x, y) \in G(T)\}.$$

Let $T \in \mathcal{LR}(H)$ and M be a subspace of H. Then the restriction of T to M, denoted by T_M, is the linear relation defined by $G(T_M) := G(T) \cap (M \times H)$. For two linear relations T and S on H, the linear relations $T + S$ and TS are defined respectively by

$$G(T + S) := \{(x, y + z) \in H \times H : (x, z) \in G(S) \text{ and } (x, y) \in G(T)\}$$

and

$$G(TS) := \{(x, y) \in H \times H : \exists z \in H \text{ such that } (x, z) \in G(S) \text{ and } (z, y) \in G(T)\}.$$

Note that $T(0) = \{0\}$ if and only if T maps the points of its domain to singletons; in this case T is said to be a *single valued operator* or a *linear operator*.

Let $T \in \mathcal{LR}(H)$. The image of T of a subset X of H and the inverse image of T^{-1} of a subset Y of H are defined, respectively, by

$$T(X) := \bigcup_{x \in \mathcal{D}(T) \cap X} Tx \text{ and } T^{-1}(Y) := \{x \in D(T) : Tx \cap Y \neq \emptyset\}.$$

The subspace $\ker(T) := T^{-1}(0)$ is called the kernel of T and the range of T is defined by $R(T) := T(\mathcal{D}(T))$. A linear relation T is said to be one-to-one if $\ker(T) = \{0\}$.

Let A, B and $C \in \mathcal{LR}(H)$. Then we know from [2, Lemma 2.5] that

i) $G((A + B)C) \subset G(AC + BC)$. If $C(0) \subset \ker(A) \cup \ker(B)$, then

$$(A + B)C = AC + BC.$$

ii) If A is everywhere defined, then $A(B + C) = AB + AC$. We know from [12, Corollary I.2.11] that $TT^{-1} = I_{R(T)} + T(0)$ and $T^{-1}T = I_{\mathcal{D}(T)} + T^{-1}(0)$.

The adjoint T^* of a linear relation T (see [19]) is defined by

$$G(T^*) := \{(y, y^{'}) \in H \times H : < x^{'}, y >=< y^{'}, x >, \text{ for all } (x, x^{'}) \in G(T)\}$$

and we have (see [19, 12])

$$\ker(T^*) = R(T)^{\perp} \text{ and } T^*(0) = \mathcal{D}(T)^{\perp}.$$

If $\overline{\mathcal{D}(T)} = H$, then T^* is a single-valued operator.

A linear relation T is called closed, if $\overline{G(T)} = G(T)$. We say that a linear relation T is continuous, if for each neighbourhood V in $R(T)$, $T^{-1}(V)$ is a neighbourhood in $\mathcal{D}(T)$. If T is continuous and $\mathcal{D}(T) = H$, then in this case, T is said bounded. The class of closed and bounded linear relations is denoted by $\mathcal{BCR}(H)$. Note that if T is closed, then $T(0)$ is closed.

For $n \in \mathbb{N} \cup \{0\}$, we let $T^0 = I$ (the identity operator in H) and if T^{n-1} is defined, then

$$T^n x := TT^{n-1}x = \bigcup_{y \in \mathcal{D}(T) \cap T^{n-1}x} Ty,$$

where

$$\mathcal{D}(T^n) := \{x \in \mathcal{D}(T^{n-1}) : \mathcal{D}(T) \cap T^{n-1}x \neq \emptyset\}.$$

By induction, we can show that $(T^n)^{-1} = (T^{-1})^n$ for all $n \in \mathbb{N}$. A linear relation $T \in \mathcal{BCR}(H)$ is said to satisfy *stabilization property* [10], if $T(0) = T^2(0)$. We also know by [18, Proposition 3.1] and [2, Lemma 3.1] that if $T \in \mathcal{BCR}(H)$ and $T(0) \subset \ker(T)$, then $T^n \in \mathcal{BCR}(H)$ for all $n \in \mathbb{N}$.

We say that a linear operator A is a *selection* of the linear relation T if $\mathcal{D}(T) = \mathcal{D}(A)$ and

$$Tx = Ax + T(0) \text{ for all } x \in \mathcal{D}(T).$$

Note that if A is continuous, then T is continuous. In addition, let $T \in \mathcal{BCR}(H)$ and $x \in H$. If A is a selection of $T \in \mathcal{BCR}(H)$, then from [7, Theorem 2.5.6], A^n is a selection of T^n. This implies that

$$T^n x = A^n x + T^n(0), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.1}$$

For more details about linear relations, we refer the reader to [2, 3, 4, 7, 10, 12, 18] and the references therein.

3. Codisk-cyclic linear relations

In this section, we present the concept of codisk-cyclicity in the context of linear relations as an extension of the notion of codisk-cyclicity for linear operators.

Definition 3.1. Let $T \in \mathcal{BCR}(H)$. We say that T is a *codisk-cyclic linear relation*, if there exists a non-zero vector $x \in H$ such that

$$\mathbb{U}Orb(T,x) := \bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{U}} \alpha T^n x$$

is dense in H. In this case, the vector x is called a *codisk-cyclic vector* for T and $\mathbb{U}Orb(T,x)$ is said the orbit of T at x.

The set of all codisk-cyclic linear relations on a separable Hilbert space H is denoted by $\mathbb{UCR}(H)$. For $T \in \mathcal{BCR}(H)$, we denoted by $\mathbb{UCR}(T)$ the set of all codisk-cyclic vectors for T. If $T \notin \mathbb{UCR}(H)$, then we set $\mathbb{UCR}(T) := \emptyset$.

Following [1], $T \in \mathcal{BCR}(H)$ is *hypercyclic* if there exists a sequence $\{x_i,\, i \in \mathbb{N}\cup\{0\}\}$ such that $\overline{\{x_i,\, i \in \mathbb{N}\cup\{0\}\}} = H$ and $\overline{\bigcup_{n\geqslant 0} T^n x_i} = H$, for each $i \geqslant 0$.

Proposition 3.2. *Let $T \in \mathcal{BCR}(H)$ be a bounded linear relation such that $T(0) \neq H$ and T satisfies the stabilization property. If T is a hypercyclic linear relation, then T is a codisk-cyclic linear relation.*

Proof. Since $T(0) \neq H$ and T satisfies the property of stabilization, we have that $T(0) = \overline{T(0)} = \overline{T^n(0)} \neq H$ for all $n \in \mathbb{N}$. Using the fact that T is hypercyclic, it follows from [1, Corollary 2.1] that there exists a non-zero vector x in H such that $\bigcup_{n\geqslant 0} T^n x$ is dense in H. Hence

$$H = \overline{\bigcup_{n\geqslant 0} T^n x} \subset \overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0} \alpha T^n x} = \overline{\mathbb{U}Orb(T,x)} \subset H,$$

which implies that the set $\mathbb{U}Orb(T,x)$ is dense in H. Finally, we deduce that T is a codisk-cyclic linear relation. □

In general, the converse of Proposition 3.2 is not true as shown by the following example.

Example 3.3. Let $\ell_2(\mathbb{Z})$ be the complex Hilbert space defined by

$$\ell_2(\mathbb{Z}) := \{(x_i)_{i\in\mathbb{Z}} \,:\, x_i \in \mathbb{C},\, \forall i \in \mathbb{Z} \text{ and } \sum_{i\in\mathbb{Z}} |x_i|^2 < \infty\}.$$

Let $\{a_n\}$ be a sequence defined by

$$a_n = \begin{cases} \frac{1}{9}, & \text{if } n \geqslant 0,\\ \frac{1}{3}, & \text{if } n < 0.\end{cases}$$

Let T be the bilateral weighted shift with weights $\{a_n\}$ defined on $\ell_2(\mathbb{Z})$ by

$$T(e_n) = a_n e_{n+1} \text{ for all } n \in \mathbb{Z},$$

where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis for $\ell_2(\mathbb{Z})$. Then T is codisk-cyclic but not hypercyclic. Indeed, since $\inf_{n\in\mathbb{Z}} a_n > 0$, T is invertible by [20, Proposition 10]. Now, let S be the inverse of T. Hence

$$Se_n = \frac{1}{a_{n-1}} e_{n-1} \quad \text{for all } n \in \mathbb{Z}.$$

We consider the sequence $\{\alpha_n\}$ defined by $\alpha_n = 4^n$ for all $n \in \mathbb{N}$. Clearly $\{\alpha_n\} \subset \mathbb{U}$. Moreover, we have

$$\begin{array}{rcl} \lim\limits_{n\to\infty} \|\alpha_n T^n e_0\| & = & \lim\limits_{n\to\infty} \alpha_n \prod\limits_{k=0}^{n-1} a_k \\ & = & \lim\limits_{n\to\infty} (\frac{4}{9})^n \\ & = & 0, \end{array}$$

and

$$\begin{array}{rcl} \lim\limits_{n\to\infty} \|\alpha_n^{-1} S^n e_0\| & = & \lim\limits_{n\to\infty} \alpha_n^{-1} \prod\limits_{k=1}^{n} \frac{1}{a_{-k}} \\ & = & \lim\limits_{n\to\infty} (\frac{3}{4})^n \\ & = & 0. \end{array}$$

Let

$$\mathcal{D}_1 = \mathcal{D}_2 := \{x \in \ell_2(\mathbb{Z}) : x \text{ has only finitely many non-zero coordinates}\}.$$

Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense sets in $\ell_2(\mathbb{Z})$. Let $x \in \mathcal{D}_1$ and $y \in \mathcal{D}_2$. Using [13, Lemma 3.1] and triangle inequality we deduce that

$$\|\alpha_n T^n x\| \longrightarrow 0 \text{ and } \|\alpha_n^{-1} S^n y\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Furthermore, we have $T^n S^n y = y$. Hence T satisfies the codisk-cyclicity criterion. This implies that T is codisk-cyclic.

On the other hand, let $\{n_k\}$ be any sequence of positive integers such that $n_k \to +\infty$. Then

$$\prod_{j=1}^{n_k} \frac{1}{a_{-j}} = 3^{n_k} \longrightarrow +\infty \text{ as } k \longrightarrow +\infty.$$

Thus, it follows from [13, Theorem 4.1] that T is not hypercyclic.

In the following we prove that every linear relation which has a codisk-cyclic selection is a codisk-cyclic linear relation.

Proposition 3.4. *Let $A \in \mathcal{B}(X)$ be a selection of a linear relation $T \in \mathcal{BCR}(H)$. If A is codisk-cyclic, then T is a codisk-cyclic linear relation.*

Proof. Let $x \in H$. Since A is a selection of T, by Equality (2.1) we obtain

$$\{\alpha A^n x : n \geqslant 0, \alpha \in \mathbb{U}\} \subset \mathbb{U}Orb(T, x)$$

Since A is a codisk-cyclic linear operator,

$$H = \overline{\{\alpha A^n x : n \geqslant 0, \alpha \in \mathbb{U}\}} \subset \overline{\mathbb{U}Orb(T, x)} \subset H.$$

This implies that the set $\mathbb{U}Orb(T, x)$ is dense in H. Finally, we conclude that T is a codisk-cyclic linear relation. ☐

Proposition 3.5. *Every non-injective codisk-cyclic linear operator is a selection for a codisk-cyclic linear relation.*

Proof. Let $A \in \mathcal{B}(H)$ be a codisk-cyclic linear operator such that $\ker(A) \neq \{0\}$. Let $T \in \mathcal{BCR}(H)$ be the linear relation defined by

$$\begin{array}{rccl} T: & H & \longrightarrow & 2^H \setminus \emptyset \\ & x & \longmapsto & A^{-1}A^2(x). \end{array}$$

Then A is a selection of T. Indeed, let $x \in \mathcal{D}(T) = H$, then we have

$$\begin{array}{rcl} Tx & = & A^{-1}A^2(x) \\ & = & A^{-1}A(Ax) \\ & = & Ax + \ker(A) \\ & = & Ax + T(0). \end{array}$$

It follows that A is a selection of T. Since A is a codisk-cyclic linear operator, by Proposition 3.4 we deduce that T is a codisk-cyclic linear relation. □

In the following, we give an example of A which satisfies the conditions of Example 3.

Example 3.6. Let A be an operator defined on $\ell_2(\mathbb{N})$ by

$$\begin{array}{cccc} A: & \ell_2 & \longrightarrow & \ell_2 \\ & x=(x_1,x_2,\ldots) & \longmapsto & 3(x_2,x_3,\ldots). \end{array}$$

Then A is a hypercyclic linear operator by Example 2.22 in [14]. Hence A is a codisk-cyclic operator and $\ker(A)\neq\{0\}$. Let T be a bounded linear relation defined by

$$\begin{array}{cccc} T: & \ell_2(\mathbb{N}) & \longrightarrow & 2^{\ell_2(\mathbb{N})}\setminus\emptyset \\ & x & \longmapsto & A^{-1}A^2(x). \end{array}$$

Therefore T is a codisk-cyclic linear relation as A is a non one-to-one selection of T.

Proposition 3.7. *Let $T\in\mathcal{BCR}(H)$, $S\in\mathcal{BCR}(K)$, and $G\in\mathcal{B}(H,K)$ be such that $SG=GT$ and $R(G)$ is dense in K. Then*

$$G(\mathbb{U}\mathcal{CR}(T))\subset\mathbb{U}\mathcal{CR}(S).$$

In particular, if T is codisk-cyclic, then S is codisk-cyclic.

Proof. If $T\notin\mathbb{U}\mathcal{CR}(\mathcal{H})$, then $G(\mathbb{U}\mathcal{CR}(T))=G(\emptyset)=\emptyset\subset\mathbb{U}\mathcal{CR}(S)$. Now assume that $T\in\mathbb{U}\mathcal{CR}(\mathcal{H})$. Let x be a codisk-cyclic vector for T, then $\mathbb{U}Orb(T,x)$ is dense in H. We have

$$\begin{aligned}
\overline{\mathbb{U}Orb(S,Gx)} &= \overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha S^nGx}\\
&= \overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha GT^nx}\\
&= \overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha G(T^nx)}\\
&= \overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}G(\alpha T^nx)}\\
&= \overline{G(\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha T^nx)}\\
&\supseteq G(\overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha T^nx})\\
&= G(H)\\
&= R(G).
\end{aligned}$$

As $R(G)$ is dense in K, we see that $\mathbb{U}Orb(S,Gx)$ is dense in K. Hence $Gx\in\mathbb{U}\mathcal{CR}(S)$. □

As immediate consequence of the preceding proposition is $\lambda\mathbb{U}\mathcal{CR}(T)=\mathbb{U}\mathcal{CR}(T)$ for all $\lambda\in\mathbb{C}\setminus\{0\}$.

Now, let $x\in H$. We define the set $\mathbb{U}_x$ as follows:

$$\mathbb{U}_x:=\{\alpha x:\alpha\in\mathbb{U}\}.$$

Lemma 3.8. *Let $T\in\mathbb{U}\mathcal{CR}(H)$. Then $x\in\mathbb{U}\mathcal{CR}(T)$ if and only if $\mathbb{U}Orb(T,x)\setminus\mathbb{U}_x$ is dense in H.*

Proof. Let x be a codisk-cyclic vector for T. Then $\mathbb{U}Orb(T,x)$ is dense in H. We set

$$B:=\bigcup_{n\in\mathbb{N}}\bigcup_{\alpha\in\mathbb{U}}\alpha T^nx.$$

Hence $B=\mathbb{U}Orb(T,x)\setminus\mathbb{U}_x$. Using the fact that $span(x)$ is a closed subspace of H, we have $\overline{span(x)}=span(x)\neq H$. Which implies that $int(span(x))=\emptyset$. Since $\mathbb{U}_x$ is a subset of $span(x)$,

$$int(\overline{\mathbb{U}_x})=\emptyset.$$

From [1, Lemma 2.1] it follows that

$$int(\overline{\mathbb{U}_x} \cup \overline{B}) = int(\overline{B}).$$

So, we get

$$\begin{array}{rcl} H & = & int(H) \\ & = & int(\overline{\mathbb{U}_x \cup B}) \\ & = & int(\overline{\mathbb{U}_x} \cup \overline{B}) \\ & = & int(\overline{B}) \\ & \subset & H. \end{array}$$

This means that B is dense in H. The converse is obvious. □

Proposition 3.9. *Let $T \in \mathcal{BCR}(H)$. If T is codisk-cyclic, then the range of T is dense in H.*

Proof. Suppose that T is a codisk-cyclic linear relation. Then there exists a non-zero vector $x \in H$ such that $\mathbb{U}Orb(T, x)$ is dense in H. Let $y \in \mathbb{U}Orb(T, x) \setminus \mathbb{U}_x$. Then there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{U}$ such that $y \in \alpha T^n x$. This means that

$$y \in \alpha T^n x = T^n(\alpha x) \subset R(T^n) \subset R(T).$$

Therefore

$$\mathbb{U}Orb(T, x) \setminus \mathbb{U}_x \subset R(T).$$

From Lemma 3.8, we obtain that $\mathbb{U}Orb(T, x) \setminus \mathbb{U}_x$ is dense in H. Finally, we deduce that the range of T is dense in H. □

Proposition 3.10. *Let $T \in \mathcal{BCR}(H)$ and $S \in \mathbb{U}\mathcal{CR}(H)$ be such that $S(0) = ST(0)$, $TS = ST$, and the range of T is dense in H. Then*

$$Tx \subset \mathbb{U}\mathcal{CR}(S)$$

for all $x \in \mathbb{U}\mathcal{CR}(S)$.

Proof. Let $x \in \mathbb{U}\mathcal{CR}(S)$, then the set $\mathbb{U}Orb(S, x)$ is dense in H. Let $y \in Tx$, then

$$\begin{array}{rcl} STx & = & S(y + T(0)) \\ & = & Sy + ST(0) \\ & = & Sy + S(0) \\ & = & Sy. \end{array}$$

Since $TS = ST$, it follows that

$$TS^n x = S^n Tx = S^n y$$

for all $n \in \mathbb{N}$. We set $\Omega = \mathbb{U}Orb(S, x) \setminus \mathbb{U}_x$. Then by Lemma 3.8, Ω is dense in H. Hence from [12, p. 33], we obtain

$$\begin{array}{rcl} R(T) & = & T(H) \\ & = & T(\overline{\Omega}) \\ & \subset & \overline{T(\Omega)} \\ & = & \overline{T(\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha S^n x)} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} T(\alpha S^n x)} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha T S^n x} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha S^n y} \\ & \subset & \overline{\mathbb{U}Orb(S, y)} \\ & \subset & H. \end{array}$$

Since the range of T is dense in H, $\mathbb{U}Orb(S,y)$ is dense in H. We conclude that y is a codisk-cyclic vector for S. Then Tx is a subset of $\mathbb{U}\mathcal{CR}(S)$. $\square$

Theorem 3.11. *Let $T \in \mathcal{BCR}(H)$ be such that $T(0) \subset \ker(T)$. Then T is a codisk-cyclic linear relation if and only if T^p is a codisk-cyclic linear relation for all $p \in \mathbb{N}$.*

Proof. Assume that T is a codisk-cyclic linear relation. Then by Proposition 3.9, the range of T is dense in H. Since $T(0) \subset \ker(T)$, we have by [10, Proposition 3.1] that $T(0) = T^n(0)$ and $T^n \in \mathcal{BCR}(H)$ for all $n \in \mathbb{N}$. Therefore, according to Proposition 3.10, we have

$$T(\mathbb{U}\mathcal{CR}(T)) \subset \mathbb{U}\mathcal{CR}(T).$$

By induction we obtain

$$T^n(\mathbb{U}\mathcal{CR}(T)) \subset \mathbb{U}\mathcal{CR}(T) \quad \text{for all } n \in \mathbb{N}.$$

Now, we prove that T^2 is a codisk-cyclic linear relation. Indeed, since T is a codisk-cyclic linear relation, there exists $x \in H$ such that $\mathbb{U}Orb(T,x)$ is dense in H. Let $n \in \mathbb{N}$ and $y \in T^n x \subset \mathbb{U}\mathcal{CR}(T)$. Using the fact that $T(0) = T^n(0)$ and [2, Lemma 2.5], we obtain

$$\begin{aligned} T^{2n}x &= T^nT^nx \\ &= T^n(y + T^n(0)) \\ &= T^ny + T^{2n}(0) \\ &= T^ny + T^n(0) \\ &= T^ny. \end{aligned}$$

This implies that

$$\mathbb{U}Orb(T^2,x) \setminus \mathbb{U}_x = \mathbb{U}Orb(T,y) \setminus \mathbb{U}_y.$$

Since y is a codisk-cyclic vector for T, from Lemma 3.8 it follows that $\mathbb{U}Orb(T,y) \setminus \mathbb{U}_y$ is also dense in H. Therefore $\mathbb{U}Orb(T^2,x)$ is dense in H. This means that T^2 is a codisk-cyclic linear relation. By induction, we can show that T^p is a codisk-cyclic linear relation for all $p \in \mathbb{N}$. $\square$

Theorem 3.12. *Let $T,S \in \mathcal{BCR}(H)$ be such that $TST(0) = TS(0) = \overline{TS(0)}$, $STS(0) = ST(0) = \overline{ST(0)}$, and the ranges of T and S are dense in H. Then TS is codisk-cyclic if and only if ST is codisk-cyclic.*

Proof. Assume that ST is codisk-cyclic. Let $n \in \mathbb{N}$. Since $TST(0) = TS(0)$, we have $(TS)^nT(0) = (TS)^n(0)$. Since ST is codisk-cyclic, there exists $x \in H = \mathcal{D}(T)$ such that $\mathbb{U}Orb(ST,x)$ is dense in H and there exists $y \in Tx$. Therefore

$$\begin{aligned} T(ST)^nx &= (TS)^nTx \\ &= (TS)^n(y + T(0)) \\ &= (TS)^ny + (TS)^nT(0) \\ &= (TS)^ny + (TS)^n(0) \\ &= (TS)^ny. \end{aligned}$$

We set $M := \mathbb{U}Orb(ST, x) \setminus \mathbb{U}_x$. By Lemma 3.8, we obtain that M is dense in H. Furthermore, from [12, p.33] it follows that

$$\begin{array}{rcl} R(T) & = & T(H) \\ & = & T(\overline{M}) \\ & \subset & \overline{T(M)} \\ & = & \overline{T(\bigcup_{n\in\mathbb{N}}\bigcup_{\alpha\in\mathbb{U}} \alpha(ST)^n x)} \\ & = & \overline{\bigcup_{n\in\mathbb{N}}\bigcup_{\alpha\in\mathbb{U}} \alpha T(ST)^n x} \\ & = & \overline{\bigcup_{n\in\mathbb{N}}\bigcup_{\alpha\in\mathbb{U}} \alpha (TS)^n y} \\ & \subset & \overline{\mathbb{U}Orb(TS, y)} \\ & \subset & H. \end{array}$$

Since $\overline{R(T)} = H$, we see that $\mathbb{U}Orb(TS, y)$ is dense in H. This means that TS is codisk-cyclic. The converse is proved similarly. □

Let $T \in \mathcal{LR}(H)$ and $S \in \mathcal{LR}(K)$. Then the linear relation $T \oplus S$ is defined by

$$\begin{array}{cccc} T\oplus S: & \mathcal{D}(T)\oplus\mathcal{D}(S) & \longrightarrow & 2^{H\oplus K}\setminus\emptyset \\ & x\oplus y & \longmapsto & Tx\oplus Sy \end{array}$$

where $H \oplus K := \{x \oplus y := (x, y) : x \in H \text{ and } y \in K\}$ and $\mathcal{D}(T \oplus S) := \mathcal{D}(T) \oplus \mathcal{D}(S)$. For $k \in \mathbb{N}$, we then have

$$(T \oplus S)^k x \oplus y = T^k x \oplus S^k y.$$

Proposition 3.13. *Let $T \in \mathcal{BCR}(H)$ and $S \in \mathcal{BCR}(K)$. If $T \oplus S$ is codisk-cyclic, then T and S are codisk-cyclic linear relations.*

Proof. Let $y := y_H \oplus y_K \in H \oplus K$. Since $T \oplus S$ is a codisk-cyclic linear relation, there exists $x := x_H \oplus x_K \in \mathbb{U}\mathcal{CR}(H \oplus K)$ such that $\mathbb{U}Orb(T \oplus S, x)$ is dense in $H \oplus K$. Therefore, there exists $\{y_k\}$ in $\mathbb{U}Orb(T \oplus S, x)$ such that $\{y_k\}$ converges to y as $k \longrightarrow \infty$. Then, for all $k \in \mathbb{N}$ there exist $\{\alpha_k\}$ in $\mathbb{U}$ and $\{n_k\}$ in $\mathbb{N}$ such that

$$y_k \longrightarrow y \text{ with } y_k \in \alpha_k (T \oplus S)^{n_k} x.$$

Let P be the bounded projection defined on $H \oplus K$ such that $R(P) = H$. Then

$$P(y_k) \in \alpha_k T^{n_k} x_H \text{ such that } P(y_k) \longrightarrow y_H$$

and so $x_H \in \mathbb{U}\mathcal{CR}(T)$. Similarly, we prove that $x_K \in \mathbb{U}\mathcal{CR}(S)$. Therefore T and S are codisk-cyclic linear relations. □

In the following, we establish a connection between orthogonal projection and the concept of codisc-cyclic linear relations.

Lemma 3.14. [6, Lemma 4.1] *Let $T \in \mathcal{LR}(H)$ and M be a non-trivial closed subspace of H such that $T(M) \subset M$ and $T(M^\perp) \subset M^\perp$. If $P \in \mathcal{B}(H)$ is the projection onto $M^\perp$, then*

$$(TP)^n = T^n P = PT^n$$

for all $n \in \mathbb{N}$.

Proposition 3.15. *Let $T \in \mathbb{U}\mathcal{CR}(H)$ and let M be a non-trivial closed subspace of H such that $T(M) \subset M$. If P is the projection onto $M^\perp$, then*

$$Px \neq 0$$

for all $x \in \mathbb{U}\mathcal{CR}(T)$.

Proof. Let x be a codisk-cyclic vector for T. For the sake of contradiction suppose that $Px = 0$, thus $x \in M$. Since $T(M) \subset M$,

$$\begin{array}{rcl} \beta T^n x & \subset & \beta T^n M \\ & \subset & \beta M \\ & = & M \end{array}$$

for all $\beta \in \mathbb{U}$ and all $n \in \mathbb{N} \cup \{0\}$. It follows that

$$H = \overline{\bigcup_{\beta \in \mathbb{U}} \bigcup_{n \geqslant 0} \beta T^n x} \subset \overline{M} = M \subset H.$$

This is a contradiction to $M \neq H$. Therefore Px is not zero. □

Proposition 3.16. *Let $T \in \mathbb{U}\mathcal{CR}(H)$ and M be a non-trivial subspace of H such that $T(M) \subset M$ and $T(M^{\perp}) \subset M^{\perp}$. Then $T_{M^{\perp}}$ and T_M are codisk-cyclic linear relations.*

Proof. Since T is a codisk-cyclic linear relation, there exists $x \in H$ such that the set $\mathbb{U}Orb(T, x)$ is dense in H. According to Lemma 3.8, $\mathbb{U}Orb(T, x) \setminus \mathbb{U}_x$ is also dense in H. As $H = M \oplus M^{\perp}$, there exit $x_1 \in M$ and $x_2 \in M^{\perp}$ such that $x = x_1 + x_2$. Now, let P be the bounded projection P onto $M^{\perp}$. Then $Px = x_2$. Using Lemma 3.14, we obtain $(TP)^n = T^n P = PT^n$ for all $n \in \mathbb{N}$. Since P is bounded, we then have

$$\begin{array}{rcl} M^{\perp} = P(H) & = & P(\overline{\mathbb{U}Orb(T, x) \setminus \mathbb{U}_x}) \\ & \subset & \overline{P(\mathbb{U}Orb(T, x) \setminus \mathbb{U}_x)} \\ & = & \overline{P(\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha T^n x)} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha P T^n x} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha T^n P x} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha (TP)^n x_2} \\ & = & \overline{\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \in \mathbb{N}} \alpha T^n_{M^{\perp}} x_2} \\ & \subset & \overline{\mathbb{U}Orb(T_{M^{\perp}}, x_2)} \\ & \subset & \overline{M^{\perp}} \\ & = & M^{\perp}. \end{array}$$

Finally, we can say that $T_{M^{\perp}}$ is a codisk-cyclic linear relation. Similarly, we obtain T_M is a codisk-cyclic linear relation. □

4. Codisk Transitive linear relations

In this section, we introduce and investigate the concept of a codisk transitive linear relation.

Definition 4.1. Let $T \in \mathcal{BCR}(H)$. We say that T is *codisk transitive*, if for any pair (U, V) of non-empty open subsets of H, there exist $\alpha \in \mathbb{U}$ and $n \geqslant 0$ such that $\alpha T^n(U) \cap V \neq 0$.

Example 4.2. Let $T \in \mathcal{BCR}(H)$ and let A be a selection of T. If A is a codisk transitive operator, then T is a codisk transitive linear relation. Indeed, since A is a codisk transitive linear operator, for any two non-empty open sets U and V of H there exist $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{U}$ such that

$$\alpha A^n(U) \cap V \neq \emptyset.$$

Therefore there exists $y \in \alpha A^n(U) \cap V$. Hence $y = \alpha A^n x$, for some $x \in U$. By (2.1) we obtain

$$\begin{array}{rcl} y & = & \alpha A^n x \\ & \in & \alpha T^n x \\ & \subset & \alpha T^n U. \end{array}$$

As a result, $\alpha T^n(U)\cap V\neq\emptyset$. Then T is a codisk transitive linear relation.

Proposition 4.3. *Let $T\in\mathcal{BCR}(H)$, $S\in\mathcal{BCR}(K)$, and $A\in\mathcal{B}(H,K)$ be such that $SA=AT$ and the range of A is dense in K. If T is a codisk transitive linear relation, then S is a codisk transitive linear relation.*

Proof. Let U and V be two non-empty open subsets of K. Since A is bounded and has dense range, $A^{-1}(U)$ and $A^{-1}(V)$ are two non-empty open subsets of H. Since T is a codisk transitive linear relation, there exist $n\in\mathbb{N}\cup\{0\}$ and $\beta\in\mathbb{U}$ such that

$$\beta T^nA^{-1}(U)\cap A^{-1}(V)\neq\emptyset.$$

So there exist $y\in A^{-1}(V)$ and $x\in A^{-1}(U)$ such that $y\in\beta T^nx$. From equality $SA=AT$, we have

$$\begin{array}{rcl}\beta S^nAx&=&\beta AT^nx\\&=&A(\beta T^nx)\\&=&A(y+\beta T^n(0))\\&=&Ay+\beta AT^n(0)\\&=&Ay+\beta S^n(0).\end{array}$$

Thus, $Ay\in\beta S^nAx\subset\beta S^n(U)$ and $Ay\in V$. Hence

$$\beta S^n(U)\cap V\neq\emptyset.$$

Finally, we conclude that S is a codisk transitive linear relation. □

Corollary 4.4. *Let $T\in\mathcal{BCR}(H)$, $S\in\mathcal{BCR}(K)$, and $A\in\mathcal{B}(H,K)$ be such that $SA=AT$ and A is bijective. Then T is a codisk transitive linear relation if and only if S is a codisk transitive linear relation.*

In the sequel, the closed unit disk in $\mathbb{C}$ is denoted by $\mathbb{D}$. The following theorem gives a characterization of a codisk transitive linear relation.

Theorem 4.5. *Let $T\in\mathcal{BCR}(H)$. Then the following assertions are equivalent.*

i) *T is codisk transitive.*

ii) *For each pair (U,V) of non-empty open subsets of H, there exist $\alpha\in\mathbb{D}\setminus\{0\}$ and $n\geqslant 0$ such that*

$$\alpha T^{-n}(U)\cap V\neq\emptyset.$$

iii) *For any non-empty open subset U of H, the set*

$$\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha T^n(U)$$

is dense in H.

iv) *For any non-empty open subset U of H, the set*

$$\bigcup_{\alpha\in\mathbb{D}\setminus\{0\}}\bigcup_{n\geqslant 0}\alpha T^{-n}(U)$$

is dense in H.

Proof. *i)* $\Longrightarrow$ *ii)*: Since T is codisk transitive, for any pair (U,V) of non-empty open subsets of H there exist $\beta\in\mathbb{U}$ and $n\geqslant 0$ such that $\beta T^n(U)\cap V\neq\emptyset$. Therefore

$$(\beta U+T^{-n}(0))\cap T^{-n}(V)\neq\emptyset.$$

Let $x\in(\beta U+T^{-n}(0))\cap T^{-n}(V)$. Then there exist $u\in U$, $y\in T^{-n}(0)$, and $v\in V$ such that $x=\beta u+y$ and $x\in T^{-n}(v)$. Hence

$$\begin{array}{rcl}T^{-n}(v)&=&x+T^{-n}(0)\\&=&\beta u+y+T^{-n}(0)\\&=&\beta u+T^{-n}(0).\end{array}$$

Thus $\beta u \in T^{-n}(v)$. We obtain $u \in \alpha T^{-n}(V) \cap U$ with $|\alpha| := |\frac{1}{\beta}| \leqslant 1$. Finally $\alpha T^{-n}(V) \cap U \neq \emptyset$ and $\alpha \in \mathbb{D} \setminus \{0\}$.

ii) $\Longrightarrow$ *i)*. This is similar to *i)* implies *ii)*.

i) $\Longleftrightarrow$ *iii)*. Let U be an open non-empty subset of H and $(O_i)_{i\in\mathbb{N}}$ be a countable basis of open sets of H. Since T is a codisk transitive linear relation, for each $i \in \mathbb{N}$ we can find $n_i \geqslant 0$ and $\alpha_i \in \mathbb{U}$ such that $\alpha_i T^{n_i}(U) \cap O_i \neq \emptyset$. It follows that the set

$$\bigcup_{\alpha\in\mathbb{U}} \bigcup_{n\geqslant 0} \alpha T^n(U)$$

is dense in H. Conversely, Let U and V be two open non-empty subsets of H. Since the set

$$\bigcup_{\alpha\in\mathbb{U}} \bigcup_{n\geqslant 0} \alpha T^n(U)$$

is dense in H, there exist $\alpha \in \mathbb{U}$ and $n \geqslant 0$ such that $\alpha T^n(U) \cap V \neq 0$. Therefore T is a codisk transitive linear relation.

ii) $\Longleftrightarrow$ *iv)*: This is similar to *i)* $\Longleftrightarrow$ *iii)*. □

In the sequel, the open ball centred at $x \in H$ and with radius $r > 0$ is denoted by $B(x, r)$.

Theorem 4.6. *Let $T \in \mathcal{BCR}(H)$. Then the following assertions are equivalent.*

i) *T is a codisk transitive linear relation.*
ii) *For each $x, y \in H$ there exist sequences $\{n_k\}$ in $\mathbb{N}$, $\{x_k\}$ in H, $\{\alpha_k\}$ in $\mathbb{U}$, and $\{y_k\}$ in H such that*

$$x_k \longrightarrow x,\ y_k \longrightarrow y \quad \textit{and} \quad \alpha_k T^{n_k} x_k = y_k + T^{n_k}(0).$$

iii) *For each $(x, y) \in H \oplus H$ and for each neighbourhood W of 0 there exist $z, t \in H$, $\alpha \in \mathbb{U}$, and $n \in \mathbb{N}$ such that*

$$x - z \in W,\ t - y \in W \quad \textit{and} \quad \alpha T^n z = t + T^n(0).$$

Proof. *i)* $\Longrightarrow$ *ii)* Let $x, y \in H$ and let $B_k := B(x, \frac{1}{k})$, $B_k^{'} := B(y, \frac{1}{k})$ for all $k \in \mathbb{N}$. Then B_k and $B_k^{'}$ are non-empty open subsets of H. Since T is a codisk transitive linear relation, there exist two sequences $\{\alpha_k\}$ in $\mathbb{U}$ and $\{n_k\}$ in $\mathbb{N}$ such that $T^{n_k}(\alpha_k B_k) \cap B_k^{'} \neq \emptyset$ for all $k \geqslant 1$. Then there exists a sequence $\{y_k\}$ in H such that

$$y_k \in T^{n_k}(\alpha_k B_k) \cap B_k^{'}$$

for all $k \geqslant 1$. Consequently, there exists a sequence $\{x_k\}$ in B_k such that $y_k \in T^{n_k}(\alpha_k x_k) \cap B_k^{'}$ for all $k \geqslant 1$. We then have

$$\begin{aligned} \alpha_k T^{n_k} x_k &= y_k + \alpha_k T^{n_k}(0) \\ &= y_k + T^{n_k}(0). \end{aligned}$$

Furthermore,

$$\|x_k - x\| < \frac{1}{k} \quad \text{and} \quad \|y_k - y\| < \frac{1}{k}$$

for all $k \geqslant 1$. Therefore $x_k \longrightarrow x$ and $y_k \longrightarrow y$ as $k \longrightarrow \infty$.

ii) $\Longrightarrow$ *iii)* Suppose that for each $(x, y) \in H \times H$ there exist sequences $\{n_k\}$ in $\mathbb{N}$, $\{x_k\}$ in H, $\{\alpha_k\}$ in $\mathbb{U}$, and $\{y_k\}$ in H such that $x_k \longrightarrow x$, $y_k \longrightarrow y$ and $\alpha_k T^{n_k} x_k = y_k + T^{n_k}(0)$. Let W be a neighbourhood of zero. Hence there exists $k_0 \in \mathbb{N}$ such that $x - x_k \in W$ and $y_k - y \in W$, for all $k \geqslant k_0$. If we take $z := x_{k_0}$ and $t := y_{k_0}$, we then have

$$x - z \in W,\ t - y \in W \text{ and } \alpha_{k_0} T^{n_{k_0}} z = t + T^{n_{k_0}}(0).$$

$iii) \Longrightarrow i)$ Let U and V be two non-empty open subsets of H. Let $(x,y) \in U \oplus V$. For all $k \geqslant 1$, $W_k := B(0, \frac{1}{k})$ is a neighbourhood of zero. By assumption there exist sequences $\{x_k\}$ in H, $\{\alpha_k\}$ in $\mathbb{U}$, $\{n_k\}$ in $\mathbb{N}$, and $\{y_k\} \subset H$ such that

$$\|x_k - x\| < \frac{1}{k}, \ \|y_k - y\| < \frac{1}{k} \ \text{ and } \ y_k \in \alpha_k T^{n_k} x_k.$$

Then $\{x_k\}$ converges to x and $\{y_k\}$ converges to y as $k \longrightarrow \infty$. Using the fact that U and V are two non-empty open subsets of H such that $(x,y) \in U \times V$, we see that for k large enough $x_k \in U$ and $y_k \in V$. Therefore

$$\emptyset \neq \alpha_k T^{n_k} x_k \cap V \subset \alpha_k T^{n_k} U \cap V.$$

This shows that T is a codisk transitive linear relation. □

Proposition 4.7. *Let $T \in \mathcal{BCR}(H)$. Then T is a codisk-transitive linear relation if and only if*

$$\mathbb{UCR}(T) = \bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(V_k)$$

is a dense G_δ-set in H, where $(V_k)_{k\in\mathbb{N}}$ is a countable basis of open subsets of H.

Proof. Let $(V_k)_{k\in\mathbb{N}}$ be a countable basis of open subsets of H. Then

$$\begin{aligned} x \in \mathbb{UCR}(T) &\iff \forall k \geqslant 1,\ V_k \cap (\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{U}} \alpha T^n x) \neq \emptyset \\ &\iff \forall k \geqslant 1,\ \exists \beta \in \mathbb{U},\ \exists n \geqslant 0 \ \text{ such that } \ V_k \cap \beta T^n x \neq \emptyset \\ &\iff \forall k \geqslant 1,\ \exists \beta \in \mathbb{U},\ \exists n \geqslant 0 \ \text{ such that } \ \beta x \in T^{-n}(V_k) \\ &\iff \forall k \geqslant 1,\ \exists \alpha \in \mathbb{D}\setminus\{0\},\ \exists n \geqslant 0 \ \text{ such that } \ x \in \alpha T^{-n}(V_k) \\ &\iff x \in \bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(V_k) \end{aligned}$$

Let T be a codisk transitive linear relation. We prove that $\mathbb{UCR}(T)$ is dense in H. Indeed, let $k \geqslant 1$ and

$$O_k := \bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(V_k).$$

By Theorem 4.5, it follows that O_k is dense in H. Since O_k is an open set of H (see [1, Remark 2.2]), by the Baire category theorem we obtain $\bigcap_{k\in\mathbb{N}} O_k = \mathbb{UCR}(T)$ is G_δ-set dense in H.

Conversely, let U and V be two non-empty open subsets of H. Since $(V_k)_{k\in\mathbb{N}}$ is a countable basis of open subsets of H and $\bigcap_{k\in\mathbb{N}} O_k = \mathbb{UCR}(T)$ is dense in H, we see that $U = \bigcup_{k\in I} V_k$ with $I \subset \mathbb{N}$ and $\bigcap_{k\in\mathbb{N}} O_k \cap V \neq \emptyset$. Therefore, for all $k \in \mathbb{N}$ we get $O_k \cap V \neq \emptyset$. For $i \in I$ we have

$$\begin{aligned} \emptyset \neq O_i \cap V &= (\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(V_i)) \cap V \\ &\subset (\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(\bigcup_{k\in I} V_k)) \cap V \\ &= (\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(U)) \cap V. \end{aligned}$$

Then $\left(\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(U)\right) \cap V \neq \emptyset$ for all non-empty open subset V of H. We deduce that $\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}\setminus\{0\}} \alpha T^{-n}(U)$ is dense in H. Finally, by virtue of Theorem 4.5, T is a codisk linear relation. □

In the following, we prove the equivalence between a codisk-cyclic liner relation and a codisk transitive linear relation.

Theorem 4.8. *Let $T \in \mathcal{BCR}(H)$. Then T is a codisk transitive linear relation if and only if T is a codisk-cyclic linear relation.*

Proof. Suppose that T is codisk transitive. Then by Proposition 4.7 it follows that the set $\mathbb{U}\mathcal{CR}(T)$ is dense in H. Hence $\mathbb{U}\mathcal{CR}(T)$ is a non-empty set of H and so T is a codisk-cyclic linear relation.

Conversely, assume that T is a codisk-cyclic linear relation. Then there exists a vector x in H such that the set $\mathbb{U}Orb(T,x)$ is dense in H. Let U and V be two non-empty open sets of H. We have

$$\mathbb{U}Orb(T,x)\cap U \neq \emptyset \quad \text{and} \quad \mathbb{U}Orb(T,x)\cap V \neq \emptyset.$$

Therefore there exist $m, n \geqslant 0$ and $\alpha, \beta \in \mathbb{U}$ such that

$$\alpha T^n x \cap U \neq \emptyset \quad \text{and} \quad \beta T^m x \cap V \neq \emptyset,$$

so, there exist two elements y_1 and y_2 such that $y_1 \in \alpha T^n x \cap U$ and $y_2 \in \beta T^m x \cap V$. It is clear that without loss of generality we can assume that $n \geqslant m$. Set $p := n - m \geq 0$ and $\gamma := \frac{\alpha}{\beta}$. We therefore have

$$\begin{array}{rcl} y_2 \in \beta T^m x & \Longleftrightarrow & y_2 \in T^m(\beta x) \\ & \Longleftrightarrow & (\beta x, y_2) \in G(T^m) \\ & \Longleftrightarrow & (y_2, \beta x) \in G((T^m)^{-1}) \\ & \Longleftrightarrow & (y_2, \beta x) \in G((T^{-m})) \\ & \Longleftrightarrow & \beta x \in T^{-m} y_2 \\ & \Longleftrightarrow & x \in \dfrac{1}{\beta} T^{-m} y_2. \end{array}$$

Hence

$$\begin{array}{rcl} y_1 \in \alpha T^n x & \subset & \dfrac{\alpha}{\beta} T^{n-m} y_2 \qquad \text{and} \qquad (y_1, y_2) \in U \oplus V \\ & \subset & \dfrac{\alpha}{\beta} T^{n-m}(V) \\ & = & \gamma T^p(V) \end{array}$$

which implies that $\gamma T^p(V) \cap U \neq \emptyset$. So, we distinguish two cases:

First case: $|\alpha| \leqslant |\beta|$. Then we obtain

$$\gamma T^p(V) \cap U \neq \emptyset \text{ with } p \in \mathbb{N} \cup \{0\} \text{ and } \gamma \in \mathbb{D} \setminus \{0\}.$$

Therefore

$$\emptyset \neq \gamma T^p(V) \cap U \subset \gamma \left(\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \geqslant 0} \alpha T^n(V) \right) \cap U.$$

Thus

$$\gamma \left(\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \geqslant 0} \alpha T^n(V)) \right) \cap U \neq \emptyset$$

for any non-empty open subset U of H. Which implies that the set

$$M := \gamma(\bigcup_{\alpha \in \mathbb{U}} \bigcup_{n \geqslant 0} \alpha T^n(V))$$

is dense in H. Now, we consider the map f_γ defined by

$$\begin{array}{rccc} f_\gamma : & H & \longrightarrow & H \\ & x & \longmapsto & \dfrac{1}{\gamma} x \end{array}$$

Clearly, f_γ is a homeomorphism. Using the fact that f_γ is closed we see that

$$\begin{array}{rcl}\overline{\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha T^n(V)} & = & \overline{\frac{1}{\gamma}M} \\ & = & \overline{f_\gamma(M)} \\ & = & f_\gamma(\overline{M}) \\ & = & f_\gamma(H) \\ & = & \dfrac{1}{\gamma}H \\ & = & H.\end{array}$$

Therefore the set $\bigcup_{\alpha\in\mathbb{U}}\bigcup_{n\geqslant 0}\alpha T^n(V)$ is dense in H. Then by Theorem 4.5 we deduce that T is a codisk transitive linear relation.

Second case: $|\beta| \leq |\alpha|$. Since $\gamma = \frac{\alpha}{\beta}$, we get

$$\gamma T^p(V) \cap U \neq \emptyset \text{ with } p \in \mathbb{N}\cup\{0\} \text{ and } \gamma \in \mathbb{U}.$$

Finally, we conclude that T is a codisk transitive linear relation. □

Lemma 4.9. *Let A and B be two non-empty subsets of H such that $\overline{A} = H$ and $\overline{B} \neq H$. Then there exists $x \in A$ such that $x \notin \overline{B}$.*

Proposition 4.10. *Let $T \in \mathbb{U}\mathcal{CR}(H)$. Then the range of $T - \lambda I$ is dense in H, for every $\lambda \in \mathbb{U}\cup\{0\}$.*

Proof. If $\lambda = 0$, then by Proposition 3.9, $R(T)$ is dense in H. Now, let $\lambda \in \mathbb{U}$. For the sake of contradiction assume that $R(T-\lambda I)$ is not dense in H. Since T is codisk-cyclic, we have by Theorem 4.8 and Proposition 4.7 that the set $\mathbb{U}\mathcal{CR}(T)$ is dense in H. By Lemma 4.9, there exists $x \in \mathbb{U}\mathcal{CR}(T)$ such that $x \notin \overline{(T-\lambda I)H}$. By the Hahn-Banach theorem, there exists a continuous linear functional S on H such that $Sx \neq 0$ and $S(\overline{R(T-\lambda I)}) = \{0\}$. In particular, $S(R(T-\lambda I)) = \{0\}$. From [18, Lemma 4.2], we obtain $R(T^n - \lambda^n I) \subset R(T-\lambda I)$ for all $n \in \mathbb{N}$. Hence

$$ST^n y = \lambda^n Sy \tag{4.2}$$

for all $n \in \mathbb{N}$ and all $y \in H$. Since $\mathbb{U}Orb(T,x)$ is dense in H, there exists $\{x_k\}$ in $\mathbb{U}Orb(T,x)$ such that $\{x_k\}$ converges to $\frac{1}{2}x$. Hence $Sx_k \longrightarrow \frac{1}{2}Sx$ as $k \longrightarrow \infty$. For each $k \in \mathbb{N}$, there exists n_k in $\mathbb{N}$ and α_k in $\mathbb{U}$ such that $x_k \in \alpha_k T^{n_k}x = T^{n_k}(\alpha_k x)$. Using Equality (4.2) and $T^{n_k}\alpha_k x = x_k + T^{n_k}(0)$ we get

$$\begin{array}{rcl}Sx_k & = & S(\alpha_k T^{n_k}x) \\ & = & \alpha_k ST^{n_k}x \\ & = & \alpha_k \lambda^{n_k} Sx.\end{array}$$

Thus $\alpha_k\lambda^{n_k}Sx \longrightarrow \frac{1}{2}Sx$ since $|\alpha_k\lambda^{n_k}| \geqslant 1$ and $Sx \neq 0$. Thus $|\alpha_k\lambda^{n_k}| \longrightarrow \frac{1}{2} \geqslant 1$ as $k \longrightarrow \infty$, which is a contradiction. Finally, we deduce that the range of $T - \lambda I$ is dense in H. □

Let $T \in \mathcal{LR}(H)$. The *point spectrum* of T, denoted by $\sigma_p(T)$, is defined by

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\}.$$

As an immediate consequence of the previous results, we obtain the following.

Corollary 4.11. *Let $T \in \mathbb{U}\mathcal{CR}(H)$. Then*

$$\sigma_p(T^*) \subset \mathbb{C} \setminus (\mathbb{U}\cup\{0\}).$$

Proof. Assume that $\sigma_p(T^*)$ is a non-empty subset of $\mathbb{C}$. Let $\lambda \in \mathbb{U} \cup \{0\}$. Then, from Proposition 4.10, it follows that $R(T-\lambda I)$ is dense in H. Consequently

$$\begin{aligned} \ker(T-\lambda I)^* &= R(T-\lambda I)^{\perp} \\ &= \overline{R(T-\lambda I)}^{\perp} \\ &= H^{\perp} \\ &= \{0\}. \end{aligned}$$

Furthermore, as λI is a bounded linear operator, we have

$$\ker(T-\lambda I)^* = \ker(T^* - \overline{\lambda} I) = \{0\}.$$

This implies that $\overline{\lambda} \notin \sigma_p(T^*)$. Since $\lambda \in \mathbb{U}$ is equivalent to $\overline{\lambda} \in \mathbb{U}$, we get $\lambda \notin \sigma_p(T^*)$. Thus $\sigma_p(T^*)$ is a subset of $\mathbb{C} \setminus \mathbb{U}$. ☐

From Theorem 3.12 and Theorem 4.8, we obtain the following corollary.

Corollary 4.12. *Let $T, S \in \mathcal{BCR}(H)$ be such that $TST(0) = TS(0) = \overline{TS(0)}$, $STS(0) = ST(0) = \overline{ST(0)}$, and the ranges of T and S are dense in H. Then TS is codisk transitive if and only if ST is codisk transitive.*

5. Codisk-cyclic criterion

This section presents two criteria for establishing the codisk-cyclicity of a linear relation.

Definition 5.1. Let $T \in \mathcal{BCR}(H)$. We say that T satisfies the *codisk-cyclicity criterion* if there exist two dense subsets X and Y of H and an increasing sequence of positive integers $\{n_k\}$, a sequence $\{\alpha_{n_k}\}$ in $\mathbb{U}$ and a sequence of maps $S_{n_k} : Y \longrightarrow H$ such that:

a) For each $x \in X$ there exists $x_{n_k} \in T^{n_k}x$ for each $k \in \mathbb{N}$ such that $\alpha_{n_k}x_{n_k} \longrightarrow 0$.
b) $\alpha_{n_k}^{-1} S_{n_k} y \longrightarrow 0$ for all $y \in Y$.
c) For each $y \in Y$ there exists $y_{n_k} \in T^{n_k} S_{n_k} y$ for each $k \in \mathbb{N}$ such that $y_{n_k} \longrightarrow y$.

Theorem 5.2. *Let $T \in \mathcal{BCR}(H)$. If T satisfies the codisk-cyclicity criterion, then T is codisk-cyclic.*

Proof. Let U and V be two non-empty open sets in H. Since T satisfies the codisk-cyclicity linear relation criterion, there exist two dense subsets X and Y of H, and three sequences $\{n_k\}$, $\{\alpha_{n_k}\}$ and $S_{n_k} : Y \longrightarrow H$ that satisfy the conditions in Definition 5.1. As X and Y two dense sets in H, it follows that

$$U \cap X \neq \emptyset \quad \text{and} \quad V \cap Y \neq \emptyset.$$

Let $x \in U \cap X$ and $y \in V \cap Y$. Now, we consider a sequence $\{z_k\}$ defined by

$$z_k := x + \alpha_{n_k}^{-1} S_{n_k} y, \quad \text{for all } k \geqslant 1.$$

Since $\{\alpha_{n_k}^{-1} S_{n_k} y\}$ converges to 0, $\{z_k\}$ converges to x. Using the fact that $x \in U$ and U is open, we see that there exists $N \in \mathbb{N}$ such that $z_k \in U$, for all $k \geqslant N$.

By assumption, there exists $x_{n_k} \in T^{n_k}x$ and $y_{n_k} \in T^{n_k} S_{n_k} y$ for all $k \in \mathbb{N}$ such that

$$\alpha_{n_k} x_{n_k} \longrightarrow 0 \quad \text{and} \quad y_{n_k} \longrightarrow y.$$

Let $\{a_k\}$ be a sequence defined by

$$a_k := \alpha_{n_k} x_{n_k} + y_{n_k}, \quad \text{for all } k \in \mathbb{N}.$$

Therefore $\{a_k\}$ converges to y. Since $y \in V$ and V is an open subset of H, there exists $N^{'} \in \mathbb{N}$ such that $a_k \in V$ for all $k \geqslant N^{'}$. Furthermore, for all $k \geqslant N$, we then have

$$\begin{array}{rcl} a_k &=& \alpha_{n_k} x_{n_k} + y_{n_k} \quad \in \quad \alpha_{n_k} T^{n_k} x + T^{n_k} S_{n_k} y \\ && \quad = \quad \alpha_{n_k}(T^{n_k} x + \alpha_{n_k}^{-1} T^{n_k} S_{n_k} y) \\ && \quad = \quad \alpha_{n_k}(T^{n_k} x + T^{n_k}(\alpha_{n_k}^{-1} S_{n_k} y)) \\ && \quad = \quad \alpha_{n_k} T^{n_k}(x + \alpha_{n_k}^{-1} S_{n_k} y) \\ && \quad = \quad \alpha_{n_k} T^{n_k}(z_k) \\ && \quad \subset \quad \alpha_{n_k} T^{n_k}(U). \end{array}$$

We set $n_0 := \max(N, N^{'})$ and obtain

$$a_k \in V \;\text{ and }\; a_k \in \alpha_{n_k} T^{n_k}(U), \;\text{for all}\; k \geqslant n_0.$$

Therefore $\alpha_{n_k} T^{n_k}(U) \cap V \neq \emptyset$ and so T is codisk transitive. Now by Theorem 4.8, we deduce that T is a codisk-cyclic linear relation. □

Theorem 5.3. *Let $T \in \mathcal{BCR}(H)$. If for any two non-empty open sets U and V of H and for each neighbourhood W of zero, in H, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{U}$ such that*

$$\alpha T^n(U) \cap W \neq \emptyset \;\text{ and }\; \alpha T^n(W) \cap V \neq \emptyset,$$

then T is a codisk-cyclic linear relation.

Proof. Let $x, y \in H$. For each $k \in \mathbb{N}$, let $U_k = B(x, \frac{1}{k})$, $V_k = B(y, \frac{1}{k})$, and $W_k = B(0, \frac{1}{k})$. Hence by assumption, for all $k \in \mathbb{N}$ there exist two sequences $\{n_k\}$ in $\mathbb{N}$ and $\{\alpha_k\}$ in $\mathbb{U}$ such that

$$\alpha_k T^{n_k}(U_k) \cap W_k \neq \emptyset \;\text{ and }\; \alpha_k T^{n_k}(W_k) \cap V_k \neq \emptyset.$$

Therefore there exist two sequence $\{b_k\}$ in W_k and $\{b_k^{'}\}$ in V_k such that $b_k \in T^{n_k}(U_k)$ and $b_k^{'} \in T^{n_k}(W_k)$ for all $k \in \mathbb{N}$. Hence, there exist two sequence $\{a_k\}$ in U_k and $\{a_k^{'}\}$ in W_k such that

$$b_k \in T^{n_k}(a_k) \;\text{ and }\; b_k^{'} \in T^{n_k}(a_k^{'}), \;\text{ for all }\; k \in \mathbb{N}.$$

Now consider two sequences x_k and y_k which are defined by $x_k := a_k + a_k^{'}$ and $y_k := b_k + b_k^{'}$, for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Hence

$$\begin{array}{rcl} T^{n_k}(x_k) &=& T^{n_k}(a_k + a_k^{'}) \\ &=& T^{n_k}(a_k) + T^{n_k}(a_k^{'}) \\ &=& b_k + T^{n_k}(0) + b_k^{'} + T^{n_k}(0) \\ &=& b_k + b_k^{'} + T^{n_k}(0) \\ &=& y_k + T^{n_k}(0) \end{array}$$

and

$$\begin{array}{rcl} \|x_k - x\| &=& \|a_k + a_k^{'} - x\| \\ &\leqslant& \|a_k - x\| + \|a_k^{'}\| \\ &\leqslant& \frac{1}{k} + \frac{1}{k} \\ &\leqslant& \frac{2}{k}. \end{array}$$

Similarly, we obtain $\|y_k - x\| \leqslant \frac{2}{k}$, which implies that

$$x_k \longrightarrow x \;\text{ and }\; y_k \longrightarrow y \;\text{ as }\; k \longrightarrow \infty.$$

As a result, according to condition *ii*) in Theorem 4.6, we can say that T is a codisk transitive linear relation. Finally, by Theorem 4.8, T is a codisk-cyclic linear relation. □

Proposition 5.4. *Let $T \in \mathcal{BCR}(H)$. If T satisfies the codisk-cyclicity criterion, then T satisfies the conditions of Theorem 5.3 and so is codisk-cyclic.*

Proof. Let U and V be two non-empty open sets in H, and W be a neighbourhood of zero in H. Assume that T satisfies the criterion of codisk-cyclicity. Hence there exist two dense subsets X, Y of H such that $U \cap X \neq \emptyset$ and $V \cap Y \neq \emptyset$, and there exist an increasing sequence of positive integers $\{n_k\}$, a sequence $\{\alpha_{n_k}\}$ of $\mathbb{U}$, and a sequence of maps $S_{n_k} : Y \longrightarrow H$ provided that

a) for each $x \in U \cap X$ there exists $x_{n_k} \in T^{n_k}x$ for each $k \in \mathbb{N}$ such that $\alpha_{n_k} x_{n_k} \longrightarrow 0$;
b) $\alpha_{n_k}^{-1} S_{n_k} y \longrightarrow 0$ for all $y \in Y \cap V$;
c) for each $y \in Y \cap V$ there exist $y_{n_k} \in T^{n_k} S_{n_k} y$ for each $k \in \mathbb{N}$ such that $y_{n_k} \longrightarrow y$.

Now, let $x \in U \cap X$. Then there exists $x_{n_k} \in T^{n_k}x$ for each $k \in \mathbb{N}$ such that $\{\alpha_{n_k} x_{n_k}\}$ converges to 0. Therefore there exists $m \in \mathbb{N}$ such that $\alpha_{n_k} x_{n_k} \in W$ for every $k \geqslant m$. Moreover, since $x \in U$ and $\alpha_{n_k} x_{n_k} \in \alpha_{n_k} T^{n_k} x$, we have

$$\alpha_{n_k} x_{n_k} \in \alpha_{n_k} T^{n_k}(U) \cap W, \text{ for all } k \geqslant m.$$

Similarly, let $y \in Y \cap V$. Then there exists $y_{n_k} \in T^{n_k} S_{n_k} y$ for each $k \in \mathbb{N}$ and $y_{n_k} \longrightarrow y$. Hence there exists m_1 in $\mathbb{N}$ such that $\alpha_{n_k}^{-1} S_{n_k} y \in W$ for every $k \geqslant m_1$. Since $y \in V$, $y_{n_k} \longrightarrow y$, and V is open, there exists m_2 in $\mathbb{N}$ such that $y_{n_k} \in V$ for every $k \geqslant m_2$. We take $k' := \max(m_1, m_2)$. Then for each $k \geqslant k'$

$$y_{n_k} \in T^{n_k} S_{n_k} y = \alpha_{n_k} T^{n_k}(\alpha_{n_k}^{-1} S_{n_k} y) \subset \alpha_{n_k} T^{n_k}(W) \text{ and } y_{n_k} \in V$$

Now we set $p = max(m, k')$. Then

$$\alpha_{n_p} T^{n_p}(U) \cap W \neq \emptyset \quad \text{and} \quad \alpha_{n_p} T^{n_p}(W) \cap V \neq \emptyset.$$

Therefore T satisfies the conditions of Theorem 5.3. □

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